

# **PHY 554 course review – transverse and longitudinal motion**

Yichao Jing

# What we learned:

Frenet-Serret coordinates  $(x,y,s)$

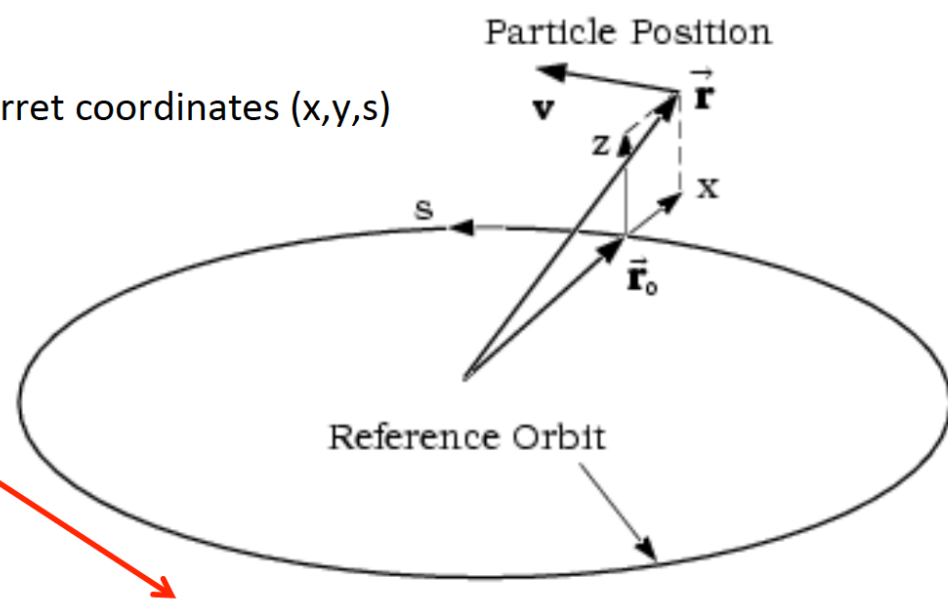
Hill's equations (derivatives w.r.t.  $s$ )

$$x'' + K_x(s)x = \pm \frac{\Delta B_z}{B\rho}, \quad y'' + K_y(s)y = \mp \frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \quad K_y(s) = \pm \frac{B_1}{B\rho}$$

Natural focusing from dipoles (curvature)

Focusing from quadrupoles



Higher order magnet, usually field errors

$$\theta = \frac{s}{R} = \frac{\beta ct}{R}$$

Solution of Hill's equations  $X(s), X'(s)$  form a coordinate set and can be transformed thru matrix representation

$$\begin{pmatrix} X(s) \\ X'(s) \end{pmatrix} = M(s, s_0) \begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}$$

$X$  can be  $x$  or  $y$

$$|M(s, s_0)| = 1$$

$$|\text{Trace}(M(s, s_0))| \leq 2$$

Stable solution conditions

## Courant-Snyder parameterization

$$M(s) = \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix} = I \cos \Phi + J \sin \Phi$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I, \quad \text{or } \beta\gamma = 1 + \alpha^2$$

Where  $\alpha, \beta, \gamma, \phi$  are functions of  $s$  and describes position dependent beam properties.

Focusing quadrupole:

$$M(s, s_0) = \begin{pmatrix} \cos \sqrt{K} \ell & \frac{1}{\sqrt{K}} \sin \sqrt{K} \ell \\ -\sqrt{K} \sin \sqrt{K} \ell & \cos \sqrt{K} \ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Defocusing quadrupole:

$$M(s, s_0) = \begin{pmatrix} \cosh \sqrt{|K|} \ell & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|} \ell \\ \sqrt{|K|} \sinh \sqrt{|K|} \ell & \cosh \sqrt{|K|} \ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

