

# PHY 564

# Advanced Accelerator Physics

# Lectures 16

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Because the parameterization and the action-angle are very useful tools for solving many standard accelerator problems, we will go through steps in a logical manner: from simple to more complicated cases.

Let's start from a simple 1D oscillator with Hamiltonian:

$$H = \frac{p^2}{2m} + k \frac{x^2}{2}; \mathbf{H} = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & k \end{bmatrix}; \mathbf{D} = \begin{bmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{bmatrix} \quad (1)$$

in our standard matrix form. The equations of motion are:

$$X = \begin{bmatrix} x \\ p_x \end{bmatrix}; \frac{dX}{dt} = \mathbf{D} \cdot X \Leftrightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad (2)$$

$$\lambda = i\omega \rightarrow -\lambda^2 = \omega^2 = \det \mathbf{D} = \frac{k}{m}$$

This is something you did studies many times before. The stability criteria is

$$k > 0$$

and general solution is

$$\mathbf{M}(t) = e^{\mathbf{D}t} = e^{i\omega t} \frac{i\omega I + D}{2i\omega} - e^{-i\omega t} \frac{i\omega I - D}{2i\omega} = I \cos \omega t + \frac{D}{\omega} \sin \omega t = \begin{bmatrix} \cos \omega t & \frac{\sin \omega t}{m\omega} \\ -\frac{k}{\omega} \sin \omega t & \cos \omega t \end{bmatrix}$$

$$X(t) = \begin{bmatrix} x \\ p_x \end{bmatrix} = \begin{bmatrix} \cos \omega t & \frac{\sin \omega t}{m\omega} \\ -\frac{k}{\omega} \sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} x_o \\ p_{x_o} \end{bmatrix}; \frac{k}{\omega} = m\omega = \sqrt{mk} \quad (3)$$

$$x = x_o \cos \omega t + \frac{\sin \omega t}{m\omega} p_{x_o}; \quad p_x = p_{x_o} \cos \omega t - x_o \frac{k}{\omega} \sin \omega t.$$

We know another form of these equations of motions:

$$x = A \cos(\omega t + \varphi); \quad p_x = -m\omega \cdot A \cdot \sin(\omega t + \varphi) \quad (4)$$

which is a consequence of

$$\mathbf{M}Y = e^{i\omega t} Y; \quad Y = \begin{bmatrix} w \\ i \\ \overline{w} \end{bmatrix}; \quad w = \sqrt{m\omega}; \quad X(t) = a \operatorname{Re} Y e^{i\omega t + \varphi} = \begin{bmatrix} \frac{a}{\sqrt{m\omega}} \cos(\omega t + \varphi) \\ -a\sqrt{m\omega} \sin(\omega t + \varphi) \end{bmatrix}; \quad (5)$$

which coincides with (4) using  $A = a\sqrt{m\omega}$ . Naturally, we know that amplitude and phase are constant of motion.

Let's make a Canonical transformation to  $\left\{ \tilde{q} = \varphi, \tilde{p} = \frac{a^2}{2} \right\}$  using

$$\begin{aligned} F(q = x, \tilde{q} = \varphi) &= -m\omega \frac{x^2}{2} \tan(\omega t + \varphi); \\ \frac{\partial F}{\partial x} &= -m\omega x \tan(\omega t + \varphi) = -m\omega \frac{a}{\sqrt{m\omega}} \sin(\omega t + \varphi) = p_x; \\ \tilde{p} &= -\frac{\partial F}{\partial \varphi} = m\omega \frac{x^2}{2 \cos^2(\omega t + \varphi)} = \frac{a^2}{2} = I \end{aligned} \quad (6)$$

Finally, the reduced Hamiltonian is

$$\begin{aligned}\tilde{H} &= H + \frac{\partial F}{\partial t} = \frac{p^2}{2m} + k \frac{x^2}{2} - m\omega^2 \frac{x^2}{2 \cos^2(\omega t + \varphi)} = \\ &\frac{a^2}{2} \left( \frac{k}{m\omega} \cos^2(\omega t + \varphi) + \omega \sin(\omega t + \varphi) \right) - \omega \frac{a^2}{2} = 0\end{aligned}\quad (7)$$

