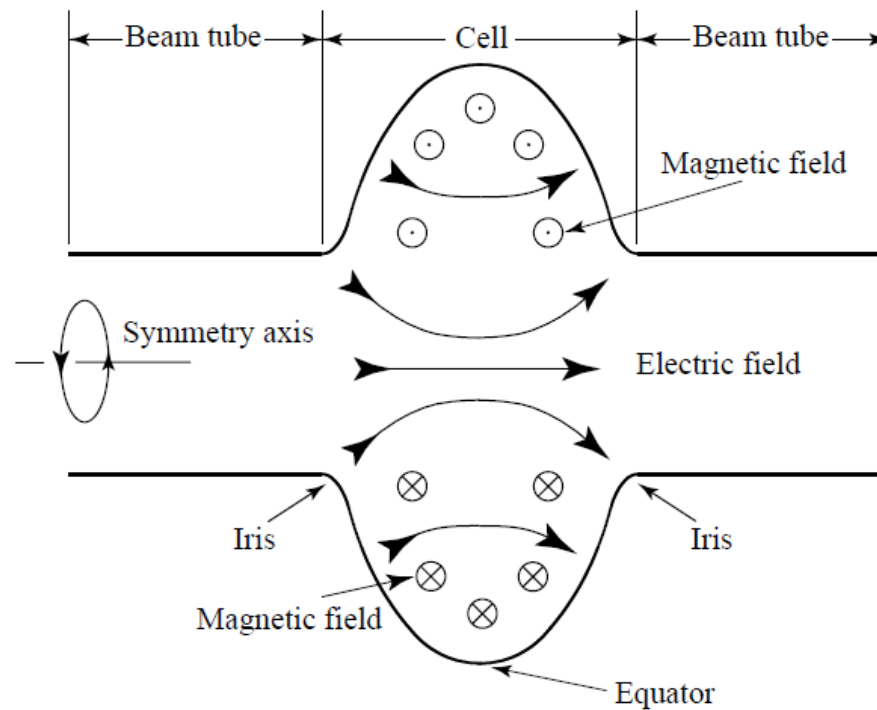# PHY 554
Fundamentals of Accelerator Physics
Lecture 10: Introduction to RF accelerators

Jun Ma

Acknowledgement

Next three lectures use some materials from courses on RF and Superconducting RF (SRF) accelerators taught by Prof. S. Belomestnykh at SBU/BNL and USPAS, which can be found on the following websites:

http://case.physics.stonybrook.edu/index.php/Courses:_P554_Fundamentals_of_Accelerator_Physics,_Spring_2014
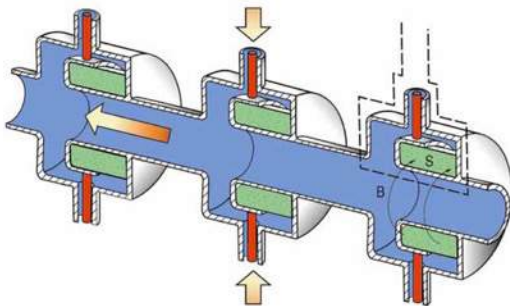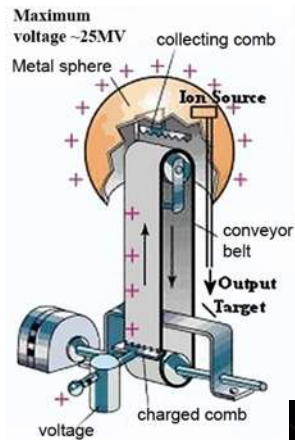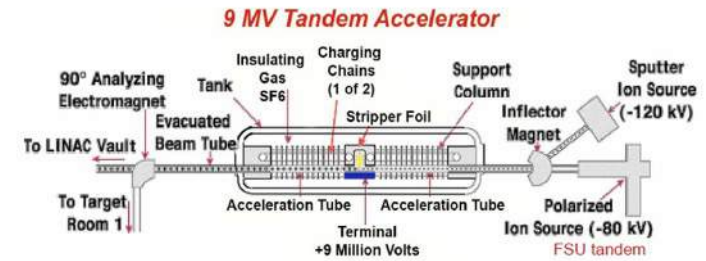*http://case.physics.stonybrook.edu/index.php/CASE:Courses*
https://sites.google.com/site/srfsbu11/
http://uspas.fnal.gov/materials/materials-table.shtml