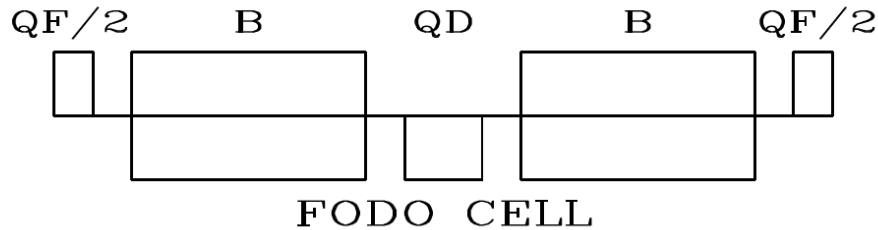
Dipole:  $K=1/\rho^2$

$$M(s, s_0) = \begin{pmatrix} \cos \frac{\ell}{\rho} & \rho \sin \frac{\ell}{\rho} \\ -\frac{1}{\rho} \sin \frac{\ell}{\rho} & \cos \frac{\ell}{\rho} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Drift space:  $K=0$

$$M(s, s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

## Example: FODO cell



A FODO cell is a basic block in beam transport, where the transfer matrices for dipoles (B) can be approximated by drift spaces, and QF and QD are the focusing and defocusing quadrupoles.

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix} \end{aligned}$$

$$\mathbf{M}(s) = \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix}$$

$$\cos \Phi = \frac{1}{2} \text{Tr}(\mathbf{M})$$

$$\cos \Phi = 1 - \frac{L_1^2}{2f^2}, \quad \sin \frac{\Phi}{2} = \frac{L_1}{2f}$$

$$\beta = \frac{2L_1(1 + \frac{L_1}{2f})}{\sin \Phi} = \frac{2L_1(1 + \sin \frac{\Phi}{2})}{\sin \Phi}$$

$$\alpha = 0$$

## Floquet Theorem

$$X'' + K(s)X = 0$$

$$K(s) = K(s + L)$$

$$X(s) = aw(s)e^{j\psi(s)}, \quad w(s) = w(s + L), \quad \psi(s + L) - \psi(s) = 2\pi n$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1 + \alpha^2}{\beta}, \quad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^s \frac{1}{\beta} ds$$

$$\begin{aligned} \begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} &= M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix} \\ M(s_2, s_1) &= \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}} (\cos \mu + \alpha_1 \sin \mu) & \sqrt{\beta_1 \beta_2} \sin \mu \\ -\frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \mu - \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos \mu & \sqrt{\frac{\beta_2}{\beta_1}} (\cos \mu - \alpha_1 \sin \mu) \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix} \end{aligned}$$

The values of the Courant–Snyder parameters  $\alpha_2, \beta_2, \gamma_2$  at  $s_2$  are related to  $\alpha_1, \beta_1, \gamma_1$  at  $s_1$  by

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

The evolution of the betatron amplitude function in a drift space is

$$\beta_2 = \frac{1}{\gamma_1} + \gamma_1 \left( s - \frac{\alpha_1}{\gamma_1} \right)^2 = \beta^* + \frac{(s - s^*)^2}{\beta^*},$$

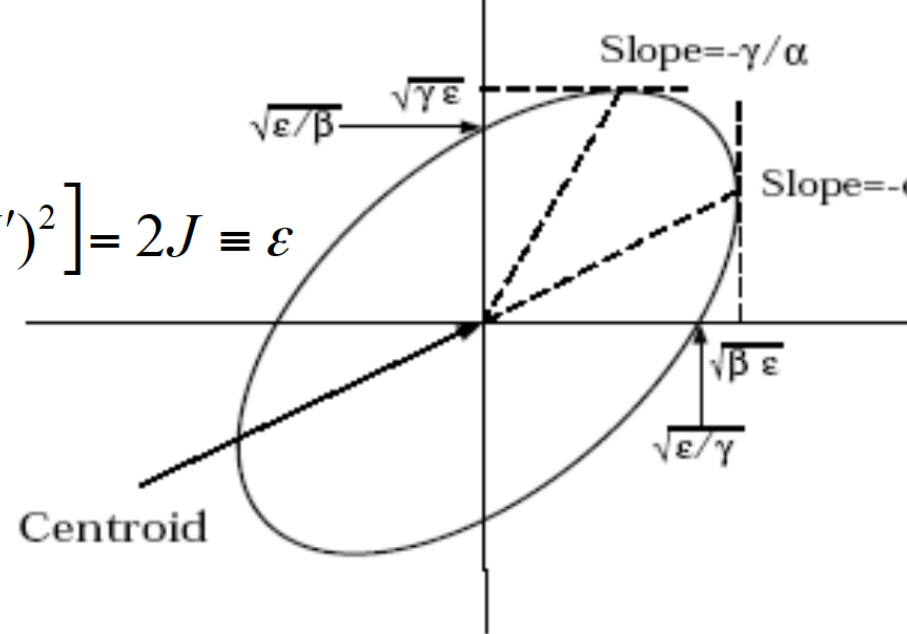
$$\alpha_2 = \alpha_1 - \gamma_1 s = -\frac{(s - s^*)}{\beta^*}, \quad \gamma_2 = \gamma_1 = \frac{1}{\beta^*}$$

