Synchrotron Radiation

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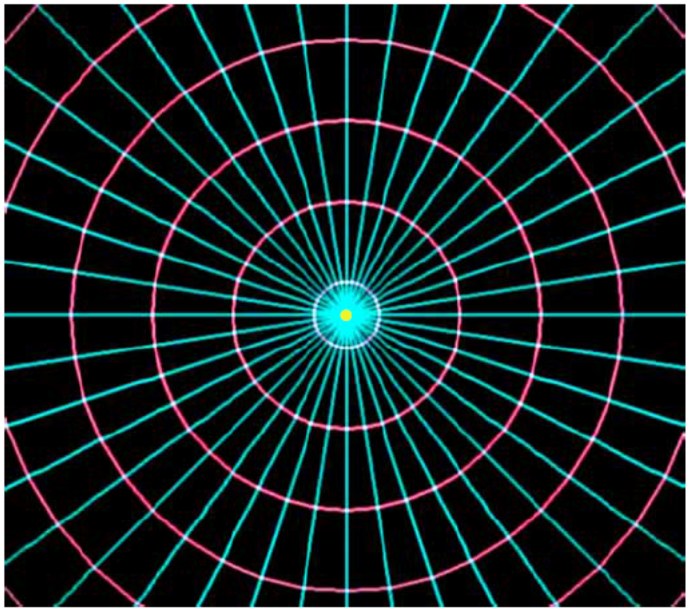
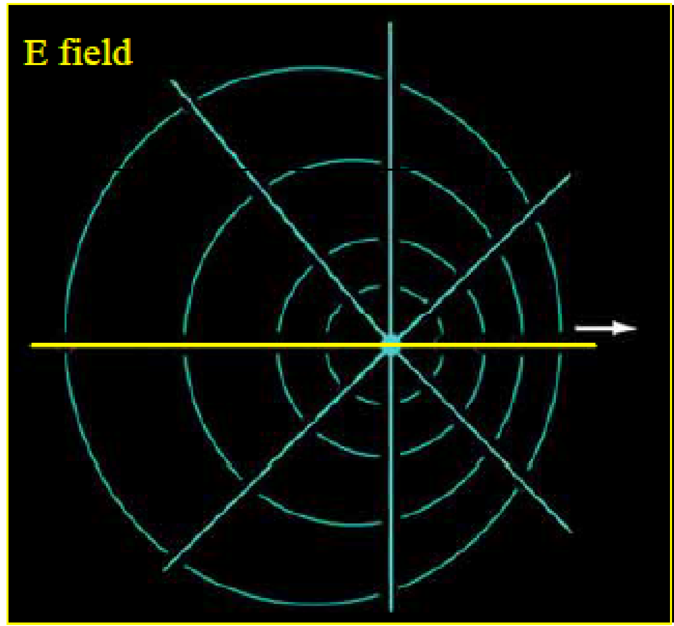
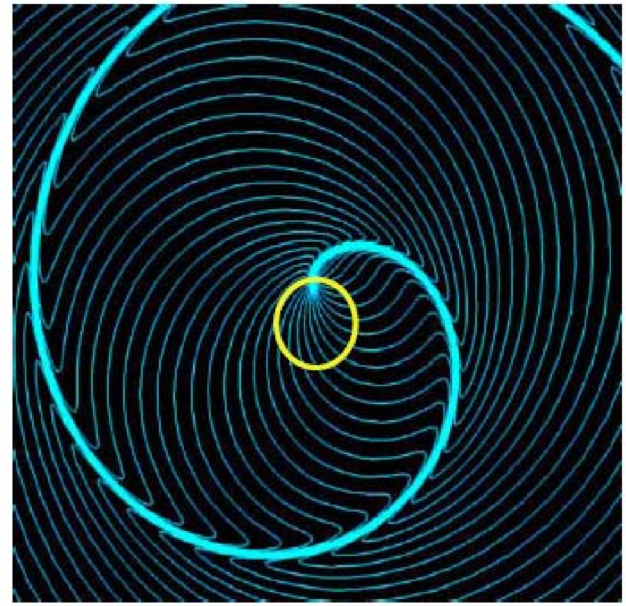
What is synchrotron radiation



Static field for a charge at rest



When a particle moves with a constant velocity, field moves with particle.

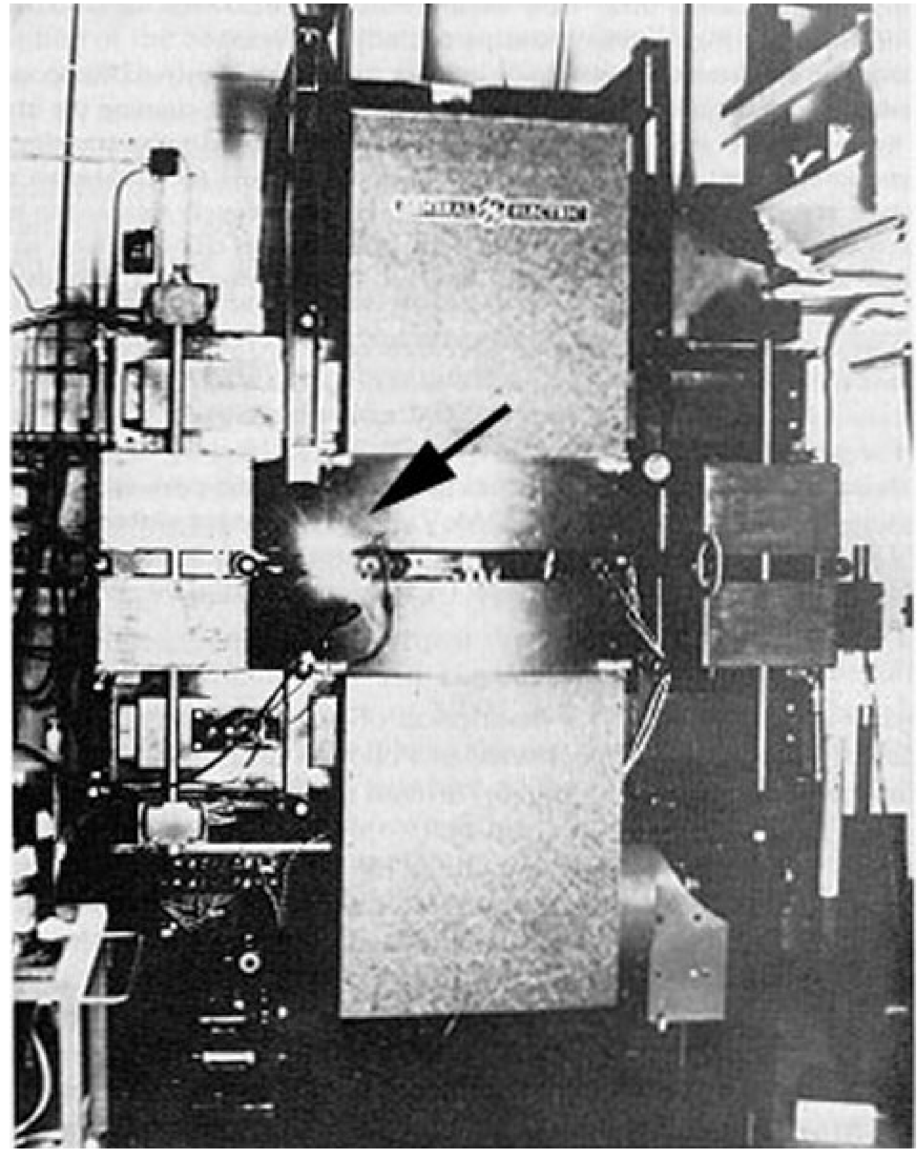


When a particle gets accelerated, some part of the field moves away from the particle to infinity: **radiation.**

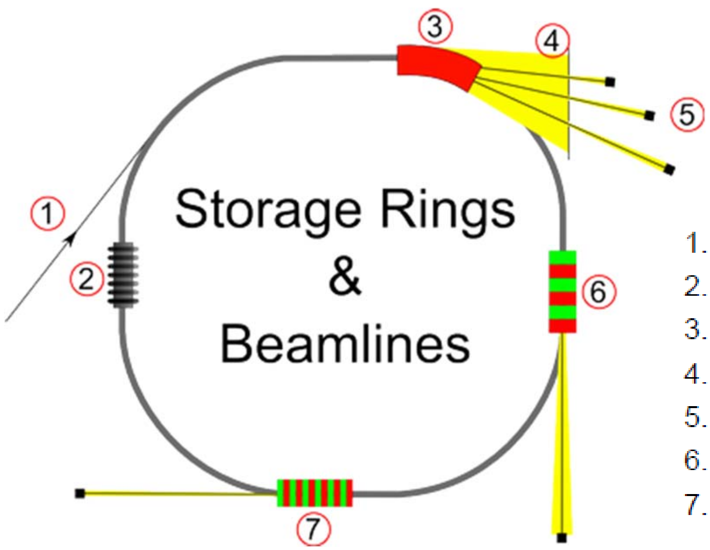
The electromagnetic radiation emitted when charged particles are accelerated radially, $\vec{a} \perp \vec{v}$, is called **synchrotron radiation**.