Thus, if

$$\begin{aligned}H = \frac{p^2}{2m} + k \frac{x^2}{2} + H_1(x, p) \rightarrow \tilde{H}(\varphi, I) = H - \frac{p^2}{2m} + k \frac{x^2}{2} = H_1(x(I, \varphi), p(I, \varphi)) \\ \frac{d\varphi}{dt} = \frac{\partial \tilde{H}}{\partial I}; \quad \frac{dI}{dt} = -\frac{\partial \tilde{H}}{\partial \varphi}.\end{aligned}\quad (8)$$

General parameterization of motion of an arbitrary linear Hamiltonian system:

$$H = \frac{1}{2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} h_{ij}(s) x_i x_j \equiv \frac{1}{2} X^T \cdot \mathbf{H}(s) \cdot X,$$

$$X^T = [q^1 \quad P_1 \quad \dots \quad \dots \quad q^n \quad P_n] = [x_1 \quad x_2 \quad \dots \quad \dots \quad x_{2n-1} \quad x_{2n}].$$

$$\frac{dX}{ds} = \mathbf{D}(s) \cdot X; \quad \mathbf{D} = \mathbf{S} \cdot \mathbf{H}(s), \quad X(s) = \mathbf{M}(s_o|s) \cdot X_o. \quad \backslash \quad (\text{I})$$

$$\mathbf{M}' \equiv \frac{d\mathbf{M}}{ds} = \mathbf{D}(s) \cdot \mathbf{M}; \quad \mathbf{M}(s_o) = \mathbf{I}, \quad \mathbf{M}^T \cdot \mathbf{S} \cdot \mathbf{M} = \mathbf{S}, \quad \mathbf{M}^{-1} = -\mathbf{S} \cdot \mathbf{M}^T \cdot \mathbf{S}. \quad (\text{II})$$

Parameterization of motion: a periodic system with period C  $\mathbf{H}(s+C) = \mathbf{H}(s)$

$$\mathbf{T}(s) = \mathbf{M}(s|s+C), \quad \det[\mathbf{T} - \lambda_i \cdot \mathbf{I}] = 0 \quad (\text{III})$$

$$\text{Stable system } |\lambda_i| = 1. \quad \lambda_k \equiv 1 / \lambda_{k+n} \equiv \lambda_{k+n}^* \equiv e^{i\mu_k}; \quad \mu_k \equiv 2\pi\nu_k, \quad \{k = 1, \dots, n\}. \quad (\text{IV})$$

We proved that

$$\tilde{Y}_k(s) = Y_k(s)e^{\psi_k(s)}; \quad Y_k(s+C) = Y_k(s); \quad \psi_k(s+C) = \psi_k(s) + \mu_k$$

$$\tilde{\mathbf{U}} = [\dots \tilde{Y}_k, \tilde{Y}_k^* \dots]; \quad \tilde{\mathbf{U}}(s) = \mathbf{U}(s) \cdot \Psi(s), \quad \Psi(s) = \begin{pmatrix} e^{i\psi_1(s)} & 0 & & 0 \\ 0 & e^{-i\psi_1(s)} & & 0 \\ & & \dots & 0 \\ 0 & 0 & 0 & e^{-i\psi_n(s)} \end{pmatrix}$$

$$X = \frac{1}{2} \sum_{k=1}^n (a_k \tilde{Y}_k + a_k^* \tilde{Y}_k^*) = \text{Re} \sum_{k=1}^n a_k \tilde{Y}_k = \frac{1}{2} \tilde{\mathbf{U}}A = \frac{1}{2} \mathbf{U}\Psi A = \frac{1}{2} \mathbf{U}\tilde{A}$$