Passing through a thin-lens quadrupole, the evolution of betatron function is

$$\beta_2 = \beta_1, \quad \alpha_2 = \alpha_1 + \frac{\beta_1}{f}, \quad \gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2}$$

# Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha XX' + \beta X'^2 = \frac{1}{\beta} \left[ X^2 + (\alpha X + \beta X')^2 \right] = 2J \equiv \varepsilon$$



## Emittance of a beam

$$\langle X \rangle = \int X \rho(X, X') dX dX', \quad \langle X' \rangle = \int X' \rho(X, X') dX dX',$$

$$\sigma_X^2 = \int (X - \langle X \rangle)^2 \rho(X, X') dX dX', \quad \sigma_{X'}^2 = \int (X' - \langle X' \rangle)^2 \rho(X, X') dX dX',$$

$$\sigma_{XX'} = \int (X - \langle X \rangle)(X' - \langle X' \rangle) \rho(X, X') dX dX' = r \sigma_X \sigma_{X'}$$

$$\varepsilon_{rms} = \sqrt{\sigma_X^2 \sigma_{X'}^2 - \sigma_{XX'}^2} = \sigma_X \sigma_{X'} \sqrt{1 - r^2}$$

The rms emittance is invariant in linear transport:

$$\frac{d\varepsilon^2}{ds} = 0$$

Normalized emittance  $\epsilon_n = \epsilon \beta \gamma$  is **invariant** when beam energy is changed.

**Adiabatic damping** – beam emittance decreases with increasing beam momentum, i.e.  $\epsilon = \epsilon_n / \beta \gamma$ , which applies to beam emittance in **linacs**.

In storage rings, the beam emittance **increases** with energy ( $\sim \gamma^2$ ). The corresponding normalized emittance is proportional to  $\gamma^3$ .

## The Gaussian distribution function

$$\rho(X, P_X) = \frac{1}{2\pi\sigma_X^2} e^{-(X^2 + P_X^2)/2\sigma_X^2}$$

$$\rho(\epsilon) = \frac{1}{2\epsilon_{rms}} e^{-\epsilon/2\epsilon_{rms}}$$

$\epsilon/\epsilon_{rms}$	2	4	6	8
Percentage in 1D [%]	63	86	95	98
Percentage in 2D [%]	40	74	90	96



# Effects of Linear Magnetic field Error

$$x'' + [K_x(s) + k(s)]x = \frac{b_0}{\rho}, \quad y'' + [K_y(s) - k(s)]y = -\frac{a_0}{\rho}$$

For a localized dipole field error:

$$\theta = \Delta B \ell / B \rho$$

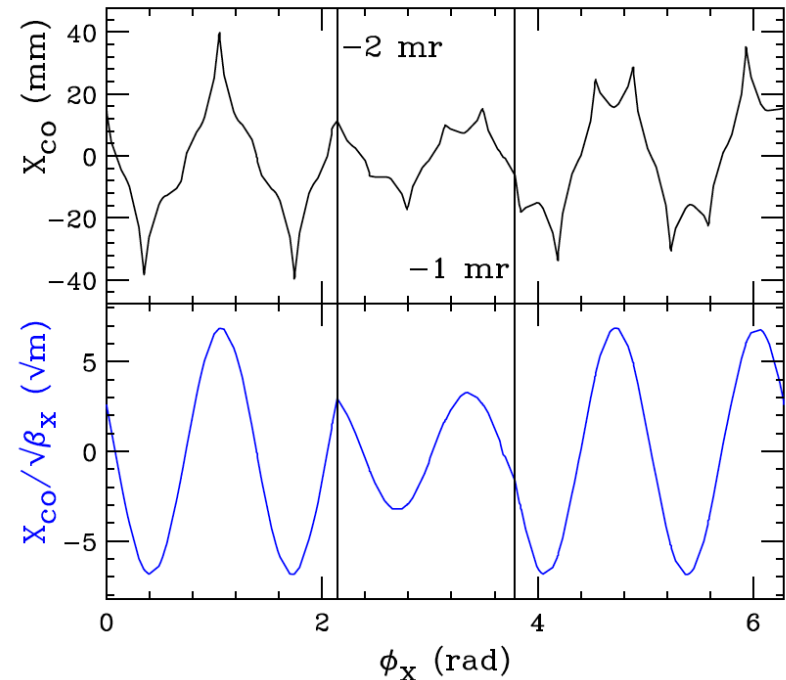
$$X'' + K_x(s)X = \theta \delta(s - s_0)$$

$$X_0 = \frac{\beta_0 \theta}{2 \sin \pi \nu} \cos \pi \nu,$$

$$X_0' = \frac{\theta}{2 \sin \pi \nu} (\sin \pi \nu - \alpha_0 \cos \pi \nu)$$

$$X_{co}(s) = G(s, s_0) \theta$$

$$G(s, s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2 \sin \pi \nu} \cos[\pi \nu - |\psi(s) - \psi(s_0)|]$$



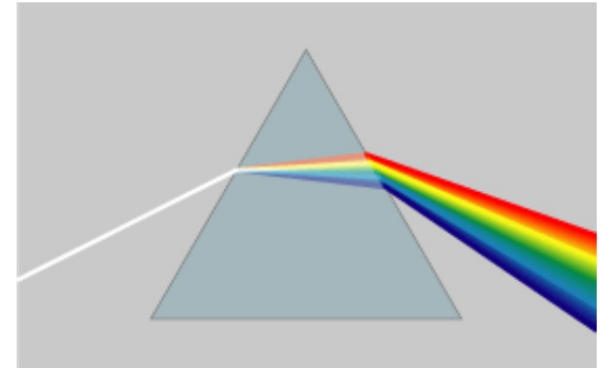
# Off-momentum and dispersion

For different particle energy  $\delta = \frac{p - p_0}{p_0}$

$$x = x_\beta + D\delta \quad x' = x'_\beta + D'\delta$$

$$x''_\beta + K_x(s)x_\beta = 0, \quad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + K_x(s)D = \frac{1}{\rho}$$



Extend the matrix representation to 3 by 3

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1) \begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

For a pure dipole (K=0):

$$M = \begin{pmatrix} \cos\theta & \rho \sin\theta & \rho(1 - \cos\theta) \\ -\frac{1}{\rho} \sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\theta \ll 1 \quad \text{i.e.} \quad L \ll \rho$$

For quadrupoles:

$$M(s, s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}} \sin\sqrt{K}\ell & 0 \\ -\sqrt{K} \sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Defocusing  
change  $K \rightarrow -K$

# Quadrupole errors

The transfer matrix of the one-turn map is  $\mathbf{M}(s_1)=\mathbf{M}_0(s_1)m(s_1)$

$$M(s_1) = \begin{pmatrix} \cos \Phi_0 + \alpha_1 \sin \Phi_0 - \beta_1 \sin \Phi_0 k(s_1) ds_1 & \beta_1 \sin \Phi_0 \\ -\gamma_1 \sin \Phi_0 - [\cos \Phi_0 + \alpha_1 \sin \Phi_0] k(s_1) ds_1 & \cos \Phi_0 - \alpha_1 \sin \Phi_0 \end{pmatrix}$$

$$\cos \Phi = \cos \Phi_0 - \frac{1}{2} \beta_1 k(s_1) ds_1 \sin \Phi_0$$

$$\Delta \Phi \approx \frac{1}{2} \beta_1 k(s_1) ds_1, \quad \Delta \nu \approx \frac{1}{4\pi} \beta_1 k(s_1) ds_1, \quad \Delta \nu \approx \frac{1}{4\pi} \oint \beta_1 k(s_1) ds_1$$

The betatron tune of the accelerator is changed by the quadrupole field error.

