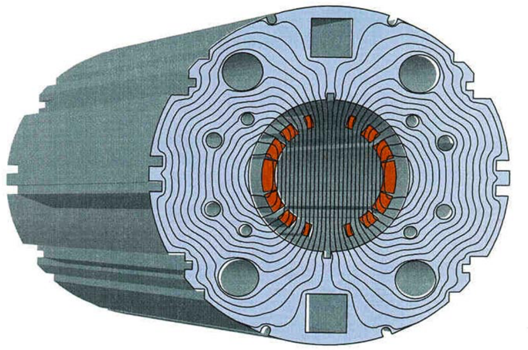
Homework 5. **Sin and Cos coils**

Majority of superconducting magnets and may air-coil magnets using coils with current distributions approximating **sin(nθ)** or **cos(nθ)** to generate n-th multipole: **sin(θ)** or **cos(θ)** for dipole field, : **sin(3θ)** or **cos(3θ)** for quadrupoles and skew-quadrupole, etc…



Typical super-conducting dipole magnet with **cos(θ)** coil (red). Current distribution approximates the **cos(θ)** with as many as possible high harmonics (sextupole, etc.) being compensated.

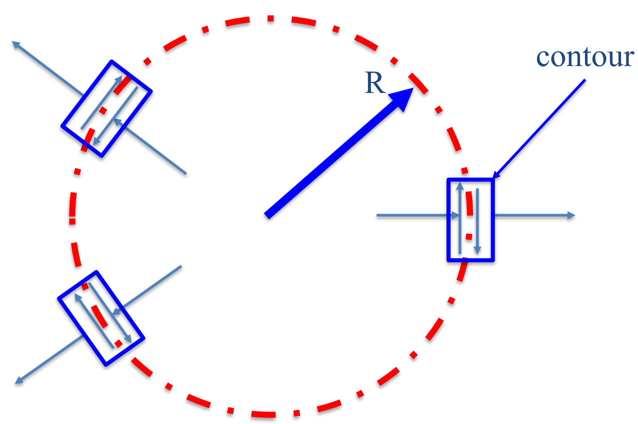
Problems:

(A) 5 points: For a long air-coil magnet with coil located at ρ=R and having current distribution

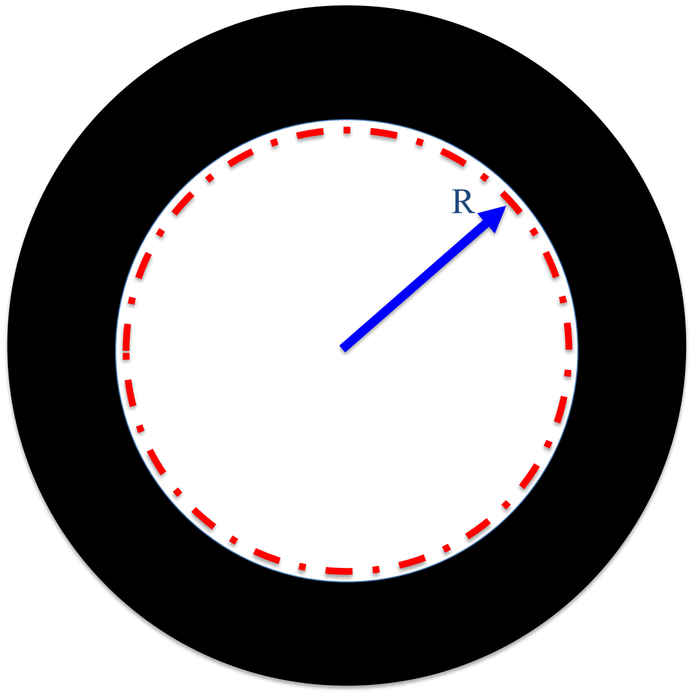
$$\vec{j} = \delta(\rho - R)\,\mathrm{Re}\left(\frac{I_o}{2\pi R} e^{in\theta}\right)$$

where is $I_o = |I_o| e^{i\phi}$ generally speaking a complex number, find magnetic fields inside and outside the coil.