$$a_i = \frac{1}{i} Y_i^{*T} S X; \quad \tilde{a}_i \equiv a_i e^{i\psi} = \frac{1}{i} Y_i^{*T} S X;$$

$$A = 2\tilde{\mathbf{U}}^{-1} \cdot X = -i\Psi^{-1} \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X; \quad \tilde{A} = \Psi A = -i \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X.$$

$$\mathbf{T}(s) = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s); \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ & & \dots & 0 \\ 0 & 0 & 0 & \lambda_n^* \end{pmatrix}; \mathbf{T} \cdot \mathbf{U} = \mathbf{U} \cdot \Lambda; \Lambda = \mathbf{U}^{-1} \cdot \mathbf{T} \cdot \mathbf{U} \dots \quad (\text{V})$$

$$\mathbf{U}(s) = [Y_1, Y_1^* \dots Y_n, Y_n^*]; \quad \mathbf{T}(s)Y_k(s) = \lambda_k Y_k(s) \Leftrightarrow \mathbf{T}(s)Y_k^*(s) = \lambda_k^* Y_k^*(s) \quad (\text{VI})$$

$$\tilde{Y}_k(s_1) = \mathbf{M}(s|s_1)\tilde{Y}_k(s) \Leftrightarrow \frac{d}{ds}\tilde{Y}_k = \mathbf{D}(s) \cdot \tilde{Y}_k; \quad \tilde{\mathbf{U}}(s_1) = \mathbf{M}(s|s_1)\tilde{\mathbf{U}}(s) \Leftrightarrow \frac{d}{ds}\tilde{\mathbf{U}} = \mathbf{D}(s) \cdot \tilde{\mathbf{U}} \quad (\text{VII})$$

$$\tilde{\mathbf{U}}(s+C) = \tilde{\mathbf{U}}(s) \cdot \Lambda, \quad \tilde{Y}_k(s+C) = \lambda_k \tilde{Y}_k(s) = e^{i\mu_k} \tilde{Y}_k(s) \quad (\text{VIII})$$

$$\tilde{Y}_k(s) = Y_k(s)e^{\psi_k(s)}; \quad Y_k(s+C) = Y_k(s); \quad \psi_k(s+C) = \psi_k(s) + \mu_k \quad (\text{IX})$$

$$\tilde{\mathbf{U}}(s) = \mathbf{U}(s) \cdot \Psi(s), \quad \Psi(s) = \begin{pmatrix} e^{i\psi_1(s)} & 0 & 0 \\ 0 & e^{-i\psi_1(s)} & 0 \\ & & \dots & 0 \\ 0 & 0 & 0 & e^{-i\psi_n(s)} \end{pmatrix} \quad (\text{X})$$

$$Y_k^{T*} \cdot \mathbf{S} \cdot Y_{j \neq k} = 0; \quad Y_k^T \cdot \mathbf{S} \cdot Y_j = 0; \quad Y_k^{T*} \cdot \mathbf{S} \cdot Y_k = 2i \quad (\text{XI})$$

$$\mathbf{U}^T \cdot \mathbf{S} \cdot \mathbf{U} \equiv \tilde{\mathbf{U}}^T \cdot \mathbf{S} \cdot \tilde{\mathbf{U}} = -2i\mathbf{S}, \quad \mathbf{U}^{-1} = \frac{1}{2i}\mathbf{S} \cdot \mathbf{U}^T \cdot \mathbf{S} \quad (\text{XII})$$

$$X(s) = \frac{1}{2} \sum_{k=1}^n (a_k \tilde{Y}_k + a_k^* \tilde{Y}_k^*) \equiv \text{Re} \sum_{k=1}^n a_k Y_k e^{i\psi_k} \equiv \frac{1}{2} \tilde{\mathbf{U}} \cdot \mathbf{A} = \frac{1}{2} \mathbf{U} \cdot \Psi \cdot \mathbf{A} = \frac{1}{2} \mathbf{U} \cdot \tilde{\mathbf{A}} \quad (\text{XIII})$$