The betatron amplitude function is also changed by the quadrupole field error. The betatron amplitude function can be obtained by a one-turn map, i.e.

$$M(s_2) = M(s_2 + C, s_1)m(s_1)M(s_1, s_2)$$

$$\begin{aligned}\Delta[M(s_2)]_{12} &= -k(s_1)ds_1\beta_1\beta_2 \sin \nu_0(\phi_1 - \phi_2) \sin \nu_0(2\pi + \phi_2 - \phi_1) \\ &= -\frac{1}{2}k(s_1)ds_1\beta_1\beta_2(\cos 2\nu_0(\pi + \phi_2 - \phi_1) - \cos 2\pi\nu_0)\end{aligned}$$

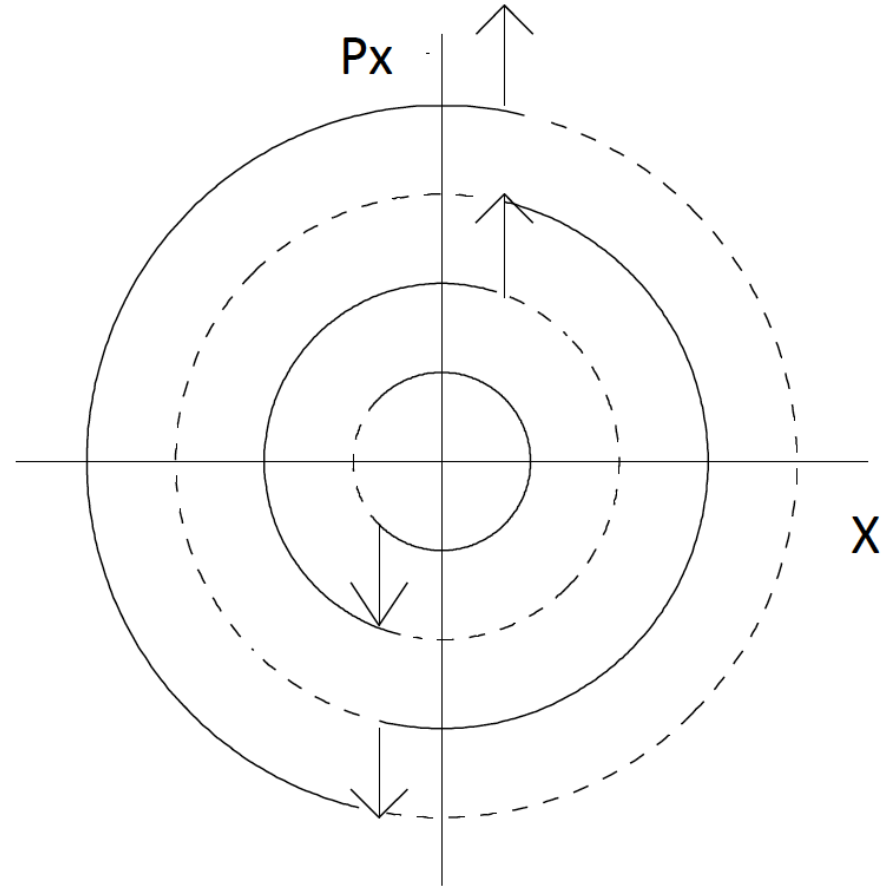
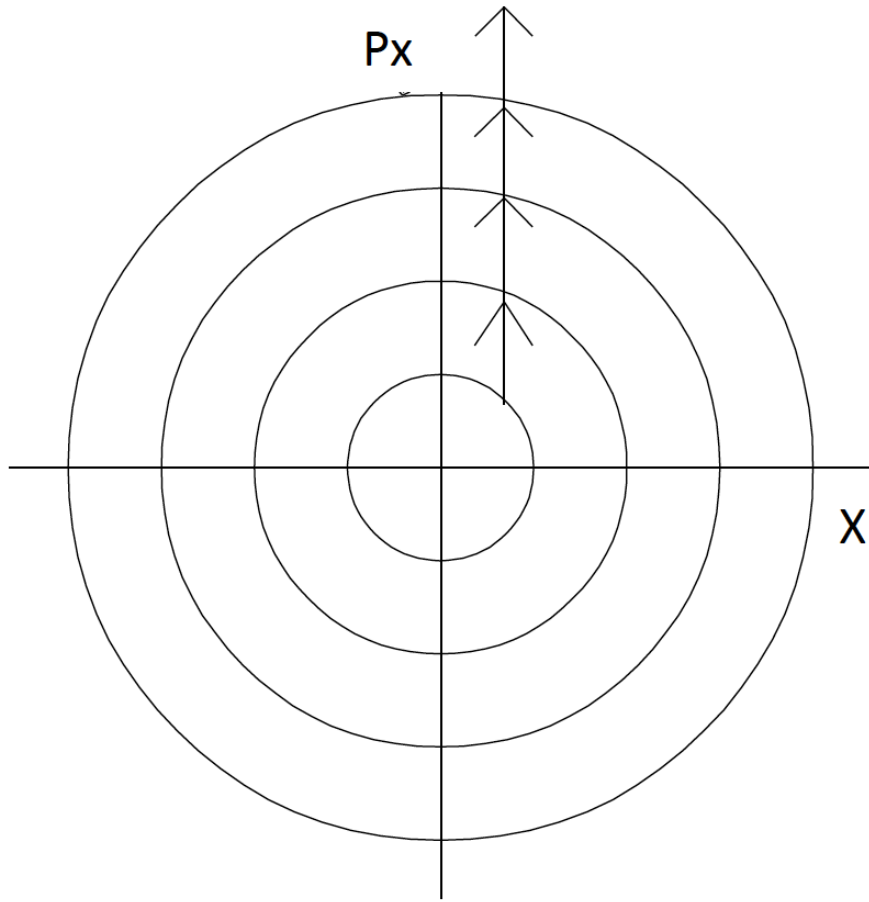
$$\begin{aligned}\Delta[M(s_2)]_{12} &= \Delta[\beta_2 \sin \Phi] \cong \Delta\beta_2 \sin \Phi_0 + \beta_2 \cos \Phi_0 \Delta\Phi \\ &= \Delta\beta_2 \sin \Phi_0 + \frac{1}{2}\beta_1 k(s_1)ds_1\beta_2 \cos \Phi_0\end{aligned}$$

$$\frac{\Delta\beta_2}{\beta_2} = -\frac{1}{2\sin \Phi_0}\beta_1 k(s_1)ds_1 \cos 2\nu_0(\pi + \phi_2 - \phi_1)$$

For a distributed quadrupole field error, the perturbation to the betatron amplitude function becomes

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{1}{2\sin \Phi_0} \int_s^{s+C} ds_1 \beta_1 k(s_1) \cos 2\nu_0(\pi + \phi_2 - \phi_1)$$

# Bad: Integer tune (dipole) & half integer (quadrupole)



# Synchrotron motion

$$\frac{d}{dt}(\phi - \phi_s) = -h\Delta\omega = h\omega_0 \frac{\Delta T}{T_0} = h\eta\omega_0 \frac{\Delta p}{p_0} = \frac{\eta h\omega_0^2}{\beta^2 E_0} \frac{\Delta E}{\omega_0}. \quad \eta = \alpha_c - \frac{1}{\gamma^2} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}.$$