Some history of Synchrotron radiation

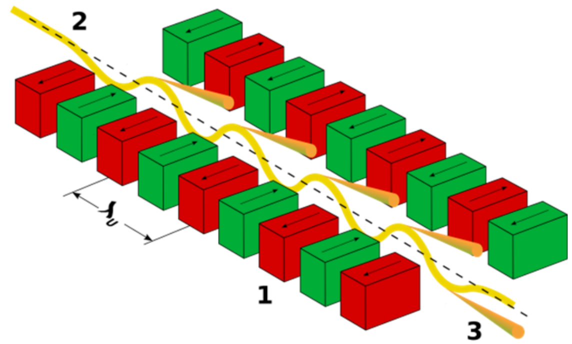
- Synchrotron radiation was named after its discovery in Schenectady, New York from a General Electric synchrotron accelerator built in 1946 and announced in May 1947 by Frank Elder, Anatole Gurewitsch, Robert Langmuir .
- Synchrotron radiation is the main constraint to accelerate electrons to very high energy and hence is bad for high energy physics application, such as colliders.
- However, it was then realized that the radiation can be so helpful for other branches of science such as biology, material science and medical applications. As a result, dedicated storage rings have been built to generate synchrotron radiation, which are called light sources.

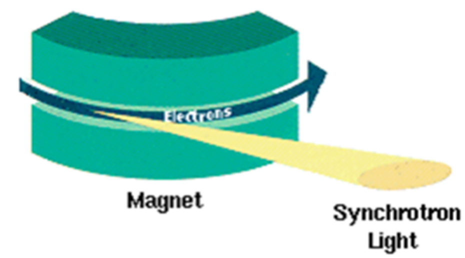


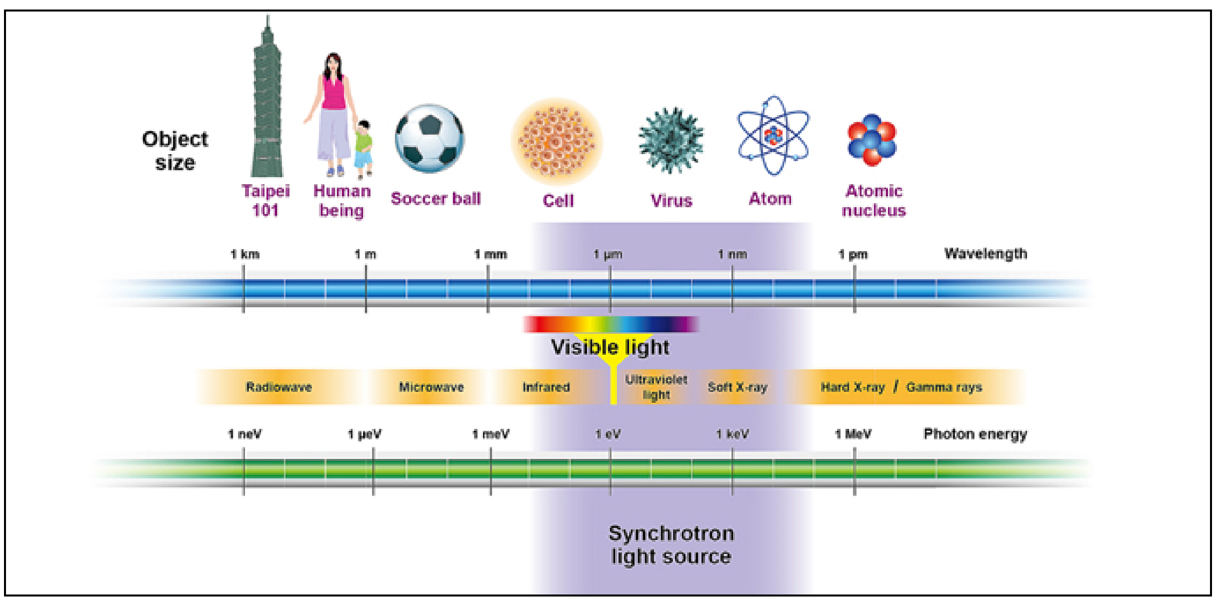
Application of Synchrotron Radiation



1. Injection
2. RF system
3. Storage Ring/Bending Magnets
4. Beamlines
5. Experiments
6. Wigglers
7. Undulators



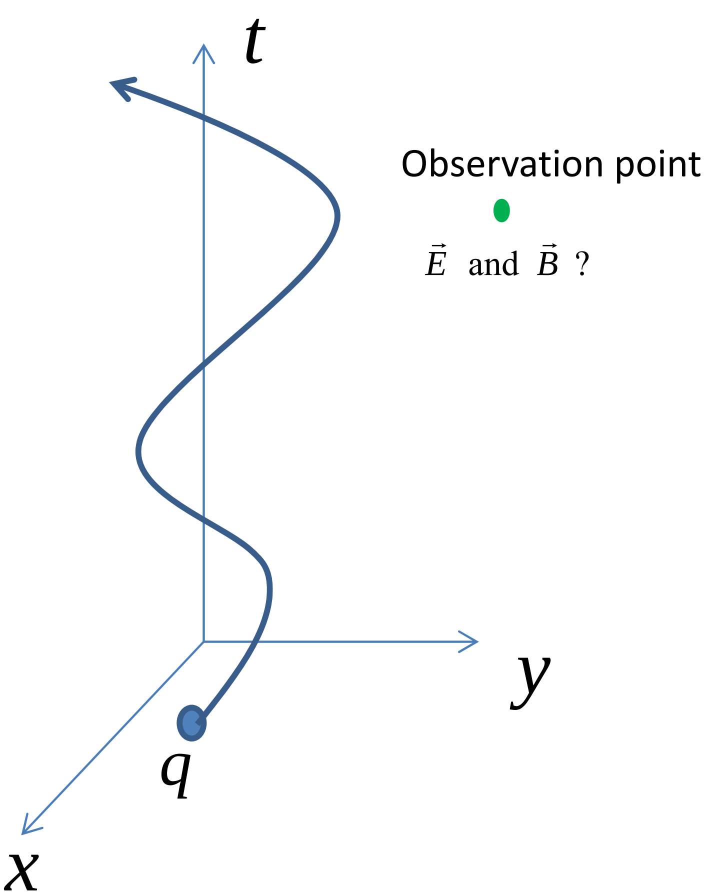




Plant Architectural view of NSLS-II

Theoretical Model: wave equation

To better understand how the synchrotron radiation is quantitatively investigated, we will try to derive formulas from 'first principle'. (refer to 'Accelerator physics' by S.Y. Lee and ' classical electrodynamics' by J.D. Jackson)



$$\Box A^\alpha \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A^\alpha - \nabla^2 A^\alpha = \mu_0 J^\alpha \qquad A^\alpha = \left(\frac{\Phi}{c}, \vec{A}\right)$$

Traditionally, the equation is solved by finding the Green function

$$A^\alpha(x) = \mu_0 \int d^4x' D(x-x') J^\alpha(x')$$

$$\Box D(\vec{x},t) \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2}D(\vec{x},t) - \nabla^2 D(\vec{x},t) = \delta^{(4)}(\vec{x},ct)$$

through Fourier transformation:

$$D(k) = -\frac{1}{k_0^2 - |\vec{k}|^2} \Rightarrow$$

The singularity is treated by treating k_0 as a complex number, which make it NOT Fourier transformation...

$$D(x) = \frac{1}{(2\pi)^4}\int d^4k D(k) e^{-ik_\mu z^\mu} = -\frac{1}{(2\pi)^4}\int d^3k e^{i\vec{k}\cdot\vec{x}} \int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0 x_0}}{k_0^2 - |\vec{k}|^2}$$

Laplace Transformation

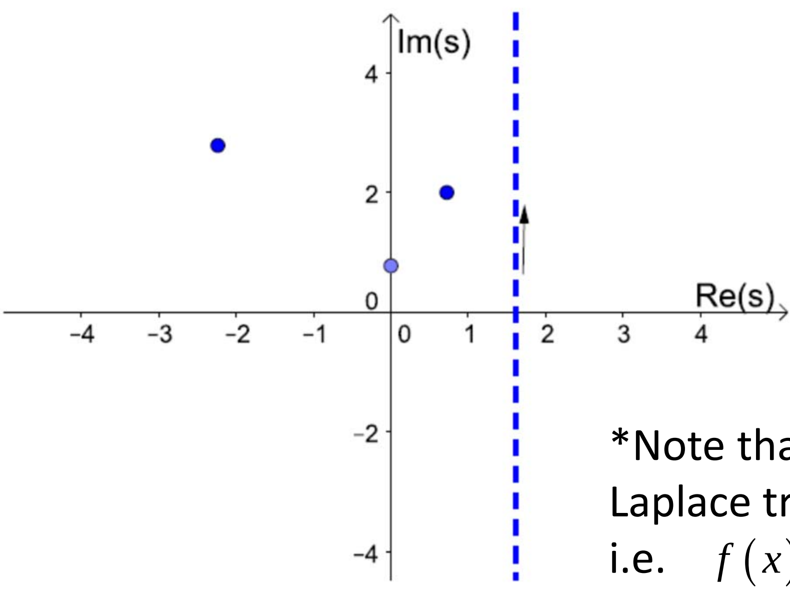
The Laplace transform of the function f(x), denoted by F(s), is defined by the integral