I added addition material as needed

Linear accelerators: from electrostatic to RF
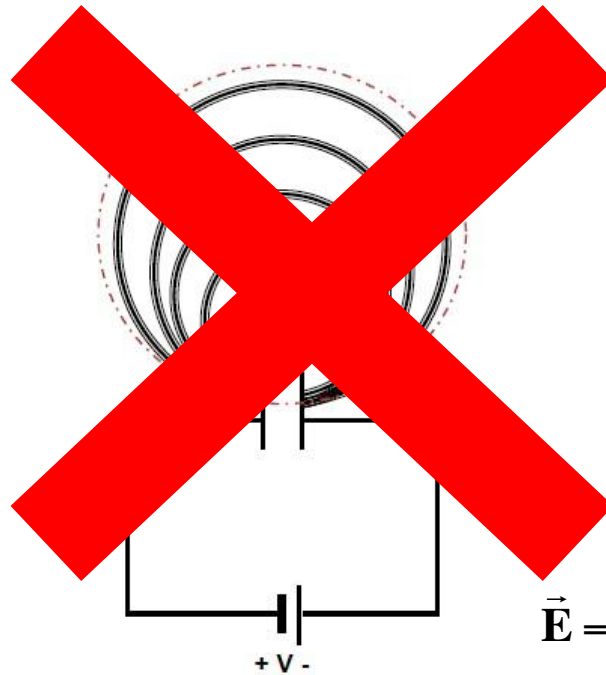

9 MV Tandem Accelerator

Can one gain the energy again and again by passing through a DC accelerating gap?

Electrostatic: what is the limit ?

Maxwell equations and energy conservation law!

$$\Delta E = e \oint \vec{\mathbf{E}} \cdot d\vec{l} = -\frac{e}{c}\frac{\partial}{\partial t}\left(\int \vec{\mathbf{H}} \cdot d\vec{s} \right)$$
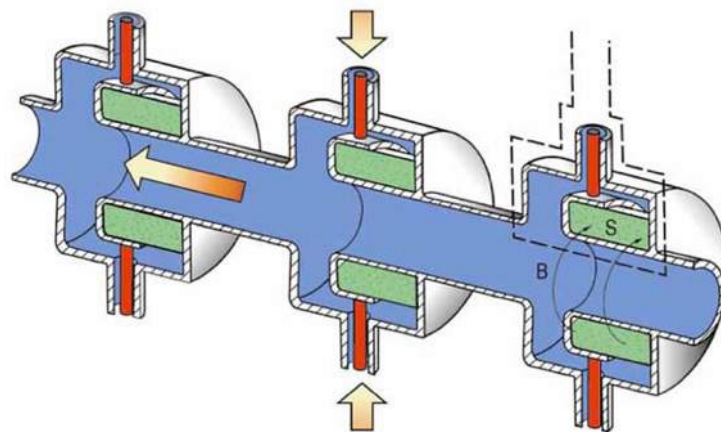
DC

$$\Delta E = e \oint \vec{\mathbf{E}} \cdot d\vec{l} = 0$$

$$\vec{\mathbf{E}} = -\vec{\nabla}\varphi \quad \to \quad E(\vec{r}) = E(0) - e\varphi(\vec{r})$$

Can not cheat the Maxwell equations

Induction linacs: linear betatrons

$$\oint \vec{\mathbf{E}} \cdot d\vec{l} = -\frac{\partial}{\partial t}\left(\int \vec{\mathbf{H}} \cdot d\vec{s}\right)$$

- Useful for high power and high current beams
- Have limited accelerating field
- By nature are pulsed, with relatedly low rep-rate (kHz)

How RF accelerator works

- It has oscillating (typically sinusoidal in time) longitudinal (along the particle's trajectory) eclectic field

- It also has longitudinal structure (cells) which alternates the direction of the filed