Hint: (1) use fact that fields are finite both inside and outside the coil; (2) use continuation of the radial component (explain why it is not changing?) and infinitesimal contour about the coil to connect amplitude of the field with the value of the current ***Io***.



(B) 5 points: Calculate fields inside the magnet for the same coil encapsulated into a perfect magnetic yoke with infinite permeability (shown as black cylinder surrounding the coil). Show that strength of magnetic field doubles when compared with air coil.



Hint: Use contour going through magnetic yoke when H=0.

(C) 5 points. Calculate ratio between the total current in the coil***, Io***. (Note that you should integrate absolute value of the current in the coil to get the total coil current) and magnetic field for dipole and gradient for quadrupole current distributions with coil encapsulated in the magnetic core (B). What current in the coil is needed for LHC type dipole with 8 T magnetic field and coil radius of 2.8 cm?

Hint: neglect finite permeability of the magnetic core.

Solutions:

As we discussed in class, by the nature of cylindrical system, it is natural that angular harmonics of current will excite the same harmonic in magnetic field:

$$\vec{j} = \delta(\rho - R)\,\mathrm{Re}\left(\frac{I}{4R}e^{in\theta}\right) \rightarrow \vec{B} = \mathrm{Re}\left(\vec{B}(\rho)e^{in\theta}\right) \quad (1)$$

Outside the coil the field satisfied equations in vacuum and we already know solutions:

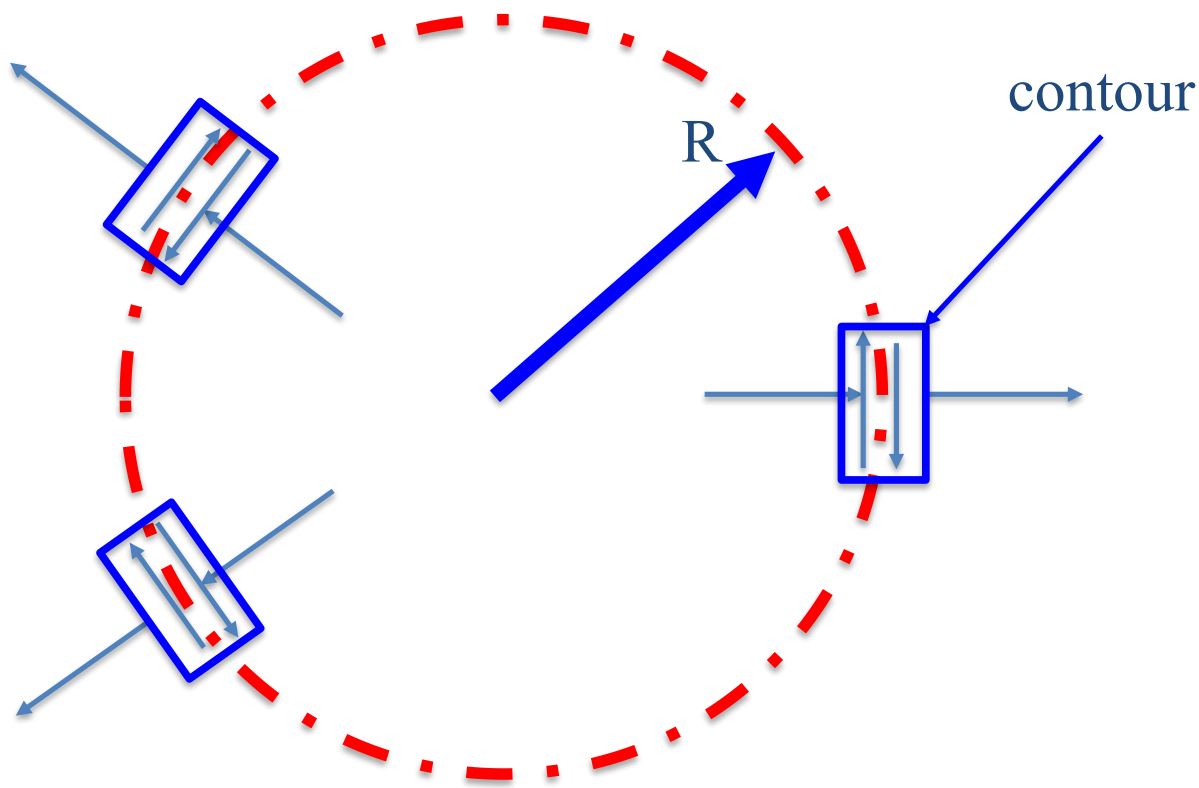
$$B_n = \mathrm{Re}\left(a_n\rho^n + \frac{b_n}{\rho^n}\right)e^{in\theta} \rightarrow \vec{H} = \vec{\nabla}\varphi_n = n\cdot\mathrm{Re}\left\{\hat{\rho}\left(a_n\rho^{n-1} - \frac{b_n}{\rho^{n+1}}\right) + i\hat{\theta}\left(a_n\rho^{n-1} + \frac{b_n}{\rho^{n+1}}\right)\right\}e^{in\theta} \quad (2)$$