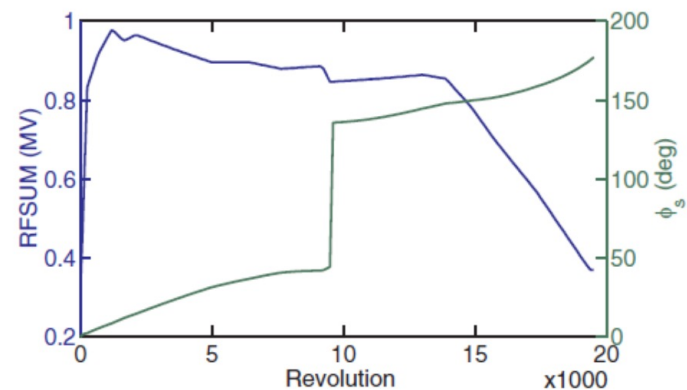
$$\frac{d}{dt} \left( \frac{\Delta E}{\omega_0} \right) = \frac{1}{2\pi} eV_0 (\sin \phi - \sin \phi_s),$$

$$\dot{\delta} = \frac{\omega_0}{2\pi\beta^2 E} eV (\sin \phi - \sin \phi_s)$$

$$\dot{\phi} = \frac{\partial H}{\partial \delta} = h\eta\omega_0 \delta \quad \dot{\delta} = -\frac{\partial H}{\partial \phi} = \frac{\omega_0 eV_0}{2\pi\beta^2 E} (\sin \phi - \sin \phi_s) \approx \frac{\omega_0 eV_0 \cos \phi_s}{2\pi\beta^2 E} (\phi - \phi_s)$$

$$\frac{d^2}{dt^2}(\phi - \phi_s) = \frac{h\omega_0^2 eV \eta_0 \cos \phi_s}{2\pi\beta^2 E} (\phi - \phi_s).$$

The stability condition for synchrotron oscillation is  $\eta_0 \cos \phi_s < 0$ , discovered by McMillan and Veksler. Below the transition energy, with  $\gamma < \gamma_T$  or  $\eta_0 < 0$ , the synchronous phase angle should be  $0 < \phi_s < \pi/2$ . Similarly the synchronous phase angle should be shifted to  $\pi - \phi_s$  above the transition energy.



$$\frac{d^2}{dt^2}(\phi - \phi_s) = \frac{h\omega_0^2 eV \eta_0 \cos \phi_s}{2\pi\beta^2 E} (\phi - \phi_s).$$

The angular synchrotron frequency is  $\omega_s = \omega_0 \sqrt{\frac{heV|\eta_0 \cos \phi_s|}{2\pi\beta^2 E}} = \frac{c}{R} \sqrt{\frac{heV|\eta \cos \phi_s|}{2\pi E}}$

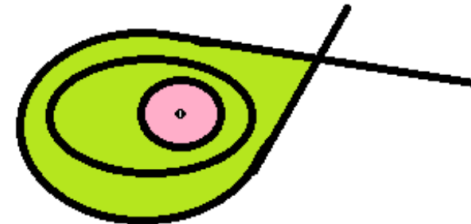
where  $c$  is the speed of light and  $R$  is the average radius of the synchrotron. The synchrotron tune, defined as the number of synchrotron oscillations per revolution, is

$$Q_s = \frac{\omega_s}{\omega_0} = \sqrt{\frac{heV|\eta_0 \cos \phi_s|}{2\pi\beta^2 E}}$$

Typically the synchrotron tune is of the order of  $\leq 10^{-3}$  for proton synchrotrons and  $10^{-1}$  for electron storage rings.

Fixed points of a Hamiltonian are located at phase space points with zero velocity field:

$$H(q, p, t): \quad \dot{q}_i = \frac{\partial H}{\partial p_i} = 0, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = 0.$$



$$H = \frac{1}{2}h\omega_0\eta_0\delta^2 + \frac{\omega_0 eV}{2\pi\beta^2 E} [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s]$$

$$\dot{\phi} = \frac{\partial H}{\partial \delta} = h\eta\omega_0\delta \quad \dot{\delta} = -\frac{\partial H}{\partial \phi} = \frac{\omega_0 eV_0}{2\pi\beta^2 E} (\sin \phi - \sin \phi_s)$$

The fixed points of the synchrotron Hamiltonian are at phase space points:  $(\phi_s, 0)$  and  $(\pi - \phi_s, 0)$ . Small amplitude motion around the stable fixed point  $(\phi_s, 0)$  is elliptical with synchrotron tune  $Q_s$ . Motion near the UFP is hyperbolic.