$$a_i = \frac{1}{2i} Y_i^{*T} \mathbf{S} X; \quad \tilde{a}_i \equiv a_i e^{i\psi_i} = \frac{1}{2i} Y_i^{*T} \mathbf{S} X; \quad (\text{XIV})$$

$$\mathbf{A} = 2\tilde{\mathbf{U}}^{-1} \cdot \mathbf{X} = -i\Psi^{-1} \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot \mathbf{X}; \quad \tilde{\mathbf{A}} = \Psi \mathbf{A} = -i \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot \mathbf{X}.$$



and finished with fact that this parameterization is equivalent to Canonical transformation to action-angle variables.

$$\left\{ \varphi_k, I_k = \frac{a_k^2}{2} \right\}; k = 1, 2, \dots, n$$

Then if  $\mathcal{H} = \frac{1}{2} X^T \mathbf{H}(s) X + \mathcal{H}_1(X, s)$

$$\tilde{H}(\varphi_k, I_k, s) = \mathcal{H}_1(X(\varphi_k, I_k, s), s);$$

$$\frac{d\varphi_k}{ds} = \frac{\partial \tilde{H}}{\partial I_k}, \quad \frac{dI_k}{ds} = -\frac{\partial \tilde{H}}{\partial \varphi_k}.$$

## 1D – ACCELERATOR

$$\tilde{h} = \frac{p^2}{2} + K_1(s) \frac{y^2}{2}; \quad \mathbf{H} = \begin{bmatrix} K_1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{D} = \mathbf{S}\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix};$$

$$\frac{d}{ds} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -K_1 x \end{bmatrix} \quad (\text{i.e. } x' \equiv p).$$

$$Y = \begin{bmatrix} w \\ w' + i/w \end{bmatrix}; \quad \psi' = \frac{1}{w^2}; \quad \tilde{Y} = Y e^{i\psi}$$

$$\mathbf{T}(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{U}(s) \Lambda \mathbf{U}^{-1}(s); \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix}$$

$$\mathbf{T} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \frac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}; \quad \mathbf{J}^2 = -\mathbf{I}$$

Note: for not normalized Hamiltonian  $H = \frac{p_y^2}{2p_o} + p_o K_1(s) \frac{y^2}{2}$  and

$$Y = \begin{bmatrix} \frac{w}{\sqrt{p_o}} \\ \sqrt{p_o} (w' + i/w) \end{bmatrix}; \psi' = \frac{1}{w^2}; \tilde{Y} = Y e^{i\psi}$$

$$\mathbf{T} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \frac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}; \quad \mathbf{J}^2 = -\mathbf{I}; \quad \gamma = (1 + \alpha^2) / \beta$$

$$\cos \mu = \text{Trace}(\mathbf{T}) / 2 = \frac{T_{11} + T_{22}}{2}$$

*Stability if* :  $-1 < \text{Trace}(\mathbf{T}) / 2 < 1$

$$w^2 \equiv \beta = \frac{T_{12}}{\sin \mu} = \frac{|T_{12}|}{\sqrt{1 - (\text{Trace}(\mathbf{T}) / 2)^2}}; \quad w = \sqrt{\frac{|T_{12}|}{\sqrt{1 - (\text{Trace}(\mathbf{T}) / 2)^2}}};$$

$$ww' \equiv -\alpha = \frac{T_{22} - T_{11}}{2 \cos \mu} = -\frac{T_{11} - T_{22}}{T_{11} + T_{22}}$$