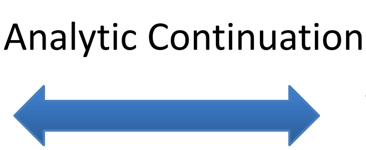
$$F(s) = \int_0^{\infty} e^{-sx} f(x)\,dx \qquad \text{for} \quad \mathrm{Re}(s) > 0$$

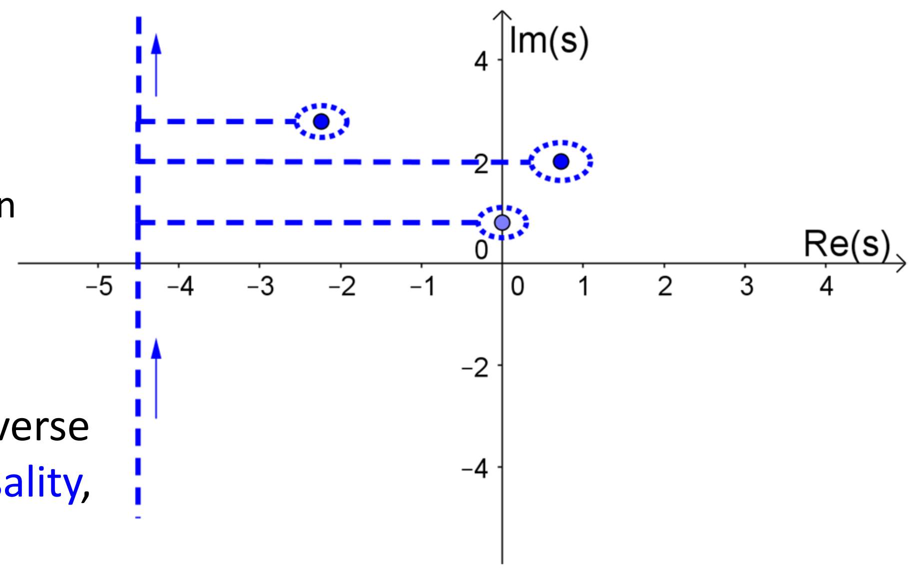
The inversion of the Laplace transform is accomplished for analytic function F(s) by means of the inversion integral*

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sx} F(s)\,ds \qquad \text{for} \quad \mathrm{Re}(s) > 0$$

where γ is a real constant that exceeds the real part of all the singularities of F(s).

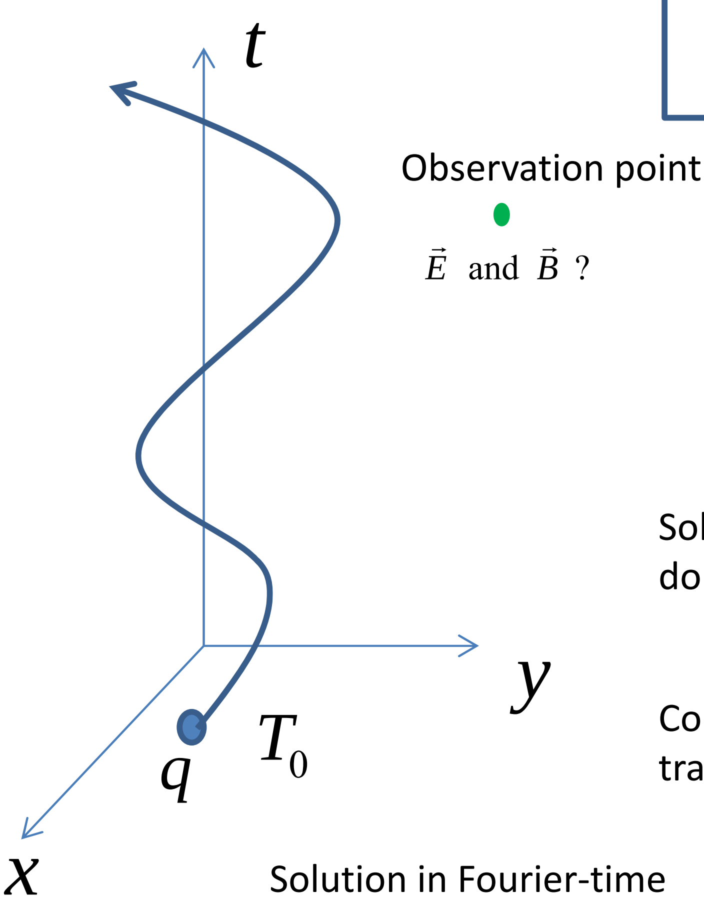






*Note that the definition of inverse Laplace transform implies causality, i.e. $f(x) = 0 \quad \text{for} \;\; x < 0$

Theoretical Model I: wave equation



$$\Box A^{\alpha} \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A^{\alpha} - \nabla^2 A^{\alpha} = \mu_0 J^{\alpha} \qquad A^{\alpha} = \left(\frac{\Phi}{c}, \vec{A}\right)$$

$$\tilde{A}^{\alpha}\left(\vec{k}, s\right) \equiv \int_0^{\infty} dx_0 \exp\left(-sx_0\right) \int_{-\infty}^{\infty} d^3x \exp\left(-i\vec{k}\cdot\vec{x}\right) A^{\alpha}\left(x_0 + T_0, \vec{x}\right)$$

$$\tilde{J}^{\alpha}\left(\vec{k}, s\right) \equiv \int_0^{\infty} dx_0 \exp\left(-sx_0\right) \int_{-\infty}^{\infty} d^3x \exp\left(-i\vec{k}\cdot\vec{x}\right) J^{\alpha}\left(x_0 + T_0, \vec{x}\right)$$

Solution in Fourier-Laplace domain →

$$\tilde{A}^{\alpha}\left(\vec{k}, s\right) = \mu_0 \frac{1}{s^2 + \kappa^2} \cdot \tilde{J}^{\alpha}\left(\vec{k}, s\right)$$

$$\kappa^2 \equiv k_1^2 + k_2^2 + k_3^2$$

Convolution theory of Laplace transformation: →

$$\mathfrak{L}^{-1}\left[F(s)G(s)\right] = \int_0^{x_0} f\left(x_0 - \xi\right) g(\xi)\, d\xi$$

Solution in Fourier-time domain →

$$\tilde{A}^{\alpha}\left(\vec{k}, x_0 + T_0\right) = \mu_0 \int_0^{x_0} \frac{\sin\left(\kappa\left(x_0 - \xi\right)\right)}{\kappa} \tilde{J}^{\alpha}\left(\vec{k}, \xi + T_0\right) d\xi$$

Convolution theory of Fourier transformation: →

$$\mathfrak{F}^{-1}\left[F\left(k_x\right)G\left(k_x\right), x\right] = \int_{-\infty}^{\infty} f\left(x - \xi\right) g(\xi)\, d\xi$$

Theoretical Model II: wave equation

$$f(\vec{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{\sin(\kappa(x_0-\eta))}{\kappa}\exp(-i\vec{k}\cdot\vec{x})d^3k = \frac{1}{4\pi|\vec{x}|}\left[\delta(x_0-\eta-|\vec{x}|) - \delta(x_0-\eta+|\vec{x}|)\right]$$

$$A^\alpha(\vec{x}, x_0+T_0) = \mu_0\int_0^{x_0} d\eta \int_{-\infty}^{\infty} \frac{\delta(x_0-\eta-|\vec{x}-\vec{\xi}|)}{4\pi|\vec{x}-\vec{\xi}|} J^\alpha(\vec{\xi}, \eta+T_0)d^3\xi$$

$$\frac{\delta(x_0-x'_0-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|} = 2\delta\left((x-x')^2\right) - \frac{\delta(x_0-x'_0+|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}$$