- When particle propagates through the RF accelerator, the field direction in each cell is synchronized with the particle arrival and the effect from all cells is added coherently
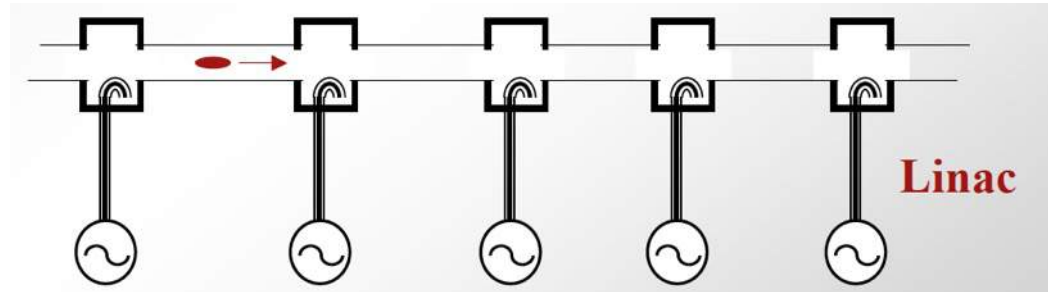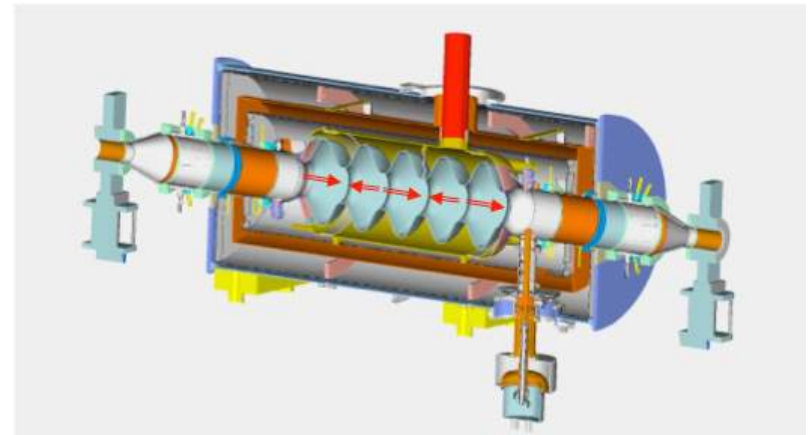
Linac

$$\frac{dE}{dt} = e\vec{\mathbf{E}} \cdot \vec{\mathrm{v}} \quad \rightarrow \mathrm{sign}(\vec{\mathbf{E}} \cdot \vec{\mathrm{v}}) = \mathrm{const}$$

Wideröe's linac: $\beta = v/c$ is changing
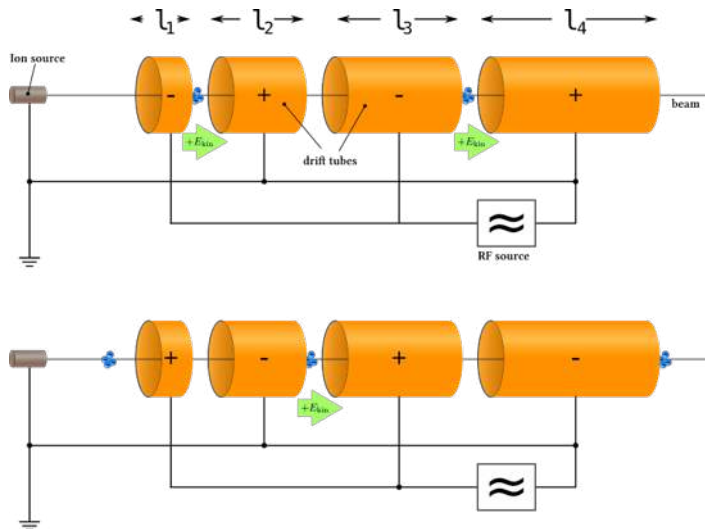
Electron linac