## 1D - ACCELERATOR

$$Y = \begin{bmatrix} w \\ w' + i/w \end{bmatrix}; \tilde{Y} = \begin{bmatrix} w \\ w' + i/w \end{bmatrix} e^{i\psi}; \mathbf{U} = \begin{bmatrix} w & w \\ w' + i/w & w' - i/w \end{bmatrix}; \tilde{\mathbf{U}} = \mathbf{U} \cdot \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$$

$$w'' + K_1(s)w = \frac{1}{w^3}, \quad \psi' = 1/w^2; \begin{bmatrix} x \\ x' \end{bmatrix} = \text{Re} \left( a e^{i\varphi} \begin{bmatrix} w \\ w' + i/w \end{bmatrix} e^{i\psi} \right);$$

$$x = a \cdot w(s) \cdot \cos(\psi(s) + \varphi)$$

$$x' = a \cdot (w'(s) \cdot \cos(\psi(s) + \varphi) - \sin(\psi(s) + \varphi) / w(s))$$

$$\beta \equiv w^2 \Rightarrow \psi' = 1/\beta; \alpha \equiv -\beta' \equiv -w w', \quad \gamma \equiv \frac{1 + \alpha^2}{\beta} - \text{definitions}$$

$$x = a \cdot \sqrt{\beta(s)} \cdot \cos(\psi(s) + \varphi)$$

$$x' = -\frac{a}{\sqrt{\beta(s)}} \cdot (\alpha(s) \cdot \cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi))$$

Complex amplitude and real amplitude and phase are easy to calculate. Expression for  $a^2$  is called Curren-Snyder invariant.

Thus, the final form of the eigen vector can be rewritten as

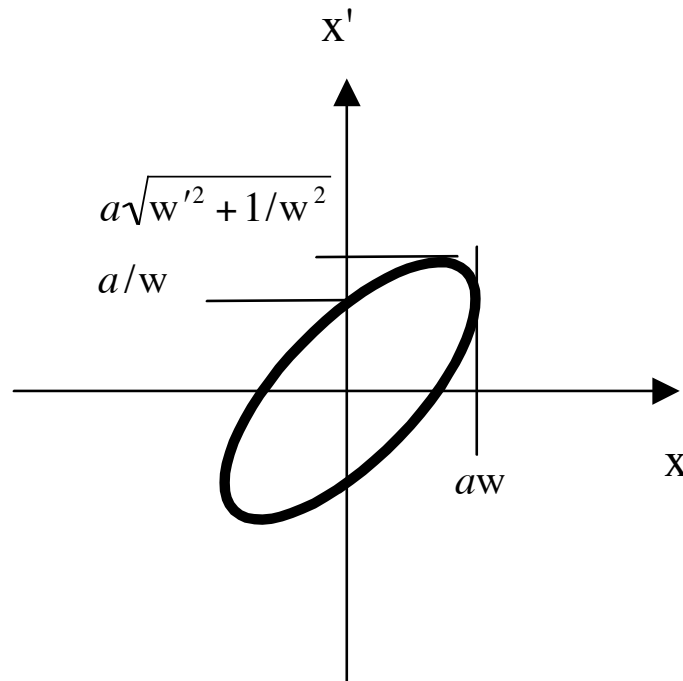
$$Y = \begin{bmatrix} w \\ w' + i/w \end{bmatrix}; \psi' = \frac{1}{w^2}; \tilde{Y} = Y e^{i\psi} \quad (34)$$

The parameterization of the linear 1D motion is

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \text{Re} \left( a e^{i\varphi} \begin{bmatrix} w \\ w' + i/w \end{bmatrix} e^{i\psi} \right);$$

$$\begin{aligned} x &= a \cdot w(s) \cdot \cos(\psi(s) + \varphi) \\ x' &= a \cdot (w'(s) \cdot \cos(\psi(s) + \varphi) - \sin(\psi(s) + \varphi) / w(s)) \end{aligned} \quad (35)$$

where  $a$  and  $\varphi$  are the constants of motion.