⇦ $\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$

$$A^\alpha(\vec{x}, x_0) = \mu_0\int_{-\infty}^{\infty} dx_0{}'\int_{-\infty}^{\infty} d^3x' D_r(x-x') J^\alpha(\vec{x}', x_0{}')$$

$$D_r(x-x') \equiv \frac{1}{2\pi}H(x_0-x_0{}')\delta\left((x-x')^2\right)$$

Theoretical Model III: Solution for point charge (Lienard-Wiechert Potential)

$$J^{\alpha}(x) = ec\int_{-\infty}^{\infty} d\tau U^{\alpha}(\tau)\,\delta^{(4)}\left[x - r(\tau)\right] = \left(ec\delta^{(3)}\left(\vec{x} - \vec{r}(t)\right), e\vec{v}(t)\,\delta^{(3)}\left(\vec{x} - \vec{r}(t)\right)\right)$$

$$A^{\alpha}(\vec{x}, x_0) = \frac{e\mu_0 c}{2\pi}\int_{-\infty}^{\infty} U^{\alpha}(\tau) H\left(x_0 - r_0(\tau)\right)\delta\left(\left(x - r(\tau)\right)^2\right) d\tau$$

$$U^{\alpha}(\tau) = \left(\gamma c, \gamma\vec{v}(\tau)\right)$$

$$t_r = \frac{r_0(\tau_0)}{c} = \frac{1}{c}\left[x_0 - R(t_r)\right]$$

$$\delta(f(x)) = \frac{\delta(x - x_0)}{\left|f'(x_0)\right|} \qquad f(x_0) = 0$$

$$R(t_r) = \left|\vec{x} - \vec{r}(t_r)\right| \qquad \vec{n} \equiv \frac{\vec{R}(\tau_0)}{R(\tau_0)} = \frac{\vec{x} - \vec{r}(\tau_0)}{\left|\vec{x} - \vec{r}(\tau_0)\right|}$$

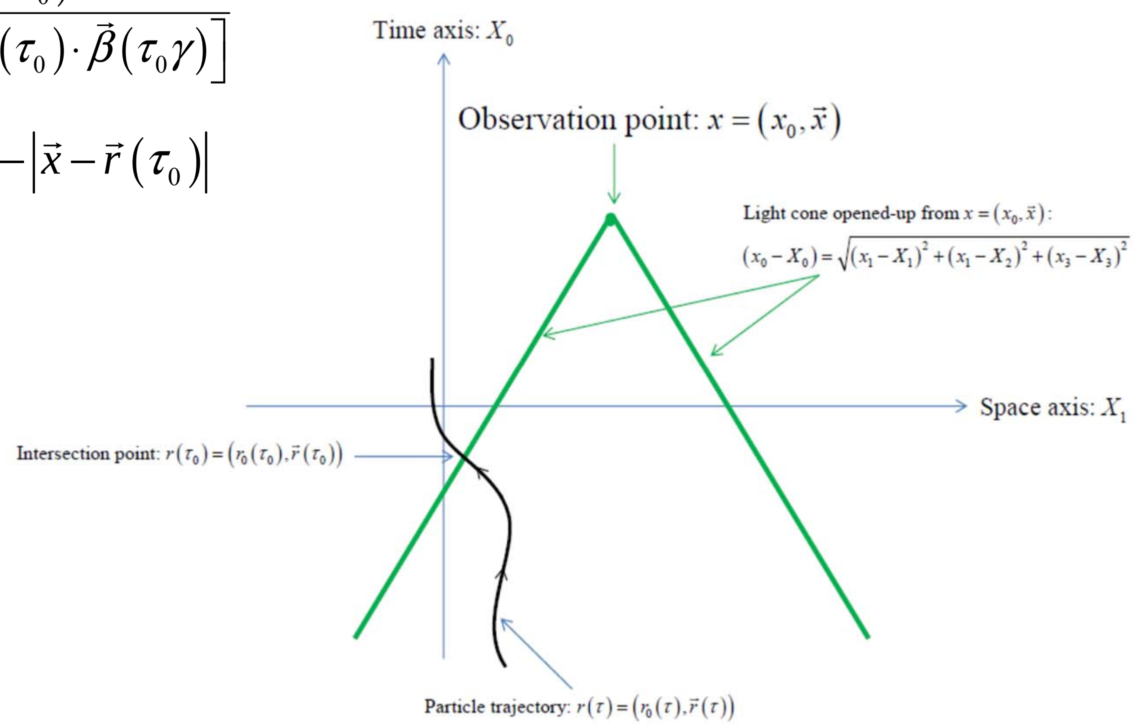
$$\delta\left(\left(x - r(\tau)\right)^2\right) = \frac{\delta(\tau - \tau_0)}{\left|\frac{d}{d\tau}\left(x - r(\tau)\right)^2\right|_{\tau=\tau_0}\Bigg|} = \frac{\delta(\tau - \tau_0)}{\gamma c R(\tau_0)\left[1 - \vec{n}(\tau_0)\cdot\vec{\beta}(\tau_0\gamma)\right]}$$

$$r_0(\tau_0) = x_0 - \left|\vec{x} - \vec{r}(\tau_0)\right|$$

Lienard-Wiechert Potential:

$$\vec{A}(\vec{x}, x_0) = \frac{e\mu_0 c}{4\pi}\frac{\vec{\beta}(t_r)}{R(t_r)\left[1 - \vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]}$$

$$\Phi(\vec{x}, x_0) = \frac{e}{4\pi\varepsilon_0}\frac{1}{R(t_r)\left[1 - \vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]}$$



Theoretical Model III: E&M field

The electric and magnetic field can be directly obtained from the following relation (notice that t_r depends on $\left(\vec{x},t\right)$.

$$\vec{E}\left(\vec{x},t\right)=-\vec{\nabla}_x\Phi\left(\vec{x},t\right)-\frac{\partial}{\partial t}\vec{A}\left(\vec{x},t\right) \qquad \vec{B}\left(\vec{x},t\right)=\vec{\nabla}_x\times\vec{A}\left(\vec{x},t\right)$$

$$\frac{dt_r}{dt}=\frac{1}{1-\vec{n}\left(t_r\right)\cdot\vec{\beta}\left(t_r\right)} \qquad \vec{\nabla}_x t_r=-\frac{\vec{n}\left(t_r\right)}{c\left[1-\left(\vec{n}\left(t_r\right)\cdot\vec{\beta}\left(t_r\right)\right)\right]} \qquad \frac{d}{dt}R\left(t_r\right)=\frac{-\vec{n}\cdot\vec{\beta}\left(t_r\right)c}{1-\vec{n}\left(t_r\right)\cdot\vec{\beta}\left(t_r\right)} \qquad \vec{\nabla}_x R\left(t_r\right)=\frac{\vec{n}\left(t_r\right)}{1-\vec{n}\left(t_r\right)\cdot\vec{\beta}\left(t_r\right)}$$

$$\begin{aligned}\vec{E}\left(\vec{x},t\right)&=\frac{e}{4\pi\varepsilon_0\gamma^2\left(t_r\right)}\frac{\left(\vec{n}\left(t_r\right)-\vec{\beta}\left(t_r\right)\right)}{R\left(t_r\right)^2\left[1-\vec{n}\left(t_r\right)\cdot\vec{\beta}\left(t_r\right)\right]^3}+\frac{e}{4\pi\varepsilon_0 c}\frac{\vec{n}\times\left[\left(\vec{n}\left(t_r\right)-\vec{\beta}\left(t_r\right)\right)\times\dot{\vec{\beta}}\left(t_r\right)\right]}{R\left(t_r\right)\left[1-\vec{n}\left(t_r\right)\cdot\vec{\beta}\left(t_r\right)\right]^3}\\&=\vec{E}_{static}\left(\vec{x},t\right)+\vec{E}_{rad}\left(\vec{x},t\right)\end{aligned}$$

$$\vec{B}\left(\vec{x},t\right)=\frac{1}{c}\vec{n}\times\vec{E}\left(\vec{x},t\right)$$

Note: Jackson follows a different approach but directly taking derivatives generate the same result.

Radiation Power I

Taking the radiation part of the field

$$\vec{E}_{rad} = \frac{e}{4\pi\varepsilon_0 c}\frac{\vec{n}\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3} \qquad \vec{B}_{rad}(\vec{x},t) = \frac{1}{c}\vec{n}\times\vec{E}_{rad}(\vec{x},t)$$