$\beta = v/c \sim 1$

Wave-form in 5-cell cavity
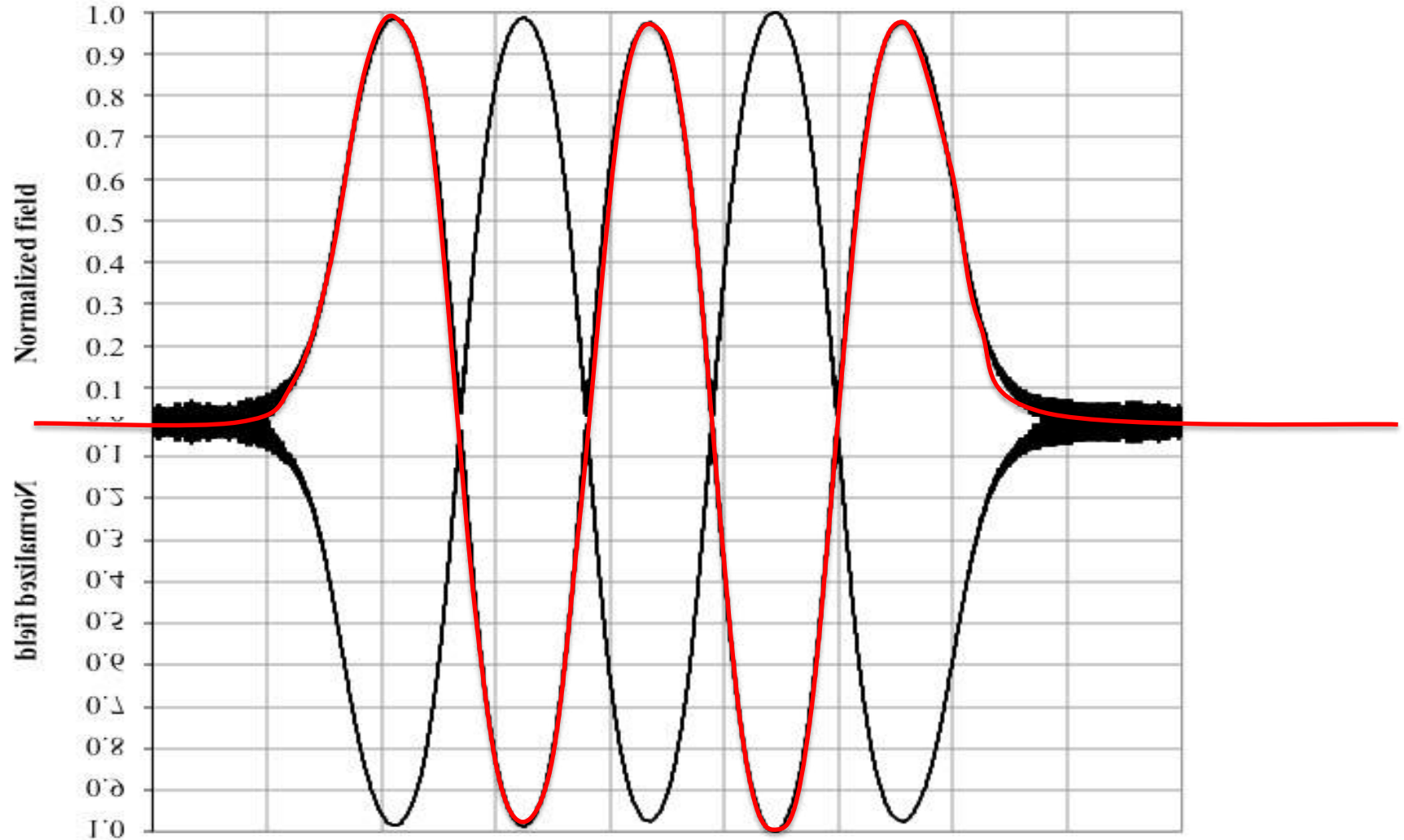
How β=1 RF linac works?
Example of 5-cell cavity