$$X = \operatorname{Re} \tilde{a} Y;$$

$$ae^{i\varphi} = -iY^{*T} SX = \begin{bmatrix} w \\ w' - i/w \end{bmatrix}^T \cdot \begin{bmatrix} x' \\ -x \end{bmatrix} = x/w + i(w'x - wx')$$

$$a^2 = \frac{x^2}{w^2} + (w'x - wx')^2 \equiv \frac{x^2 + (\alpha x + \beta x')^2}{\beta} \equiv \gamma x^2 + 2\alpha x x' + \beta x'^2$$

$$\varphi = \arg(x/w + i(w'x - wx')) = \tan^{-1} \frac{ww'x - w^2x'}{x} = -\tan^{-1} \frac{\alpha x + \beta x'}{x};$$

$$\varphi = \sin^{-1} \frac{w'x - wx'}{\sqrt{x^2 + w^2(w'x - wx')^2}} = -\sin^{-1} \frac{\alpha x + \beta x'}{\sqrt{\gamma x^2 + 2\alpha x x' + \beta x'^2}}$$

Inverse ratios – matrices through parameterization: reverse of eq (VII), where U is propagated by M.

$$\mathbf{M}(s_1|s_2) = \tilde{\mathbf{U}}(s_2)\tilde{\mathbf{U}}^{-1}(s_1) = \frac{i}{2}\tilde{\mathbf{U}}(s_2) \cdot \mathbf{S} \cdot \tilde{\mathbf{U}}^T(s_1) \cdot \mathbf{S} = \frac{i}{2}\mathbf{U}(s_2) \cdot \Psi(s_2) \cdot \mathbf{S} \cdot \Psi^{-1}(s_1) \cdot \mathbf{U}^T(s_1) \cdot \mathbf{S} \quad (\text{XV})$$

$$\Psi(s_2) \cdot \mathbf{S} \cdot \Psi^{-1}(s_1) \equiv \Psi(s_2 - s_1) \cdot \mathbf{S}; \quad \mathbf{M}(s_1|s_2) = \frac{i}{2}\mathbf{U}(s_2) \cdot \Psi(s_2 - s_1) \cdot \mathbf{S} \cdot \mathbf{U}^T(s_1) \cdot \mathbf{S}$$

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \frac{i}{2}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^T\mathbf{S} \quad \text{Specific case of } s_I = s + \mathbf{C}$$

$$\mathbf{T} = \frac{1}{2i}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^T\mathbf{S} = \frac{1}{2i} \begin{bmatrix} w & w \\ w' + \frac{i}{w} & w' - \frac{i}{w} \end{bmatrix} \cdot \begin{bmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} w & w' + \frac{i}{w} \\ w & w' - \frac{i}{w} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =$$

$$\frac{1}{2i} \begin{bmatrix} we^{i\mu} & we^{-i\mu} \\ \left(w' + \frac{i}{w}\right)e^{i\mu} & \left(w' - \frac{i}{w}\right)e^{-i\mu} \end{bmatrix} \cdot \begin{bmatrix} \frac{i}{w} - w' & w \\ \frac{i}{w} + w' & -w \end{bmatrix} =$$

$$\begin{bmatrix} \frac{e^{i\mu} + e^{-i\mu}}{2} - \frac{e^{i\mu} - e^{-i\mu}}{2i} ww' & w^2 \frac{e^{i\mu} - e^{-i\mu}}{2i} \\ -\left(w'^2 + \frac{1}{w^2}\right) \frac{e^{i\mu} - e^{-i\mu}}{2} & \frac{e^{i\mu} + e^{-i\mu}}{2} + \frac{e^{i\mu} - e^{-i\mu}}{2i} ww' \end{bmatrix}$$



$$\mathbf{T} = \frac{1}{2i} \mathbf{U} \Lambda \mathbf{S} \mathbf{U}^T \mathbf{S} = \begin{bmatrix} \cos \mu - ww' \cdot \sin \mu & w^2 \cdot \sin \mu \\ -\left(w'^2 + \frac{1}{w^2}\right) \cdot \sin \mu & \cos \mu + ww' \cdot \sin \mu \end{bmatrix} = \mathbf{I} \cos \mu + \mathbf{J} \sin \mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \frac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix};$$