and the energy flow is determined by the Poynting vector

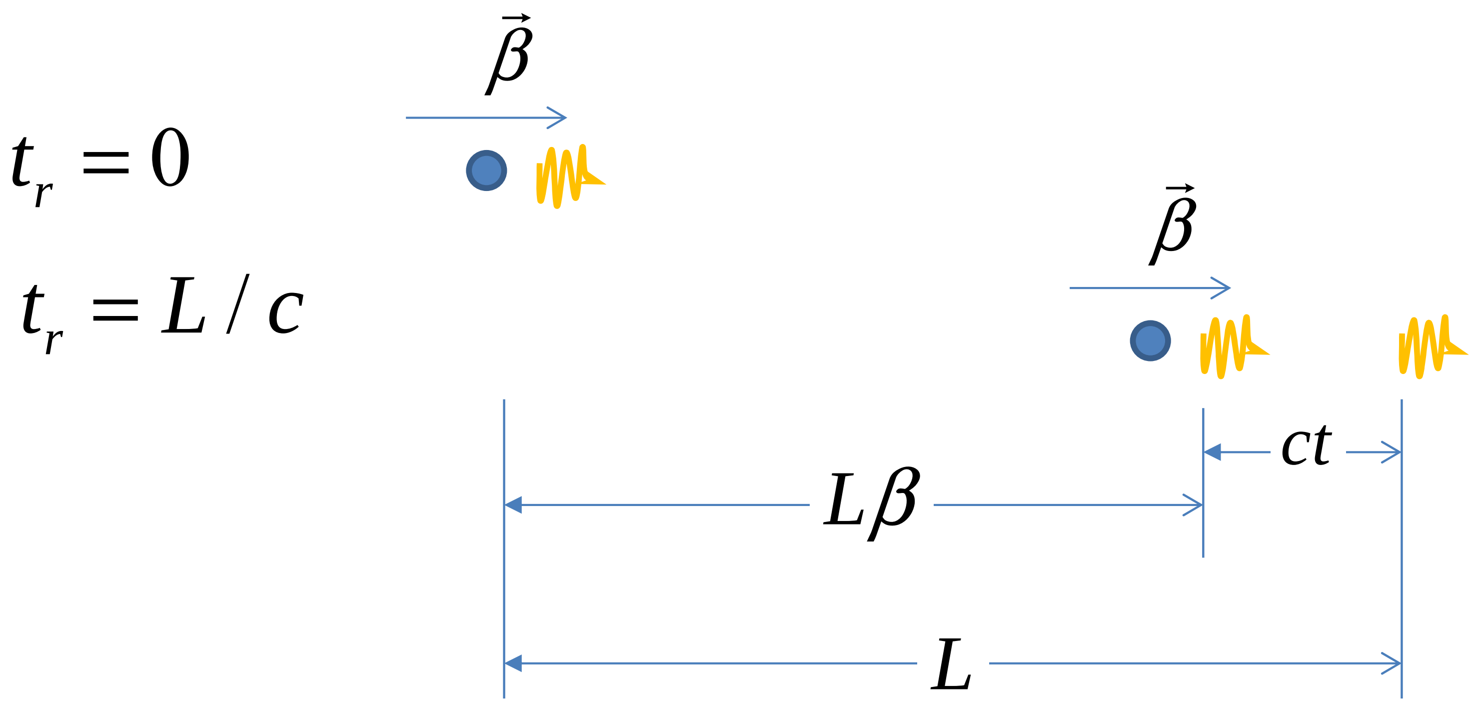
$$\vec{S}(\vec{x},t) = \frac{1}{\mu_0}\vec{E}_{rad}(\vec{x},t)\times\vec{B}_{rad}(\vec{x},t) = \frac{1}{c\mu_0}E_{rad}^2(\vec{x},t)\vec{n}$$

The radiated power per solid angle is then given by

$$\frac{dP(t_r)}{d\Omega} = \left(\vec{n}\cdot\vec{S}\right)R(t_r)^2\frac{dt}{dt_r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\left|\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]\right|^2}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^5}$$

Time interval difference between radiation and observation. See the next slide

Time interval at radiation point and the observation point



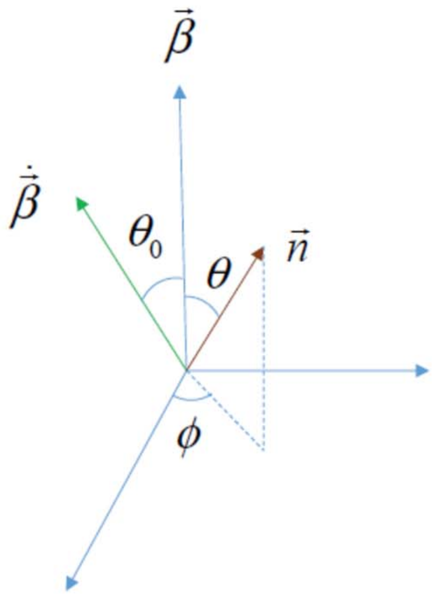
$$ct = L - L\beta \Rightarrow t = t_r\left(1-\beta\right)$$

Time interval at observation point (t)

Time interval at the radiation point (t_r)

Radiation Power II

$$\frac{dP(t_r)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\left|\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]\right|^2}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^5}$$



$$\dot{\vec{\beta}} = \dot{\beta}\left(\sin\theta_0, 0, \cos\theta_0\right)$$

$$\vec{\beta} = \beta(0,0,1)$$

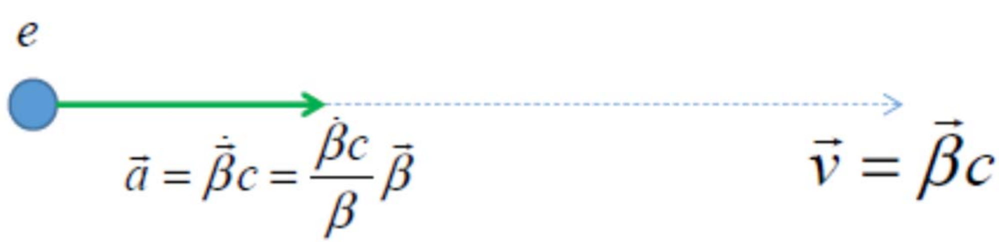
$$\vec{n} = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$$

$$\vec{n}\cdot\dot{\vec{\beta}} = \dot{\beta}\left(\sin\theta\sin\theta_0\cos\phi + \cos\theta\cos\theta_0\right)$$

$$P(t_r) = \int\frac{dP(t_r)}{d\Omega}d\Omega = \int\frac{dP(t_r)}{d\Omega}\sin\theta d\theta d\phi = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c}\gamma^6\left[\dot{\beta}^2-\left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right]$$

Note: Jackson uses Lorentz transformation to derive this from non-relativistic result. Here, we take a more tedious but straightforward approach.

Parallel acceleration (Linac)

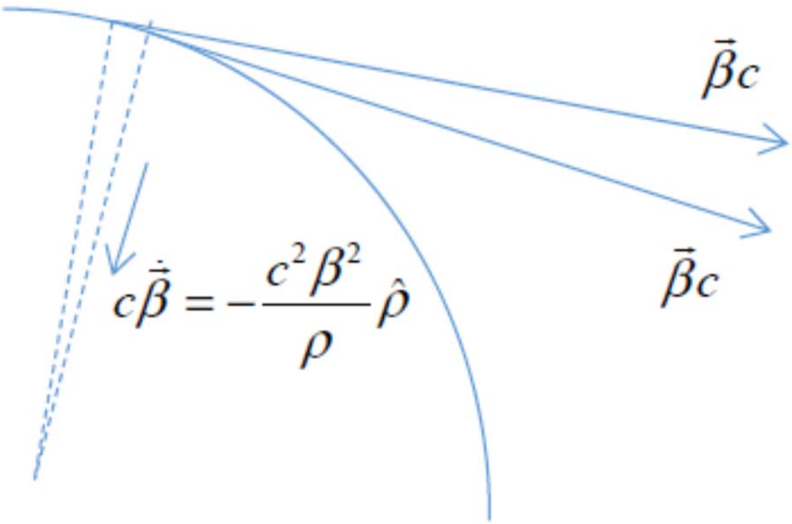


$$P(t_r) = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{e^2}{c} \gamma^6 \dot{\beta}^2$$

$$\frac{P(t_r)}{dE/dt} = \frac{2}{3\beta} \frac{dE/dx}{mc^2/r_e} \qquad \frac{mc^2}{r_e} = \frac{0.55MeV}{2.8 \cdot 10^{-15} m} = 1.9 \times 10^{14} \frac{MeV}{m}$$

The state of art accelerating rate at the moment is below 100 MeV/m and hence synchrotron radiation is negligible in linear accelerators.

Circular orbit



$$\dot{a} = -\frac{v^2}{\rho}\hat{\rho} \Rightarrow \dot{\vec{\beta}} = -\frac{\beta^2 c}{\rho}\hat{\rho}$$

$$P(t_r) = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c}\gamma^6\dot{\beta}^2\left(1-\beta^2\right) = \frac{1}{4\pi\varepsilon_0}\frac{2e^2c\beta^4\gamma^4}{3\rho^2}$$

For a storage ring, the energy loss per turn: $U_0 = \int_C P(t_r)\,dt = \frac{1}{\beta c}\int_C P(t_r)\,ds = \frac{1}{4\pi\varepsilon_0}\frac{2e^2\beta^3\gamma^4}{3}\int_C \frac{1}{\rho^2}ds$

If all dipoles in the storage ring has the same bending radius (iso-magnetic case): $U_0 = \frac{1}{4\pi\varepsilon_0}\frac{2e^2\beta^3\gamma^4}{3}\frac{2\pi\rho}{\rho^2} = \frac{e^2\beta^3\gamma^4}{3\varepsilon_0\rho}$

Power radiated by a beam of average current I_b: $P_{beam} = U_0\frac{I_b}{e} = \frac{e\beta^3\gamma^4}{3\varepsilon_0\rho}I_b$