$t=0$

$t=T_o$

$t=2T_o$

$t=1/4T_o$

$t=5/4T_o$

$t=9/4T_o$

$t=1/2T_o$

$t=3/2T_o$

$t=5/2T_o$

$t=3/4T_o$

$t=7/4T_o$

Electrons are out
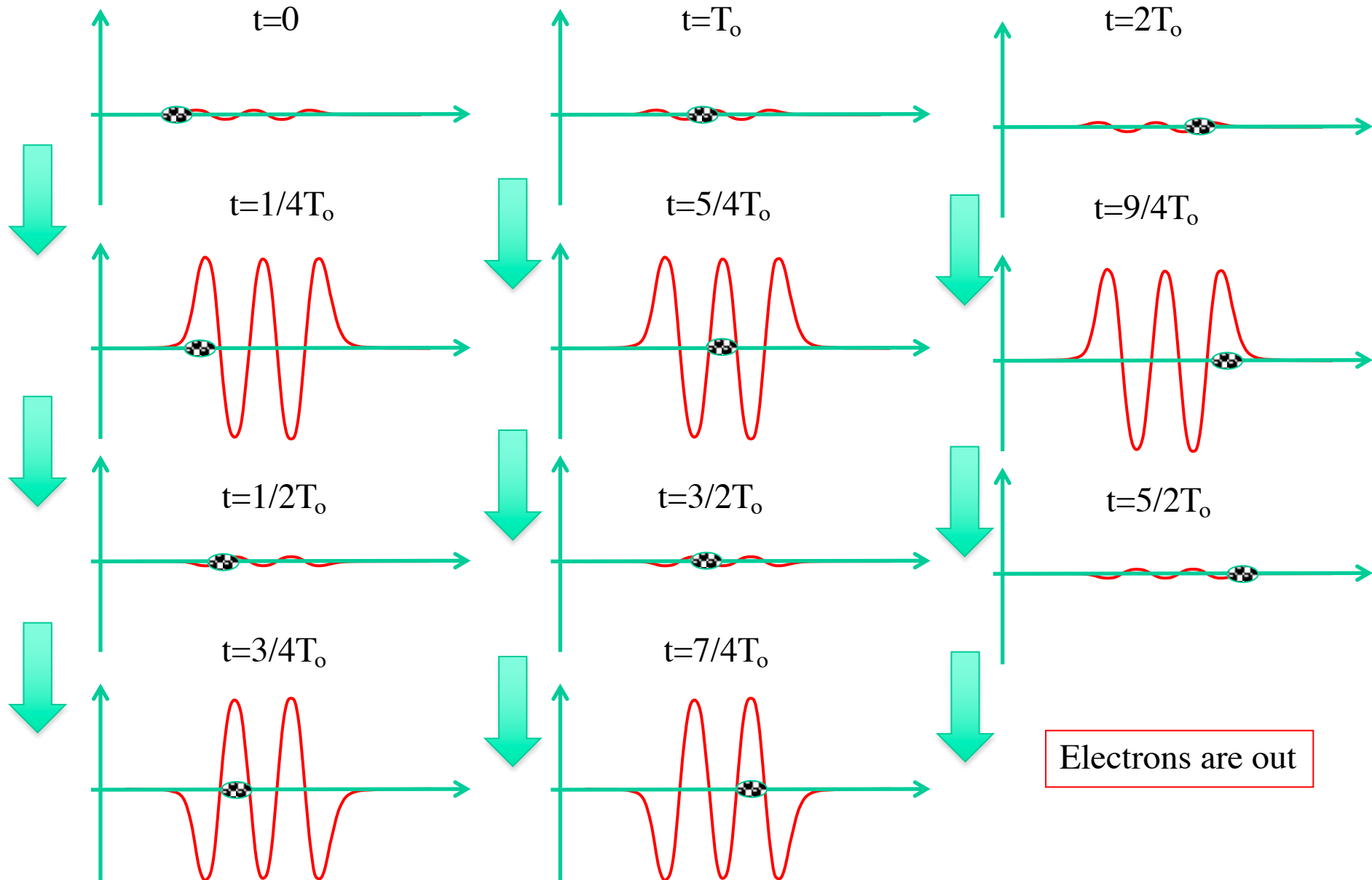
Simple things to remember

- Acceleration in DC electrostatic is limited to the difference in terminal potential (e.g. voltage between the ground and the cathode)

- RF linear accelerators (RF linacs or simply linacs) are not limited in beam energy

- In RF linacs, the coherent addition/subtraction of the energy gain from cell to cell happens by design: period of the electric field oscillation is matched to the travel time of electron between the cells.

- Accurate synchronization of RF linac is important task for any linear accelerator

