Homework 16. Due November 4

Problem 1. 2 x 12 points. Sextupoles and octupoles, or un-harmonic oscillators

For an simple harmonic oscillator

$$H = \frac{p^2}{2} + \omega^2 \frac{x^2}{2}$$

consider adding and additional term:

(a) 4th order (octupole):

$$\delta H = \varepsilon \frac{x^4}{4}$$

(b) 3rd order (sextupole)

$$\delta H = \varepsilon \frac{x^3}{3}$$

1. For both case write reduces equations of motion using action (or amplitude – whatever you prefer) and angle variables. Use perturbation method and find first order variation of the "slow" variables. Show that there is a first order frequency dependence (e.g. a constant time-averaged phase advance) on the action of the oscillator (amplitude of oscillation squared) for 4th order perturbation but there is no such term. Show that simply averaging the δH over time gives the same result.
2. For sextupole case, go to the second order of the perturbation theory. Calculate $\varepsilon\left\langle F(\xi,s)+\varepsilon\tilde{F}\right\rangle$ from Bogolyubov and Metropolsky method. Show that there is a frequency dependence on the amplitude of oscillation $\sim \varepsilon^2 a^2$.

Solution:

Since solution of the linear (unperturbed) equations of motion are

$$x = A\cos(\omega t+\varphi);\, p = -A\omega\sin(\omega t+\varphi);$$

$$A = \mathrm{w}\sqrt{2I};\;\; A\omega = \sqrt{2I}\,/\,\mathrm{w};\;\; \mathrm{w} = \frac{1}{\sqrt{\omega}};$$

$$x = \sqrt{\frac{2I}{\omega}}\cos(\omega t+\varphi);\, p = \sqrt{2I\omega}\cos(\omega t+\varphi);$$

where φ, I are Canonical angle and action pair. Linear part of Hamiltonian vanishes and

$$\delta H_o = \varepsilon\frac{x^4}{4} = \frac{\varepsilon I^2}{\omega^2}\cos^4(\omega t+\varphi);$$

$$\delta H_s = \varepsilon\frac{x^3}{3} = \varepsilon\frac{2\sqrt{2}}{3}\frac{I^{3/2}}{\omega^{3/2}}\cos^3(\omega t+\varphi);$$

Let's start from easy octupole:

$$\frac{dI}{dt} = -\frac{\partial \delta H_o}{\partial \varphi} = 4\frac{\varepsilon I^2}{\omega^2}\sin(\omega t+\varphi)\cos^3(\omega t+\varphi);$$
$$\frac{d\varphi}{dt} = \frac{\partial \delta H_o}{\partial I} = 2\frac{\varepsilon I}{\omega^2}\cos^4(\omega t+\varphi);$$
$$\left\langle \frac{dI}{dt} \right\rangle = 2\frac{\varepsilon I^2}{\omega^2}\left\langle \sin 2(\omega t+\varphi)\cos^2(\omega t+\varphi)\right\rangle = 0;$$
$$\Delta\omega = \left\langle \frac{d\varphi}{dt} \right\rangle = \varepsilon\frac{2I}{\omega^2}\left\langle \cos^4(\omega t+\varphi)\right\rangle = \varepsilon\frac{3}{4}\frac{I}{\omega^2} = \varepsilon\frac{3}{8}\frac{A^2}{\omega};$$

where we used

$$\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1+\cos 2\theta+\cos^2 2\theta}{2} = \frac{3}{8}+\frac{2\cos 2\theta+\cos 4\theta}{4}.$$

Thus, there is a dependence of the oscillation frequency on the square of the amplitude of oscillations, which is linearly proportional to the strength of octupole.

It is worth noting that the same result can be obtained by simply averaging the δH - it would yield the same result.

In sextupole case the average gives zeros:

$$\frac{dI}{dt} = -\frac{\partial \delta H_o}{\partial \varphi} = \frac{\varepsilon(2I)^{3/2}}{\omega^{3/2}}\sin(\omega t+\varphi)\cos^2(\omega t+\varphi);$$
$$\frac{d\varphi}{dt} = \frac{\partial \delta H_o}{\partial I} = \frac{\varepsilon\sqrt{2I}}{\omega^{3/2}}\cos^3(\omega t+\varphi);$$
$$\left\langle \frac{dI}{dt} \right\rangle = \frac{\varepsilon(2I)^{3/2}}{\omega^{3/2}}\left\langle \sin(\omega t+\varphi)\cos^2(\omega t+\varphi)\right\rangle = 0;$$
$$\Delta\omega = \left\langle \frac{d\varphi}{dt} \right\rangle = \frac{\varepsilon\sqrt{2I}}{\omega^{3/2}}\left\langle \cos^3(\omega t+\varphi)\right\rangle = 0;$$