Compare parallel with perpendicular

$$P_{//}(t_r) = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2\dot{\beta}_{//}^{\,2}}{c}\gamma^6$$

$$P_{\perp}(t_r) = \frac{1}{4\pi\varepsilon_0}\frac{2e^2 c\beta^4\gamma^4}{3\rho^2} = \frac{1}{4\pi\varepsilon_0}\frac{2e^2\dot{\beta}_{\perp}^{\,2}}{3c}\gamma^4$$

It looks as if the longitudinal acceleration cause more radiation for the same values of acceleration… However what really matters is the force.

$$\dot{\beta}_{\perp} = \frac{a_{\perp}}{c} = \frac{\beta^2 c}{\rho}$$

$$F_{//} = \frac{dp_{//}}{dt} = c\frac{d}{dt}(m\gamma\beta_{//}) = cm\frac{1}{\beta_{//}}\frac{d\gamma}{dt} = m\gamma^3 c\dot{\beta}_{//}$$

$$F_{\perp} = \frac{dp_{\perp}}{dt} = c\frac{d}{dt}(m\gamma\beta_{\perp}) = m\gamma c\dot{\beta}_{\perp}$$

$$P_{//}(t_r) = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2\gamma^6}{c}\left(\frac{F_{//}}{mc\gamma^3}\right)^2 = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{m^2c^3}F_{//}^{\,2}$$

$$P_{\perp}(t_r) = \frac{1}{4\pi\varepsilon_0}\frac{2e^2\gamma^4}{3c}\left(\frac{F_{\perp}}{m\gamma c}\right)^2 = \frac{1}{4\pi\varepsilon_0}\frac{2e^2\gamma^2}{3mc^3}F_{\perp}^{\,2}$$

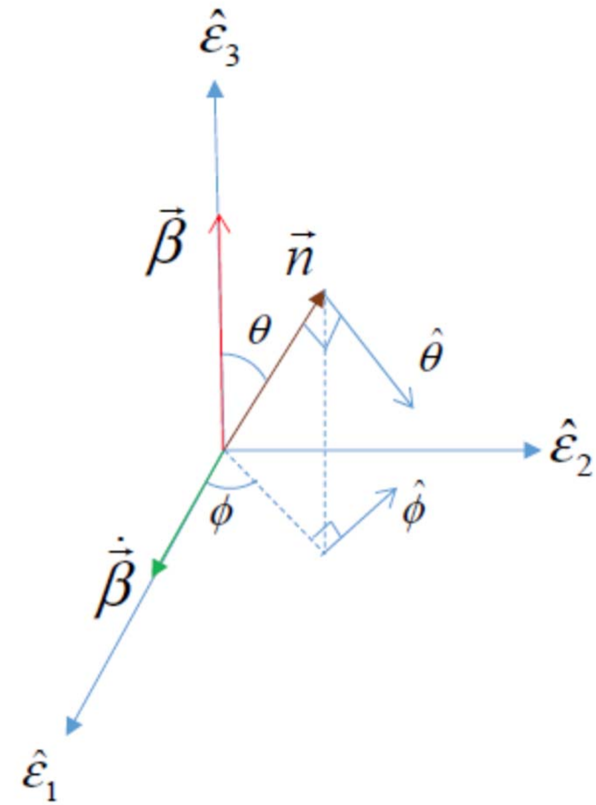
Therefore, for similar accelerating force, the radiation power from perpendicular acceleration is larger than that from parallel acceleration by a factor of γ^2 .

Angular distribution (Circular orbit)

$$\frac{dP(t_r)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\left|\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]\right|^2}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^5}$$

$$\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right] = -\dot{\beta}(\cos\theta-\beta)\cos\phi\hat{\theta}+\dot{\beta}\sin\phi(1-\beta\cos\theta)\hat{\phi}$$

$$\frac{dP(t_r)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$



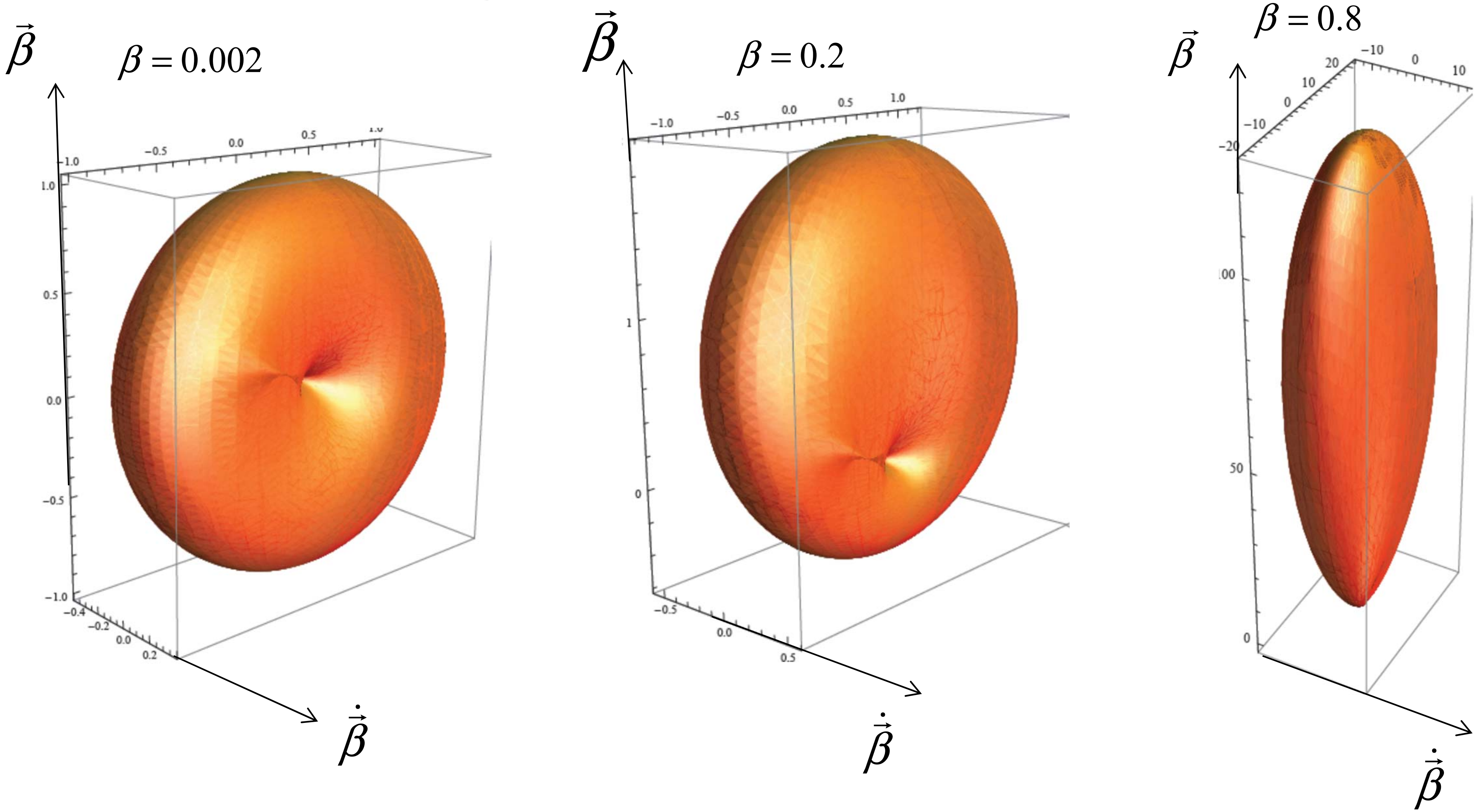
For $\gamma^{-4} << \theta << 1$ and $\gamma >> 1$, it can be shown that the angular spread of the radiation power is ~ γ^{-1}.

$$\dot{\vec{\beta}} = \dot{\beta}\hat{\varepsilon}_1 = \dot{\beta}\left(\vec{n}\sin\theta\cos\phi+\hat{\theta}\cos\theta\cos\phi-\hat{\phi}\sin\phi\right)$$

$$\vec{\beta} = \beta\hat{\varepsilon}_3 = \beta\left(\vec{n}\cos\theta-\hat{\theta}\sin\theta\right)$$

$$\vec{n} = \hat{r}$$

Angular distribution



* These plots show how the length of a vector, r, depends on its direction (θ,ϕ) . Since the length has the same directional dependance as the power, we can see the angular distribution of power by looking at the length of the vector along all directions. (Spherical 3D plot in Mathematica)

$$r = \frac{1}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

Spectrum

The power per solid angle at the observation time reads

$$\frac{dP(t)}{d\Omega} = \left(\vec{n}\cdot\vec{S}\right)R(t_r)^2 = c\varepsilon_0 E_{rad}^2(\vec{x},t)R(t_r)^2$$

In order to get the frequency contents of the radiation, or the spectrum, we need to do Fourier transformations.