which have to be finite both inside and outside the coil:

$$\vec{H}_{in} = \operatorname{Re} A_n \left(\hat{\rho} + i\hat{\varphi}\right) \rho^{n-1} e^{in\theta}, \vec{H}_{out} = \operatorname{Re} B_n \left(\hat{\rho} - i\hat{\varphi}\right) \rho^{-n-1} e^{in\theta} \tag{3}$$

Next we need to use continuity for radial component of magnetic field and integral equation for the angular component we can connect to the current in the coil:

(A) - Air coil



$$\hat{\rho} \cdot \vec{H}_{in}(R) = \hat{\rho} \cdot \vec{H}_{in}(R) \Rightarrow \operatorname{Re} A_n R^{n-1} e^{in\theta} = \operatorname{Re} B_n R^{-n-1} e^{in\theta} \rightarrow B_n = A_n R^{2n} = H_o R^{n+1}$$

$$\vec{B}_{in} = \operatorname{Re} H_o \left(\hat{\rho} + i\hat{\theta}\right) \left(\frac{\rho}{R}\right)^{n-1} e^{in\theta}, \vec{B}_{out} = \operatorname{Re} H_o \left(\hat{\rho} - i\hat{\theta}\right) \left(\frac{R}{\rho}\right)^{n+1} e^{in\theta}$$

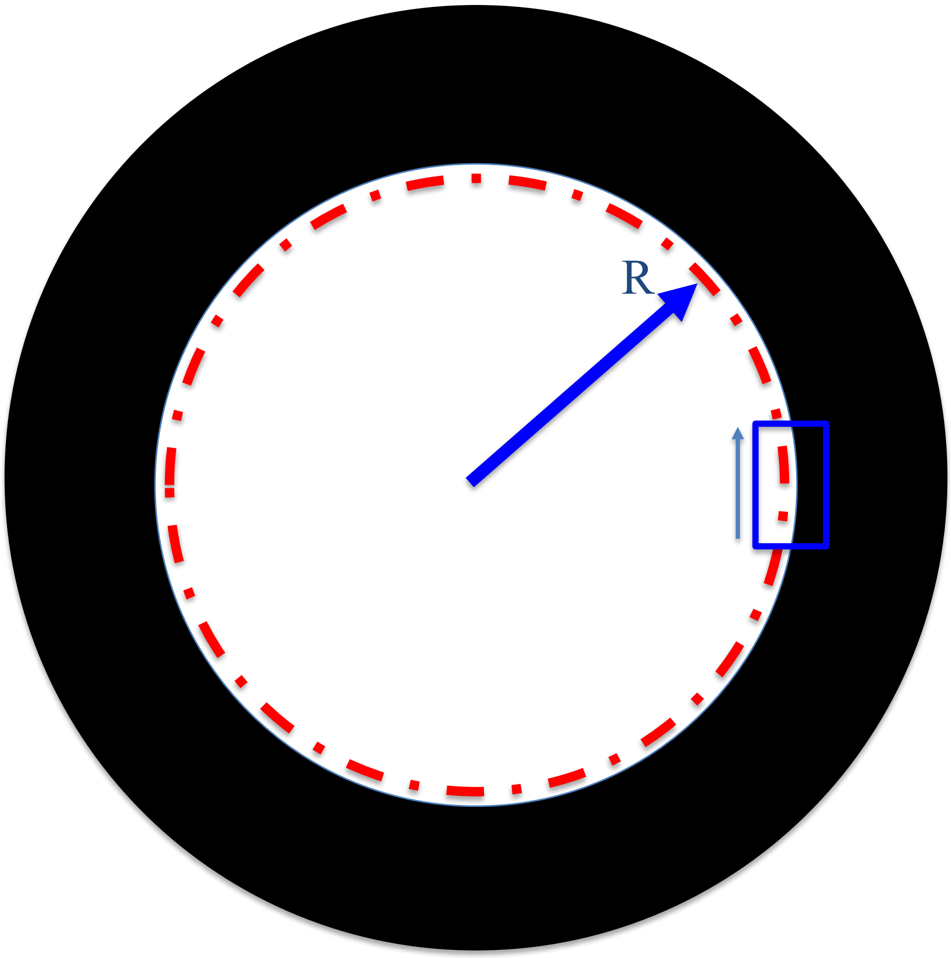
$$\oint \vec{B} d\vec{l} = \frac{4\pi}{c} \oint \vec{j} \cdot d\vec{A};\ d\vec{A} = \hat{z}\rho d\theta d\rho = \hat{z} R d\theta d\rho \rightarrow \frac{4\pi}{c} \oint \vec{j} \cdot d\vec{A} = \frac{\pi}{c} \Delta\theta \cdot \operatorname{Re} I_o e^{in\theta} \tag{4}$$