$$\mathbf{J}^2 = \begin{bmatrix} -ww' & w^2 \\ -\left(w'^2 + \frac{1}{w^2}\right) & ww' \end{bmatrix} \begin{bmatrix} -ww' & w^2 \\ -\left(w'^2 + \frac{1}{w^2}\right) & ww' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}$$

$$M(s_1|s_2) = \frac{1}{2i} \begin{bmatrix} w_2 & w_2 \\ w_2' + i/w_2 & w_2' - i/w_2 \end{bmatrix} \cdot \begin{pmatrix} e^{i\Delta\psi} & 0 \\ 0 & e^{-i\Delta\psi} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{bmatrix} w_1 & w_1' + i/w_1 \\ w_1 & w_1' - i/w_1 \end{bmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Delta\psi = \psi(s_2) - \psi(s_1);$$

$$M(s_1|s_2) = \frac{1}{2i} \begin{bmatrix} w_2 & w_2 \\ w_2' + i/w_2 & w_2' - i/w_2 \end{bmatrix} \cdot \begin{bmatrix} -(w_1' - i/w_1)e^{i\Delta\psi} & w_1 e^{i\Delta\psi} \\ (w_1' + i/w_1)e^{-i\Delta\psi} & -w_1 e^{-i\Delta\psi} \end{bmatrix} =$$

$$\begin{bmatrix} w_2/w_1 \cos\Delta\psi - w_2 w_1' \sin\Delta\psi & w_1 w_2 \sin\Delta\psi \\ (w_2'/w_1 - w_1'/w_2) \cos\Delta\psi & w_1/w_2 \cos\Delta\psi + w_1 w_2' \sin\Delta\psi \\ -(w_1' w_2' + 1/(w_2 w_1)) \sin\Delta\psi & \end{bmatrix}$$

use  $w = \sqrt{\beta}$ ;  $w' = -\alpha/\sqrt{\beta}$  to get a standard

$$M(s_1|s_2) = \begin{bmatrix} \frac{\cos\Delta\psi + \alpha_1 \sin\Delta\psi}{\sqrt{\beta_1/\beta_2}} & \sqrt{\beta_1\beta_2} \sin\Delta\psi \\ -\frac{(\alpha_2 - \alpha_1) \cos\Delta\psi + (1 + \alpha_1\alpha_2) \sin\Delta\psi}{\sqrt{\beta_1\beta_2}} & \frac{\cos\Delta\psi - \alpha_2 \sin\Delta\psi}{\sqrt{\beta_2/\beta_1}} \end{bmatrix}$$

with obvious simplification for one-turn matrix:

$$\mathbf{T} = M(s|s+C) = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\frac{(1+\alpha^2)\sin\mu}{\beta} & \cos\mu - \alpha \sin\mu \end{bmatrix} \mathbf{I} \cos\mu + \mathbf{J} \sin\mu; \mathbf{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$$

from where one can get easily all functions and constants:

$$\mu = \cos^{-1}\left(\frac{t_{11} + t_{22}}{2}\right) \text{ with } \text{sign}(\sin\mu) = \text{sign}(t_{12}); \beta = t_{12}/\sin\mu; \alpha = \frac{t_{11} - t_{22}}{2\sin\mu}$$

A little bit more complex is fully coupled 2D case.

