Home work 3

Problem 1 – 5 points

We proved already that for an arbitrary set of m (does not matter odd or even!) ordinary first order linear differential equation

$$\frac{d}{dt}X = \mathbf{D}(t) \times X; \qquad (1)$$

solution can be written in form of transport matrix

$$X(t) = \mathbf{M}(t) \times X; \ \frac{d}{dt}\mathbf{M} = \mathbf{D}(t) \times \mathbf{M} \qquad (2)$$

show that

$$\frac{d}{dt}(det\mathbf{M}) = Trace(\mathbf{D}(t) \times det\mathbf{M} \qquad (3)$$

Hint: consider a infinitesimal step in t:

$$\mathbf{M}(t+dt) = (\mathbf{I} + \mathbf{D}(t)dt) \times \mathbf{M}(t) + O(dt^2)$$

and show that

$$det(\mathbf{I} + \mathbf{D}(t)ds) = 1 + \mathrm{Trace}\mathbf{D}(t) \times dt + O(dt^2)$$

Note: there was typo and 1 was missing in the above equation

Solution: Let's start from obvious expansion:

$$\mathbf{M}(t+\delta t) = \mathbf{M}(t) + \frac{d^n\mathbf{M}(t)}{dt^n}\frac{\delta t^n}{n!} = \mathbf{M}(t) + \frac{d\mathbf{M}(t)}{dt}\delta t + O(\delta t^2);$$

to arrive to

$$\mathbf{M}(t+dt) = \mathbf{M}(t) + \mathbf{D}(t)\cdot\mathbf{M}(t)\cdot dt + \mathbf{O}(dt^2) = (\mathbf{I} + \mathbf{D}(\mathbf{t})\cdot\mathbf{dt})\cdot\mathbf{M}(t) + \mathbf{O}(\mathbf{dt^2});$$

$$\mathbf{detM}(t+dt) = \mathbf{det}(\mathbf{I} + \mathbf{D}(\mathbf{t})\cdot\mathbf{dt})\cdot\mathbf{detM}(t) + O(dt^2).$$

where we used that $det(\mathbf{A}\cdot\mathbf{B}) = det\mathbf{A}\cdot deat\mathbf{B}$. Now we need to show that

$$\det(\mathbf{I} + \mathbf{D}\cdot dt) = 1 + \mathrm{Trace}(\mathbf{D})\cdot dt + O(dt^2)$$

There are three ways of proving this: you can write this matrix and observe that contribution of any non-diagonal term is in order dt^2 or higher. It means that only product of diagonal terms contributes first order of dt. Second, more mathematical approach will be to write expression for determinant using totally antisymmetric tensor:

$$\det(\mathbf{I} + \mathbf{D}\cdot dt) = \sum e^{i_1 i_2 .. i_n} \cdot \prod_{k=1}^{n} (\mathbf{I}_{\mathbf{k}\mathbf{i}_\mathbf{k}} + \mathbf{D}_{ki_k}\cdot dt)$$

with $e^{i_1 i_2 .. i_n}$ non-zero (i.e. +1 or -1) only when its indices are permutations of 1,2,…n. First, lets look and the most obvious diagonal product:

$$\det(\mathbf{I}+\mathbf{D}\cdot dt) = \sum e^{i_1 i_2 .. i_n} \cdot \prod_{k=1}^{n} (\mathbf{I}_{\mathbf{k i_k}} + \mathbf{D}_{k i_k} \cdot dt);$$

$$d = e^{12..n} \prod_{k=1}^{n} (\mathbf{I}_{\mathbf{kk}} + \mathbf{D}_{kk} \cdot dt) = \prod_{k=1}^{n} (1 + \mathbf{D}_{kk} \cdot dt) = 1 + dt \cdot Trace\mathbf{D} + O(dt^2)$$

where we used obvious

$$(1 + a \cdot dt) \cdot (1 + b \cdot dt) = 1 + (a+b) \cdot dt \ + a \cdot b \cdot dt^2 = 1 + (a+b) \cdot dt \ + O\left(dt^2\right).$$

Now, let's assume that product contains one of non-diagonal term $dt \cdot \mathbf{D}_{ki_k}$, $i_k \neq k$, then at least index of $e^{i_1 i_2 .. i_n}$ located in position i_k is not equal to its numbering (position), otherwise the totally asymmetric tensor contains two identical indices and is equal zero. Since unit matrix *I* has zeros outside of the diagonals, the product would contain $dt^2 \cdot \mathbf{D}_{ki_k} \cdot \mathbf{D}_{i_k m}$, $i_k \neq k, m$, which is already second order of *dt*. The last stroke is trivial:

$$det\mathbf{M}(t+dt) = (1 + dt \cdot Trace\mathbf{D}) \cdot det\mathbf{M}(t) + O(dt^2);$$

$$\frac{d}{dt} det\mathbf{M} = Trace\mathbf{D} \cdot det\mathbf{M}.$$

The third, most elegant way, is to use transformation of the matrix to its normal, in general case Jordan, form with eigen values λ_k on the diagonal and zeros below the diagonal (and I case of Jordan form some 1 above the diagonal):

$$\mathbf{D} = \mathbf{U} \cdot \mathbf{\Lambda} \cdot \mathbf{U}^{-1} \Rightarrow \mathbf{I} + dt \cdot \mathbf{D} = \mathbf{U} \cdot (\mathbf{I} + dt \cdot \mathbf{\Lambda}) \mathbf{U}^{-1}$$

$$det\left(\mathbf{U} \cdot (\mathbf{I} + \mathbf{dt} \cdot \mathbf{\Lambda}) \mathbf{U}^{-1}\right) = det(\mathbf{I} + dt \cdot \mathbf{\Lambda}) = \prod_{k=1}^{n} \left(1 + \lambda_k \cdot dt\right) = 1 + dt \cdot \sum_{k=1}^{n} \lambda_k + O\left(dt^2\right)$$