For the second order perturbation we need oscillating part of the first order of perturbation $\tilde{F}$: remember we have to assume φ, I constant when integraring:

$$A = \xi(s)+\varepsilon\tilde{F}(\xi,s); \frac{d}{ds}\xi(s) = \langle F(\xi,s)\rangle;$$
$$\langle F(A,s)\rangle = \frac{1}{S}\int_s^{s+S}\langle F(A=const,s)\rangle ds; \quad \tilde{F} = \int\left(F-\langle F\rangle\right)ds; \tag{19-7}$$
$$\tilde{I} = -\frac{(2I)^{3/2}}{\omega^{5/2}}\frac{\cos^3(\omega t+\varphi)}{3};$$
$$\tilde{\varphi} = \frac{\sqrt{2I}}{\omega^{5/2}}\left(\frac{3\sin(\omega t+\varphi)}{4}+\frac{1}{12}\sin 3(\omega t+\varphi)\right);$$

and use second order of perturbation:

$$A = \xi(s) + \varepsilon \tilde{F}(\xi,s) + \varepsilon^2 \overbrace{\left\{\left(\tilde{F}\frac{\partial}{\partial \xi}\right)F\right\}}^{\sim} - \varepsilon^2 \frac{\partial \tilde{\tilde{F}}}{\partial \xi}\langle F(\xi,s)\rangle;$$

$$\frac{d}{ds}\xi(s) = \varepsilon\left\langle F(\xi,s) + \varepsilon\tilde{F}\right\rangle \approx \varepsilon\left\langle\left(1 + \varepsilon\left(\tilde{F}\frac{\partial}{\partial\xi}\right)\right)F(\xi,s)\right\rangle.$$

$$\frac{dI}{dt} = -2\sqrt{2}\frac{\varepsilon\left(I_o + \varepsilon\tilde{I}\right)^{3/2}}{\omega^{3/2}}\frac{\partial}{\partial\varphi}\cos^3(\theta + \varepsilon\tilde{\varphi});$$

$$\frac{d\varphi}{dt} = \sqrt{2}\frac{\varepsilon\sqrt{I_o + \varepsilon\tilde{I}}}{\omega^{3/2}}\cos^3(\theta + \varepsilon\tilde{\varphi}); \theta = \omega t + \varphi;$$

$$\tilde{I} = -\frac{(2I)^{3/2}}{\omega^{5/2}}\frac{\cos^3\theta}{3}; \tilde{\varphi} = \frac{\sqrt{2I}}{\omega^{5/2}}\left(\frac{3\sin\theta}{4} + \frac{1}{12}\sin 3\theta\right);$$

$$\cos(\theta + \varepsilon\tilde{\varphi}) \cong \cos\theta - \varepsilon\tilde{\varphi}\sin\theta$$

$$\cos^3(\theta + \varepsilon\tilde{\varphi}) \cong \cos^3\theta - 3\varepsilon\tilde{\varphi}\sin\theta\cos^2\theta;$$

$$\frac{\partial}{\partial\varphi}\cos^3(\theta + \varepsilon\tilde{\varphi}) \cong 3\sin\theta\cos^2\theta + 3\varepsilon\tilde{\varphi}\left(\cos\theta + 3\cos^3\theta\right);$$

$$\left\langle 3\sin\theta\cos^2\theta\right\rangle = 0; \left\langle\cos\theta\sin^2\theta\right\rangle$$

$$\left(I_o + \varepsilon\tilde{I}\right)^{3/2} \cong I_o^{\,3/2}\left(1 + \varepsilon\frac{3\tilde{I}}{2I_o}\right); \sqrt{I_o + \varepsilon\tilde{I}} \cong \sqrt{I_o}\left(1 + \varepsilon\frac{\tilde{I}}{2I_o}\right);$$

Writing down a long expressions of the action change

$$\left\langle\frac{dI}{dt}\right\rangle = \frac{\varepsilon(2I)^{3/2}}{\omega^{3/2}}\left(1 - \varepsilon\frac{\sqrt{2I_o}}{\omega^{5/2}}\cos^3\theta\right)\times$$

$$\left\{3\sin\theta\cos^2\theta + \varepsilon\frac{\sqrt{2I}}{\omega^{5/2}}\left(\cos\theta + 3\cos^3\theta\right)\left(\frac{3\sin\theta}{4} + \frac{1}{12}\sin 3\theta\right)\right\} = 0$$

is equal zero. It easy to see since the expression in figure brackets is odd function of θ, which is multiplied by even function of θ. A bit more work for phase:

$$\frac{d\varphi}{dt} = \frac{\varepsilon\sqrt{2I}}{\omega^{3/2}}\left(1 - \varepsilon\frac{\sqrt{2I}}{\omega^{5/2}}\frac{\cos^3\theta}{3}\right)$$

$$\left(\cos^3\theta - 3\varepsilon\frac{\sqrt{2I}}{\omega^{5/2}}\sin\theta\cos^2\theta\left(\frac{3\sin\theta}{4} + \frac{1}{12}\sin 3\theta\right)\right);$$

$$\left\langle\frac{d\varphi}{dt}\right\rangle = -\varepsilon^2\frac{2I}{\omega^4}\left(\frac{\left\langle\cos^6\theta\right\rangle}{3} + 3\left\langle\cos^2\theta\left(\frac{3\sin^2\theta}{4} + \frac{\sin\theta\sin 3\theta}{12}\right)\right\rangle\right)$$

It is boring but the coefficient is equal 5/12 – just a lot manipulations. The result is”

$$\Delta\omega_s = \left\langle \frac{d\varphi}{dt} \right\rangle = -\frac{5\varepsilon^2}{12\omega^3}\frac{2I}{\omega} = -\frac{5\varepsilon^2}{12\omega^3}A^2$$

e.g. sextupole term gives negative frequency shift proportional to the amplitude of oscillation squared. It is proportional to the strength of sextupole squared.