The Hamiltonian torus that passes through the unstable fixed point is called the separatrix.

$$H(\phi, \delta) = H_{sx} = H(\pi - \phi_s, 0) = \frac{\omega_0 e V_0}{2\pi\beta^2 E} [-2\cos\phi_s + (\pi - 2\phi_s)\sin\phi_s]$$

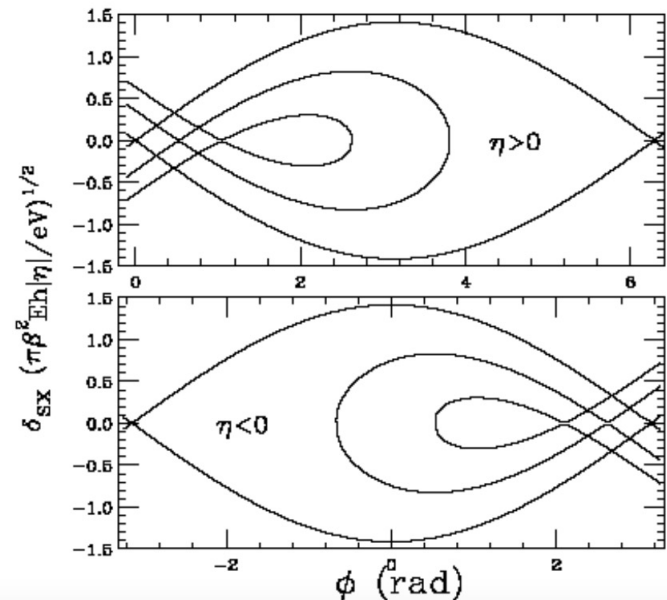
$$\delta_{sx}^2 + \frac{eV_0}{\pi\beta^2 E h \eta} [\cos\phi + \cos\phi_s - (\pi - \phi - \phi_s)\sin\phi_s] = 0$$

The separatrix has two turning points,  $\phi_u$  and  $\pi - \phi_s$ , where  $\phi_u$  is

$$\cos\phi_u + \phi_u \sin\phi_s = -\cos\phi_s + (\pi - \phi_s)\sin\phi_s.$$

The phase space area enclosed by the separatrix is called the bucket, where particle motion around the stable fixed point is elliptical. The motion around the unstable fixed point is hyperbolic.

The bucket length is  $|(\pi - \phi_s) - \phi_u|$ .



The phase-space area enclosed by the separatrix orbit is called the bucket area.

$$\tilde{A}_B = \oint \delta_{sx}(\phi) d\phi = 16 \sqrt{\frac{eV_0}{2\pi\beta^2 E h |\eta|}} \alpha_b(\phi_s) = \frac{16 v_s}{h |\eta|} \alpha_b(\phi_s)$$

$$\alpha_b(\phi_s) = \frac{1}{4\sqrt{2}} \int_{\phi_u}^{\pi - \phi_s} \left[ -\frac{|\eta|}{\eta} [\cos \phi + \cos \phi_s - (\pi - \phi - \phi_s) \sin \phi_s] \right]^{1/2} d\phi.$$

Here  $\alpha_b$  is the moving bucket factor.

The maximum momentum deviation of the separatrix orbit is called the bucket height.

$$\delta_B = \left( \frac{2eV}{\pi\beta^2 E h |\eta|} \right)^{1/2} Y(\phi_s) = \frac{2Q_s}{h|\eta|} \tilde{Y}(\phi_s).$$

$$Y(\phi_s) = \left| \cos \phi_s - \frac{\pi - 2\phi_s}{2} \sin \phi_s \right|^{1/2}$$

$$\tilde{Y}(\phi_s) = \left| 1 - \frac{\pi - 2\phi_s}{2} \tan \phi_s \right|^{1/2}$$