$$\vec{a}(\vec{x},t) = \sqrt{c\varepsilon_0}R(t_r)\vec{E}_{rad}(\vec{x},t) \qquad \vec{a}(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{a}}(\vec{x},\omega)e^{-i\omega t}d\omega$$

$$\frac{dP(t)}{d\Omega} = \vec{a}(\vec{x},t)\cdot\vec{a}(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega d\omega'\tilde{\vec{a}}(\vec{x},\omega)\cdot\tilde{\vec{a}}^*(\vec{x},\omega')e^{i\omega' t}e^{-i\omega t}$$

Now we calculate the total energy per solid angle received at the **observation point**

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty}\frac{dP(t)}{d\Omega}dt = 2\int_0^{\infty}\left|\tilde{\vec{a}}(\vec{x},\omega)\right|^2 d\omega \equiv \int_0^{\infty}\frac{d^2I(\omega)}{d\omega d\Omega}d\omega \qquad \tilde{\vec{a}}(\vec{x},\omega) = \tilde{\vec{a}}^*(\vec{x},-\omega)$$

⇓

Spectrum intensity: energy received per solid angle per frequency interval ←

$$\frac{d^2I(\omega)}{d\omega d\Omega} = 2\left|\tilde{\vec{a}}(\vec{x},\omega)\right|^2$$

Spectrum II

To proceed, we need to calculate the Fourier components of the electric field:

$$\tilde{\vec{a}}(\vec{x},\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{a}(\vec{x},t)e^{i\omega t}dt \qquad \vec{a}(\vec{x},t)=\sqrt{c\varepsilon_0}R(t_r)\vec{E}_{rad}(\vec{x},t) \qquad \vec{E}_{rad}=\frac{e}{4\pi\varepsilon_0 c}\frac{\vec{n}\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}$$

$$t_r=\frac{1}{c}\left[x_0-R(t_r)\right]\Rightarrow t=\frac{x_0}{c}=t_r+R(t_r)/c \qquad dt=\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]dt_r$$

$$\frac{d}{dt_r}\left(\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)\approx\frac{\vec{n}\times\left(\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}$$

Far field approximation, $\left|\vec{r}(\tau_0)\right|<<\left|\vec{x}\right|$

$$R(\tau_0)^2=\left(\vec{x}-\vec{r}(\tau_0)\right)\cdot\left(\vec{x}-\vec{r}(\tau_0)\right)$$

$$\approx\left|\vec{x}\right|^2\left(1-2\frac{\vec{n}\cdot\vec{r}(\tau_0)}{\left|\vec{x}\right|}\right)$$

$$\Rightarrow R(\tau_0)\approx\left|\vec{x}\right|-\vec{n}\cdot\vec{r}(t_r)$$

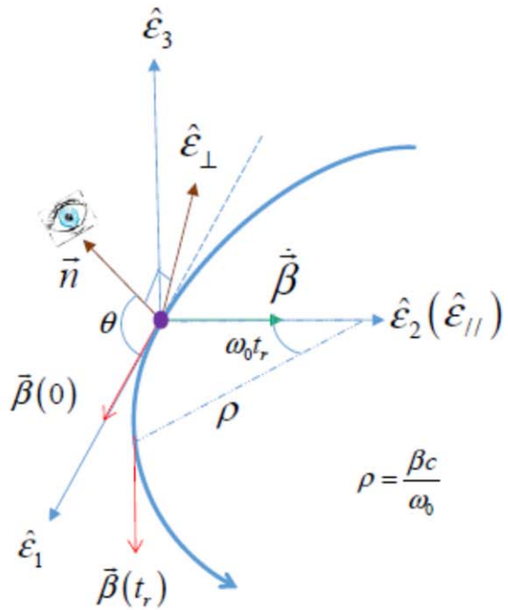
$$\tilde{\vec{a}}(\vec{x},\omega)=\frac{e}{4\pi\sqrt{2\pi\varepsilon_0 c}}\int_{-\infty}^{\infty}\frac{\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^2}e^{i\omega\left(t_r+R(t_r)/c\right)}dt_r$$

$$=-\frac{i\omega e}{4\pi\sqrt{2\pi\varepsilon_0 c}}e^{i|\vec{x}|\omega/c}\int_{-\infty}^{\infty}\left[\vec{n}(t_r)\times\left(\vec{n}(t_r)\times\vec{\beta}(t_r)\right)\right]e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}(t_r)/c\right)}dt_r$$

$$\frac{d^2I(\omega)}{d\omega d\Omega}=2\left|\tilde{\vec{a}}(\vec{x},\omega)\right|^2=\frac{1}{4\pi\varepsilon_0}\frac{\omega^2e^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}\left(\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)\right)e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}(t_r)/c\right)}dt_r\right|^2$$

Spectrum III

$$\frac{d^2 I(\omega)}{d\omega d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}\left(\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)\right)e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}(t_r)/c\right)}dt_r\right|^2$$



$$\dot{\vec{\beta}} = \dot{\beta}(0,1,0)$$

$$\vec{\beta}(0) = \beta(1,0,0)$$

$$\vec{n} = (\cos\theta, 0, \sin\theta) \qquad \rho = \frac{\beta c}{\omega_0}$$

$$\vec{\beta}(t_r) = \beta\left[\cos(\omega_0 t_r)\hat{\varepsilon}_1 + \sin(\omega_0 t_r)\hat{\varepsilon}_2\right]$$

$$\approx \beta\hat{\varepsilon}_1 + \beta\omega_0 t_r\hat{\varepsilon}_2 \qquad \omega_0 t_r << 1$$

$$\hat{\varepsilon}_\perp = \vec{n}\times\hat{\varepsilon}_2 \qquad \hat{\varepsilon}_{//} = \hat{\varepsilon}_2$$

Polarization of electric field is decomposed into

$$\tilde{\vec{a}}(\vec{x},\omega) = \tilde{a}_\perp(\vec{x},\omega)\hat{\varepsilon}_\perp + \tilde{\vec{a}}_{//}(\vec{x},\omega)\hat{\varepsilon}_{//}$$

$$\tilde{\vec{E}}(\vec{x},\omega) = \tilde{E}_\perp(\vec{x},\omega)\hat{\varepsilon}_\perp + \tilde{E}_{//}(\vec{x},\omega)\hat{\varepsilon}_{//}$$

$\hat{\varepsilon}_{//}$: inside the orbit plane of particle

$\hat{\varepsilon}_\perp$: nearly perpendicular to the orbit plane

$$\vec{n}(t_r)\times\left[\vec{n}(t_r)\times\vec{\beta}(t_r)\right] = \beta\sin\theta\hat{\varepsilon}_\perp - \beta\omega_0 t_r\hat{\varepsilon}_{//} \approx \beta\theta\hat{\varepsilon}_\perp - \beta\omega_0 t_r\hat{\varepsilon}_{//}$$

$$\theta \sim \frac{1}{\gamma} << 1$$

$$\vec{r}(t_r) = \frac{\beta c}{\omega_0}\left[\sin(\omega_0 t_r)\hat{\varepsilon}_1 - \cos(\omega_0 t_r)\hat{\varepsilon}_2\right] + \frac{\beta c}{\omega_0}\hat{\varepsilon}_2 \approx \frac{\beta c}{\omega_0}\left(\omega_0 t_r - \frac{(\omega_0 t_r)^3}{6}\right)\hat{\varepsilon}_1 + \frac{\beta c}{\omega_0}\frac{(\omega_0 t_r)^2}{2}\hat{\varepsilon}_2$$

$$\omega\left[t_r - \vec{n}(t_r)\cdot\vec{r}(t_r)/c\right] \approx \frac{\omega}{2\omega_0}\left[\omega_0 t_r\left(\frac{1}{\gamma^2} + \theta^2\right) + \frac{(\omega_0 t_r)^3}{3}\right]$$

Spectrum IV

$$\frac{d^2I(\omega)}{d\omega d\Omega}=\frac{1}{4\pi\varepsilon_0}\frac{\omega^2e^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}(\beta\theta\hat{\varepsilon}_\perp-\beta\omega_0t_r\hat{\varepsilon}_{//})\exp\left\{i\frac{\omega}{2\omega_0}\left[\omega_0t_r\left(\frac{1}{\gamma^2}+\theta^2\right)+\frac{(\omega_0t_r)^3}{3}\right]\right\}dt_r\right|^2$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{\omega^2e^2}{4\pi^2c\omega_0^2}\left|\sqrt{\frac{1}{\gamma^2}+\theta^2}\,\beta\theta\hat{\varepsilon}_\perp I_\perp(\eta)-\left(\frac{1}{\gamma^2}+\theta^2\right)\beta\hat{\varepsilon}_{//}I_{//}(\eta)\right|^2$$

$$\eta=\frac{1}{3}\frac{\omega}{\omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{\frac{3}{2}}=\frac{\omega}{2\omega_c}\left(1+\gamma^2\theta^2\right)^{\frac{3}{2}}$$