A bit of EM and conducting media
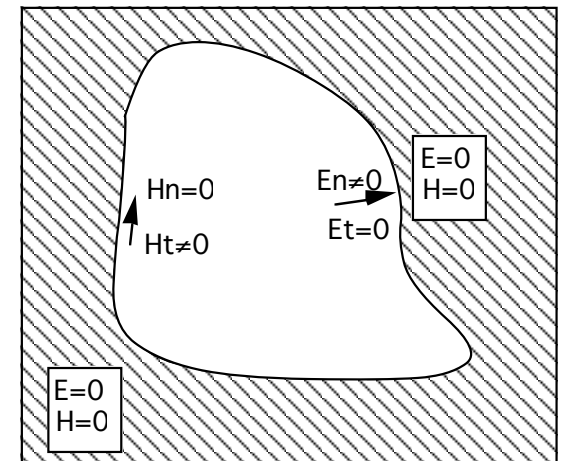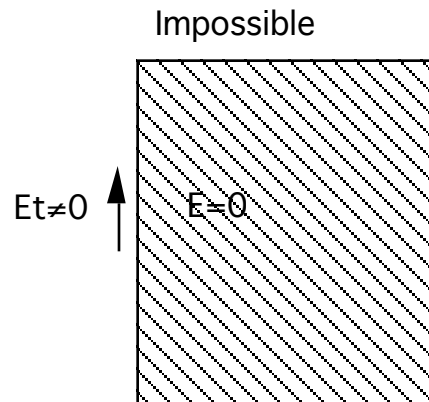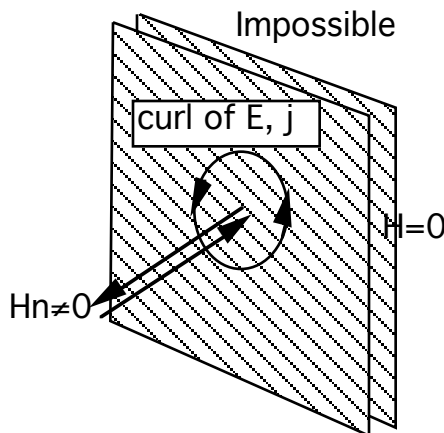
$$\vec{j} = \sigma \vec{E};$$

• Assuming oscillating field we can use Coulomb gauge for EM field

$$\vec{A} = \text{Re}\left\{\vec{A}(\vec{r})\exp(i\omega t)\right\}; \varphi = 0;$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = curl\vec{A}.$$

$$\left|\vec{H}\right| \propto \left|\frac{(\alpha + i\beta)}{k_o}\right|\left|\vec{E}\right| = \left|\sqrt{1 + \frac{4\pi i\sigma}{\omega}}\right|\left|\vec{E}\right|$$

$$\sigma \rightarrow \infty$$

Impossible

curl of E, j

H=0

Hn≠0

Impossible

Et≠0   E=0

Hn=0
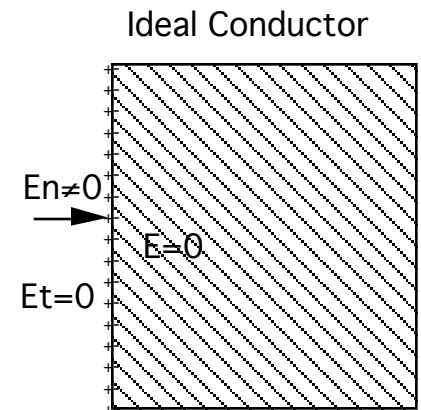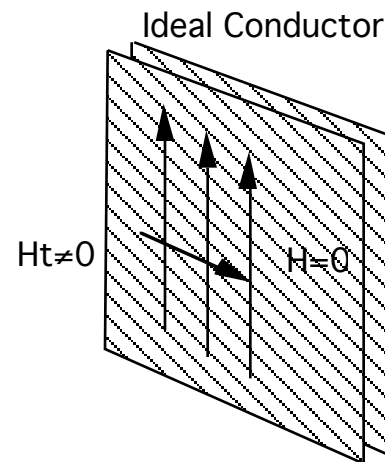
Ht≠0