$$\mathbf{T}(s) = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}; \det[\mathbf{T} - \lambda I] = \lambda^4 - \lambda^3 \text{Tr}[T] + \lambda^2 (2 + a) - \lambda \text{Tr}[T] + 1 =$$

$$a = \text{Tr}[N_1] \cdot \text{Tr}[N_4] - \text{Tr}[N_2 N_3] - 2 \det N_2$$

$$(\text{note } \det N_2 = \det N_3 = 1 - \det N_1 = 1 - \det N_4)$$

Finding roots:

$$z_k = \lambda_k + \lambda_k^{-1}; \quad z_k = \frac{\text{Tr}[N_1 + N_4]}{2} \pm \left\{ \frac{\text{Tr}^2[N_1 - N_4]}{4} + \text{Tr}[N_2 N_3] + 2 \det N_2 \right\}$$

$$X = \begin{bmatrix} x \\ P_x \\ y \\ P_y \end{bmatrix} = \operatorname{Re} \tilde{a}_1 Y_1 + \operatorname{Re} \tilde{a}_2 Y_2 = \operatorname{Re} a_1 \tilde{Y}_1 + \operatorname{Re} \tilde{a}_2 \tilde{Y}_2$$

$$Y_k = R_k + iQ_k; \quad \tilde{Y}_k = \begin{bmatrix} w_{kx} e^{i\psi_{kx}} \\ (u_{kx} + iv_{kx}) e^{i\psi_{kx}} \\ w_{ky} e^{i\psi_{ky}} \\ (u_{ky} + iv_{ky}) e^{i\psi_{ky}} \end{bmatrix};$$

$$\psi_{kx}(s + C) = \psi_{kx}(s) + \mu_k; \quad \psi_{ky}(s + C) = \psi_{ky}(s) + \mu_k;$$

$$w_{kx} v_{kx} + w_{ky} v_{ky} = 1;$$

Conditions: there are

$$Y_k^{*T} SY_k = 2i; \quad Y_1^{*T} SY_2 = 0; \quad Y_1^T SY_2 = 0; \quad \theta_k = \psi_{kx} - \psi_{ky}$$

$$a) \quad w_{1x} v_{1x} = w_{2y} v_{2y} = 1 - q \quad \Rightarrow \quad v_{1x} = \frac{1 - q}{w_{1x}}; \quad v_{2y} = \frac{1 - q}{w_{2y}}$$

$$b) \quad w_{1y} v_{1y} = w_{2x} v_{2x} = q \quad \Rightarrow \quad v_{2x} = \frac{q}{w_{2x}}; \quad w_{1y} = \frac{q}{w_{1y}}$$

$$c) \quad c = w_{1x} w_{1y} \sin \theta_1 = -w_{2x} w_{2y} \sin \theta_2$$

$$d) \quad d = w_{1x} (u_{1y} \sin \theta_1 - v_{1y} \cos \theta_1) = -w_{2x} (u_{2y} \sin \theta_2 - v_{2y} \cos \theta_2)$$

$$e) \quad e = w_{1y} (u_{1x} \sin \theta_1 + v_{1x} \cos \theta_1) = -w_{2y} (u_{2x} \sin \theta_2 + v_{2x} \cos \theta_2)$$

Conditions are result of symplecticity. Conditions a) and b) are equivalent to Poincaré's invariants conserving sum of projections on (x-px) and (y-py) planes.

$$Y_1 = \left[ \begin{array}{c} w_{1x} e^{i\varphi_{1x}} \\ \left( u_{1x} + i \frac{q}{w_{1x}} \right) e^{i\varphi_{1x}} \\ w_{1y} e^{i\varphi_{1y}} \\ \left( u_{1y} + i \frac{1-q}{w_{1y}} \right) e^{i\varphi_{1y}} \end{array} \right] ; \quad Y_2 = \left[ \begin{array}{c} w_{2x} e^{i\varphi_{2x}} \\ \left( u_{2x} + i \frac{1-q}{w_{2x}} \right) e^{i\varphi_{2x}} \\ w_{2y} e^{i\varphi_{2y}} \\ \left( u_{2y} + i \frac{q}{w_{2y}} \right) e^{i\varphi_{2y}} \end{array} \right]$$

One should note for completeness that there is another way of parameterization of coupled motion proposed by Edward and Teng parameterization (*D.A.Edwards, L.C.Teng, IEEE Trans. Nucl. Sci. NS-20 (1973) 885*), which differs from what we are discussing here.