Critical frequency

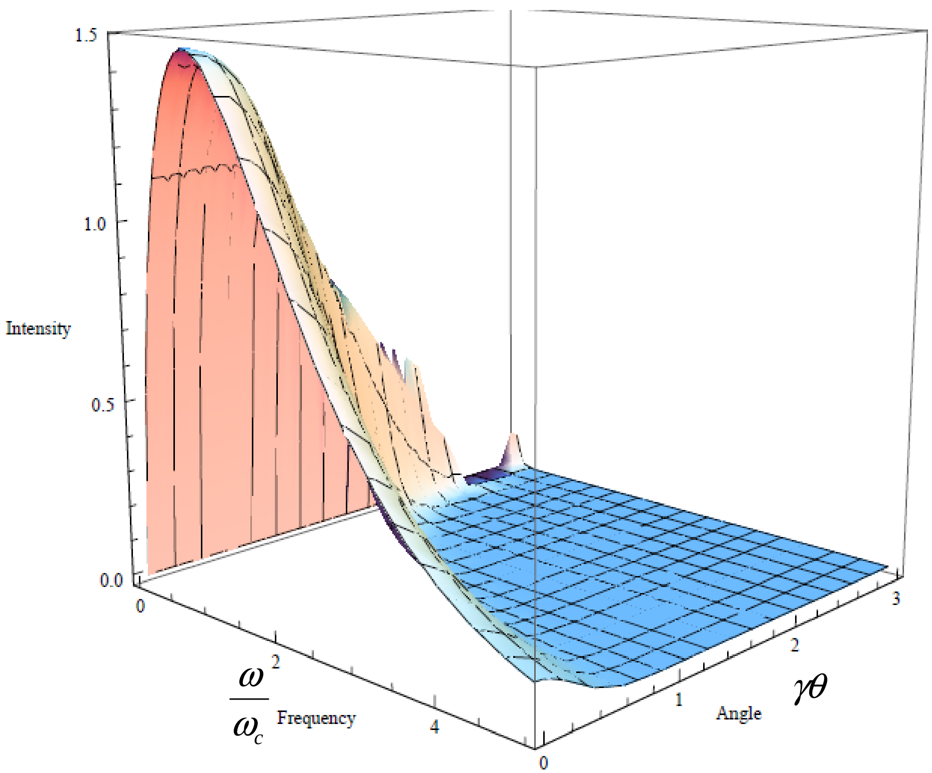
$$\omega_c=\frac{3}{2}\gamma^3\frac{c}{\rho}\approx\frac{3}{2}\gamma^3\omega_0$$

$$I_\perp(\eta)\equiv\int_{-\infty}^{\infty}\exp\left[i\frac{\eta}{2}\left(3x+x^3\right)\right]dx=\frac{2}{\sqrt{3}}K_{\frac{1}{3}}(\eta)\qquad I_{//}(\eta)\equiv\int_{-\infty}^{\infty}x\exp\left[i\frac{\eta}{2}\left(3x+x^3\right)\right]dx=-\frac{2}{i\sqrt{3}}K_{\frac{2}{3}}(\eta)$$

Contribution from $\tilde{E}_{//}(\vec{x},\omega)\hat{\varepsilon}_{//}$

$$\frac{d^2I(\omega)}{d\omega d\Omega}=\frac{1}{4\pi\varepsilon_0}\frac{3e^2\gamma^2}{4\pi^2c}\frac{\omega^2}{\omega_c^2}\left(1+\gamma^2\theta^2\right)^2\left\{\frac{\theta^2\gamma^2}{\left(1+\theta^2\gamma^2\right)}K_{\frac{1}{3}}^{\;2}(\eta)+K_{\frac{2}{3}}^{\;2}(\eta)\right\}$$

Contribution from $\tilde{E}_\perp(\vec{x},\omega)\hat{\varepsilon}_\perp$



For $\eta<<1$, $K_\nu(\eta)\sim\frac{\Gamma(\nu)}{2}\left(\frac{2}{\eta}\right)^\nu$ for $\eta>>1$, $K_\nu(\eta)\sim\sqrt{\frac{\pi}{2\eta}}e^{-\eta}$.

For $\theta=0$, using the asymptotic approximation of Bessel function,

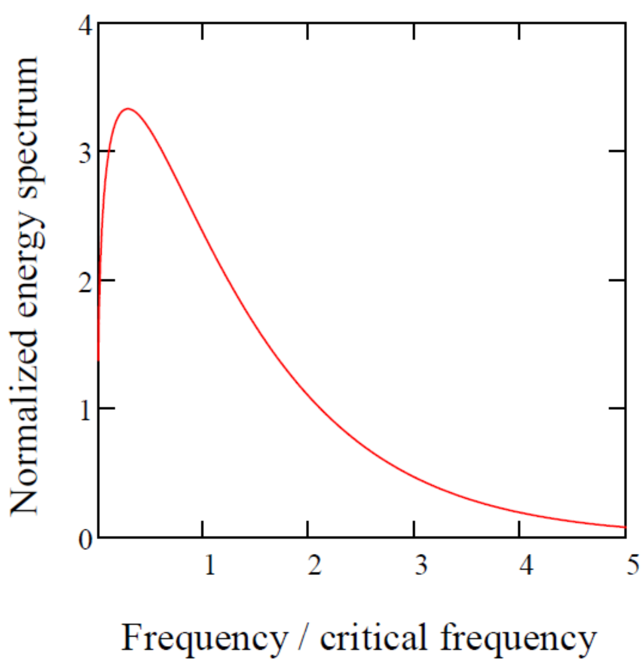
$$\frac{d^2I(\omega)}{d\omega d\Omega}=\frac{1}{4\pi\varepsilon_0}\frac{3e^2\gamma^2}{4\pi^2c}\frac{\omega^2}{\omega_c^2}K_{\frac{2}{3}}^{\;2}(\eta)=\begin{cases}\frac{1}{4\pi\varepsilon_0}\frac{3e^2\gamma^2 4^{\frac{1}{3}}}{4\pi^2c}\Gamma\left(\frac{2}{3}\right)^2\left(\frac{\omega}{\omega_c}\right)^{\frac{2}{3}}; & \omega<<\omega_c\\ \frac{1}{4\pi\varepsilon_0}\frac{3e^2\gamma^2}{4\pi c}\frac{\omega}{\omega_c}e^{-\frac{\omega}{\omega_c}} & ;\ \omega>>\omega_c\end{cases}$$

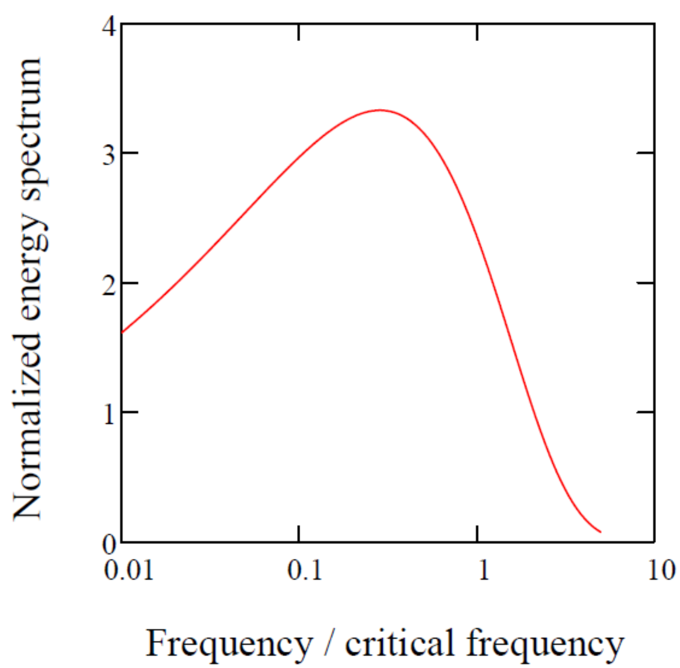
Energy spectrum V

- The total energy spectrum is obtained by integrating over the solid angle:

$$\frac{dW}{d\omega} = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d^2 I(\omega)}{d\omega d\Omega} \cos\theta d\theta = \frac{2\pi}{\gamma} \int_{-\gamma\frac{\pi}{2}}^{\gamma\frac{\pi}{2}} \frac{d^2 I(\omega)}{d\omega d\Omega} d(\gamma\theta)$$

$$\approx \frac{1}{4\pi\varepsilon_0} \frac{3e^2\gamma}{2\pi c} \frac{\omega^2}{\omega_c^2} \int_{-\infty}^{\infty} \left(1+y^2\right)^2 \left\{ \frac{y^2}{\left(1+y^2\right)} K_{\frac{1}{3}}^{\;2}\left(\frac{\omega}{2\omega_c}\left(1+y^2\right)^{\frac{3}{2}}\right) + K_{\frac{2}{3}}^{\;2}\left(\frac{\omega}{2\omega_c}\left(1+y^2\right)^{\frac{3}{2}}\right) \right\} dy$$





A more concise and popular expression for the energy spectrum:

$$\frac{dW}{d\omega} = \frac{1}{4\pi\varepsilon_0} \sqrt{3} \frac{e^2\gamma}{c} \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{\frac{5}{3}}(x)\, dx$$