$$\left(\hat{\theta} \vec{B}_{in}(R-\varepsilon) - \hat{\theta} \cdot \vec{B}_{in}(R+\varepsilon)\right) R\Delta\theta = 2 \operatorname{Re} H_o R \Delta\theta e^{in\theta} = \frac{\pi}{c} \Delta\theta \cdot \operatorname{Re} I_o e^{in\theta};$$

$$H_o = \frac{\pi}{2Rc} I_o$$

(B) – Ideal magnetic core

Axial component parallel to the surface of the perfect magnetic yoke is zero"



$$\hat{\theta}\cdot\vec{B}_{in}(R+\varepsilon)=0$$

$$\left(\hat{\theta}\vec{B}_{in}(R-\varepsilon)-\hat{\theta}\cdot\vec{B}_{in}(R+\varepsilon)\right)R\Delta\theta=\mathrm{Re}\,H_o R\Delta\theta e^{in\theta}=\frac{\pi}{c}\Delta\theta\cdot\mathrm{Re}\,I_o e^{in\theta}; \qquad (5)$$

$$H_o=\frac{\pi}{Rc}I_o$$

(C) The total current in the coil is

$$I_{tot}=\oint|\vec{j}|\rho d\rho d\theta=\frac{|I_o|}{2\pi}\oint|\cos\theta|d\theta=\frac{|I_o|}{\pi}\int_0^{\pi}d\theta\sin\theta=2\frac{|I_o|}{\pi};$$

Applying "practical' unit formula from lecture 8,

$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I \Leftrightarrow \oint \vec{B}[T]\cdot d\vec{l}[m] = 0.4\pi \cdot I[MA];$$

$$H_o = \frac{\pi}{Rc} I_o \Leftrightarrow H_o[T] = \frac{0.1\pi \cdot I_o[MA]}{R[m]} \Rightarrow I_o[MA] = \frac{H_o[T]R[m]}{0.1\pi};$$

$$I_{tot}[MA] = 2\frac{|I_o|}{\pi} = 20\frac{H_o[T]R[m]}{\pi^2}$$

Plugging bnumber gives us total coli current of 0.454 meagaampreas.

Naturally magetic field is an usniform dipole field

$$\vec{B}_{in} = \mathrm{Re}\, H_o\left(\hat{\rho} + i\hat{\theta}\right)e^{i\theta} = \hat{y}\cdot H_o$$