En≠0

Et=0

E=0
H=0

E=0
H=0

Boundary conditions

- We are considering oscillating EM fields in RF structures
- RF structures are built from highly conducting material, both to contain EM filed inside and to provide low losses
- In first approximation we can consider an ideal boundary conditions and take finite conductivty as a perturbation later
- Q-factor: $Q_{room\ temp} \sim 10^3$-10^5, $Q_{SRF} \sim 10^9$-10^{10}
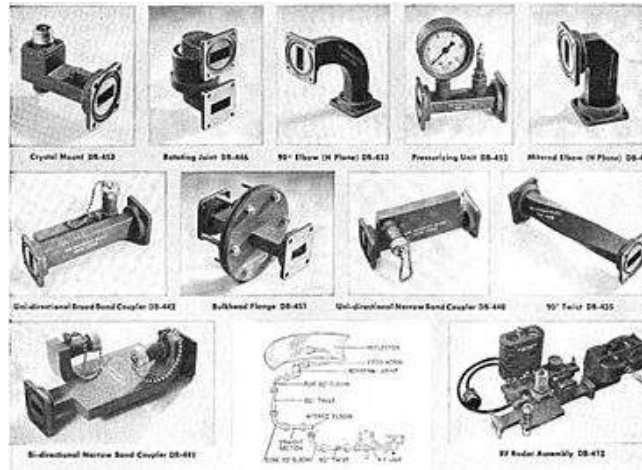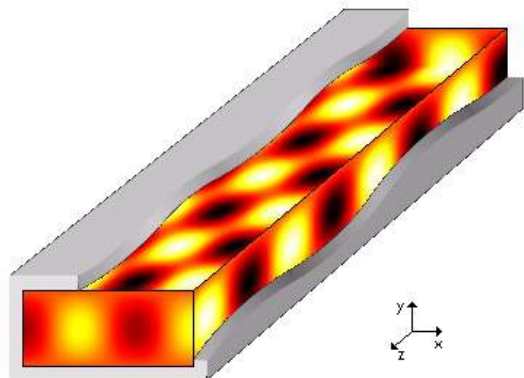
$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r})\exp(i\omega t - \alpha t)\right\};$$

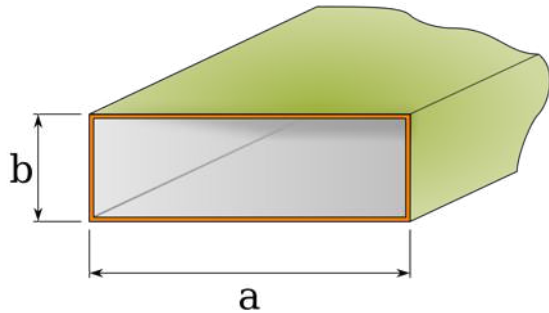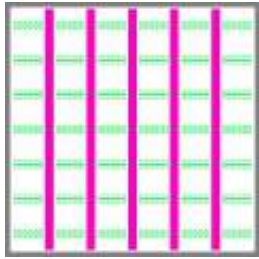$$\alpha = \frac{2\pi\omega}{Q}$$

Ideal Conductor

Ht≠0   H=0

Ideal Conductor

En≠0
E=0
Et=0

$$\vec{E} = \vec{n}\left(\vec{n}\vec{E}\right) + \vec{E}_{//}; \quad \vec{B} = \vec{n}\left(\vec{n}\vec{B}\right) + \vec{B}_{//};$$

Waveguides

Rectangular

Circular
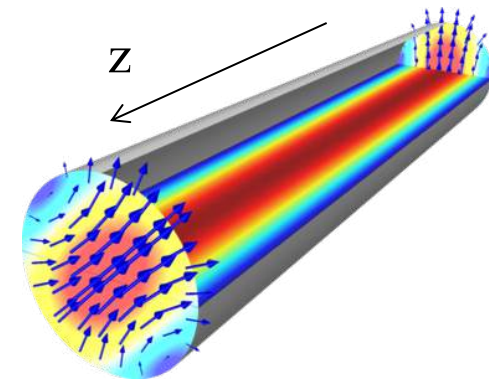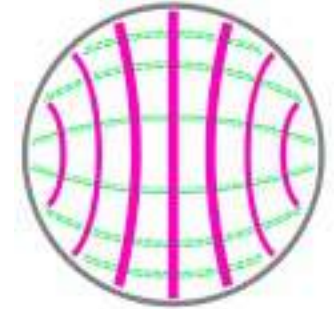
$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = 0; \ \ \Delta \equiv \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2};$$

$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r}_\perp)\exp\left(i(\omega t - k_z z)\right)\right\};$$

$$\vec{\nabla}_\perp^2 \vec{A} + \left(k_o^2 - k_z^2\right)\vec{A} = 0; \quad k_o = \frac{\omega}{c}.$$
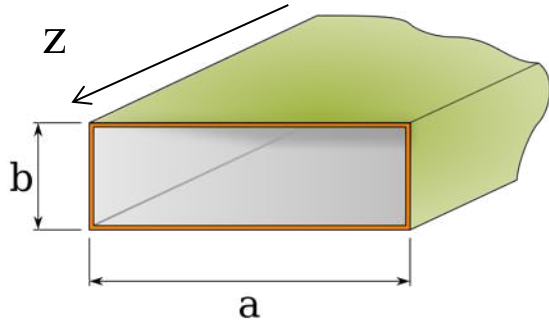
$$\vec{\nabla}_\perp^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