The only remaining part is to remember that

$$Trace\mathbf{D} = Trace \ (\mathbf{U} \cdot \mathbf{\Lambda} \cdot \mathbf{U}^{-1}) = Trace\mathbf{\Lambda} = \sum_{k=1}^{n} \lambda_k$$

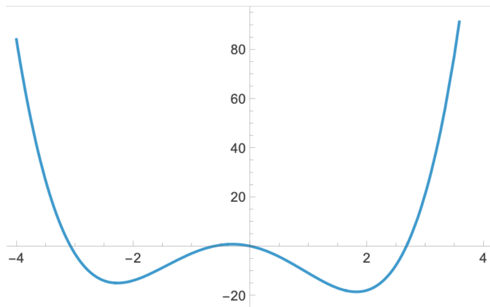
It is important theorem which we will use in the course when we are considering non-conservative effects (such as radiation) and damping of particle's oscillations.

Problem 2 – 5 points

For time (or s – does not matter) independent Hamiltonian of

$$H = \frac{p^2}{2} + U(x); \ U(x) = -5x - 8x^2 + x^3 + x^4 \qquad (1)$$

The graph of U(x) looks like and has extrema at approximately at x=-1.82; x=-0.321 and x=2.14



Write equations for stationary point and draw (either by hand or using computer) trajectories in x-p plane in x range from -3 to +3 and p from -8 to +8. The most important is designating stable and unstable points at the phase plot, drawing a separatrix which goes through a local maximum at x=-0.321 and approximating trajectories for *H > U(-0.321)* and *H < U(-0.321)*.

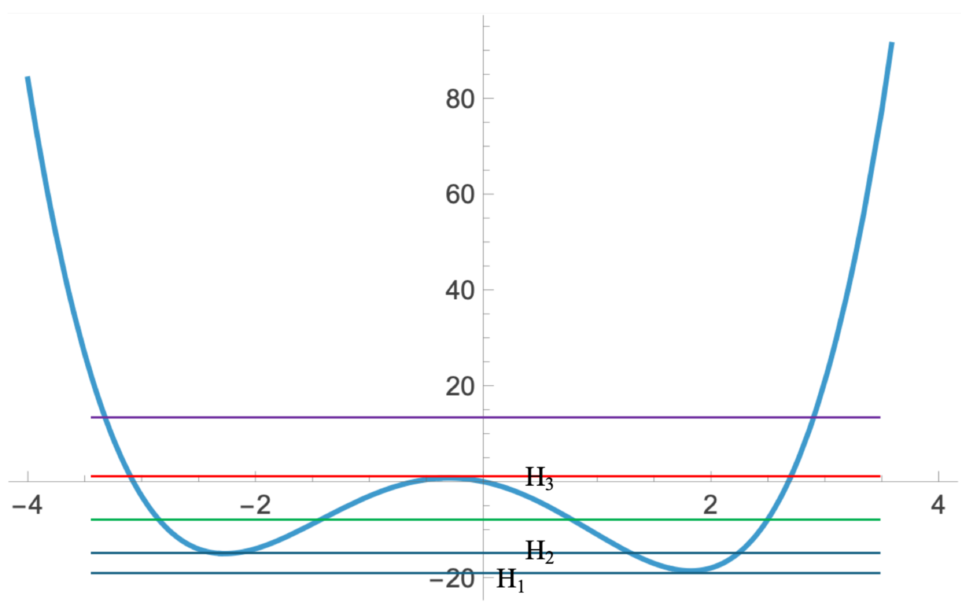
Solution: Since the Hamiltonian is time- (or *s*-) independent, it is a constant of motion and trajectories are nothing else that its equipotentials:

$$p = \pm\sqrt{2(H - U(x))} \tag{1}$$

Stationary points are determined by

$$\frac{dx}{ds} = \frac{\partial H}{\partial p} = p = 0; \quad \frac{dp}{ds} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x} = 0;$$

i.e. located at *p=0* and extrema of *U(x)*. There is three extrema of *U(x)* with positive second derivatives at the both sides of the local maxima with negative second derivative. It is easy to show that positive second derivatives result in stable stationary points with small oscillations in their proximity. Vice versa: negative derivative is indication of unstable stationary point with exponential growth of small deviations.



There is three distinct values of Hamiltonian H_1, H_2, and H_3 determining allowable solutions for trajectories(1): $H<H_1$, no solutions; $H_2>H>H_1$ – a single trajectory at x>-0.321; $H3>H>H_2$ - two solutions around stationery points at x=-1.82 and x=2.14; $H>H_3$ - single trajectory. The most interesting trajectory is at $H=H_3$, called a separatrix. It "separate" two typographically distinct areas in the phase space. I used Mathematica to plot *H=constant*, with special attention to plot the separatrix. It is done by plotting levels of $H-H_3$.