At the surfaces

$$\vec{n} \times \vec{E}\Big|_s = 0; \ \vec{n}\cdot\vec{B} = 0 \rightarrow E_z\Big|_s = 0; \ \frac{\partial B_z}{\partial n}\Big|_s = 0$$

TE and TM waves

Rectangular

Circular

- There is simplification
  - The modes are divided into two types: TE (transverse electric) and TM (transverse magnetic)

$$\vec{E} = \vec{E}_z + \vec{E}_\perp; \vec{B} = \vec{B}_z + \vec{B}_\perp; \ \vec{A}_z \equiv \hat{z}A_z;$$

$$\vec{\nabla} \times \vec{E} = ik_o\vec{B}; \ \vec{\nabla} \times \vec{B} = -ik_o\vec{E}; \ \Rightarrow$$

$$ik_z\vec{E}_\perp + ik_o\left[\hat{z} \times \vec{B}_\perp\right] = \vec{\nabla}_\perp\vec{E}_z;$$

$$ik_z\vec{B}_\perp - ik_o\left[\hat{z} \times \vec{E}_\perp\right] = \vec{\nabla}_\perp\vec{B}_z;$$

At the surfaces

$$\vec{n} \times \vec{E}\Big|_s = 0; \vec{n} \cdot \vec{B} = 0 \rightarrow E_z\Big|_s = 0; \ \frac{\partial B_z}{\partial n}\Big|_s = 0$$

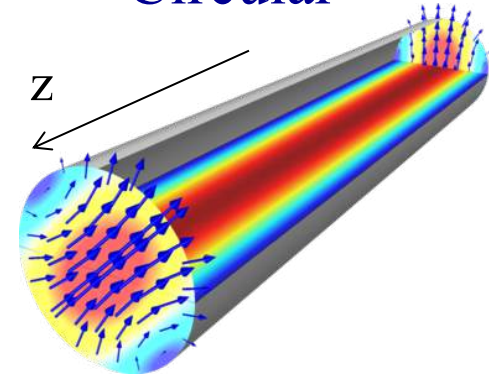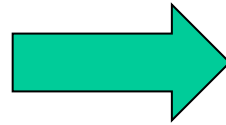$$TM : B_z \equiv 0; \ E_z\Big|_s = 0;$$

$$TE : E_z \equiv 0; \ B_z\Big|_s = 0;$$

- Last two equations indicated that E_z and B_z fully determine transverse component of the EM field
- It means that we can always consider a linear combination of the fields with $E_z = 0$ *everywhere* (TE) and $B_z = 0$ *everywhere* (TM)
- Naturally, when we interested in accelerating particles, we will need TM mode with $E_z \neq 0$.

$$\vec{B}_\perp = \pm\frac{k_z}{k_o}\left[\hat{z} \times \vec{E}_\perp\right] \ \text{for both TE and TM modes}$$

TM: $\vec{E}_\perp = \vec{\nabla}_\perp\psi_1(\vec{r}_\perp);$ TE: $\vec{B}_\perp = \vec{\nabla}_\perp\psi_2(\vec{r}_\perp);$

Cut-off frequency

Rectangular
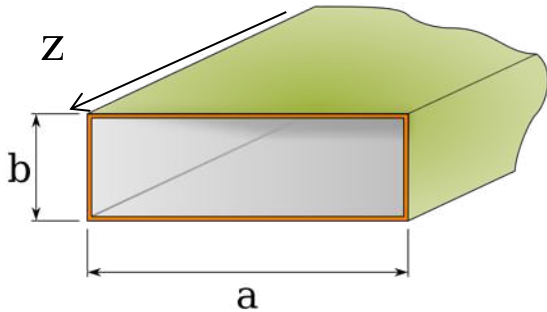
- EM field is a linear combination of modes with $E_z = 0$ *everywhere* (TE) and $B_z = 0$ *everywhere* (TM)

Circular

At the surfaces

$$\vec{n} \times \vec{E}\big|_s = 0; \; \vec{n} \cdot \vec{B} = 0 \rightarrow E_z\big|_s = 0; \; \frac{\partial B_z}{\partial n}\bigg|_s = 0$$
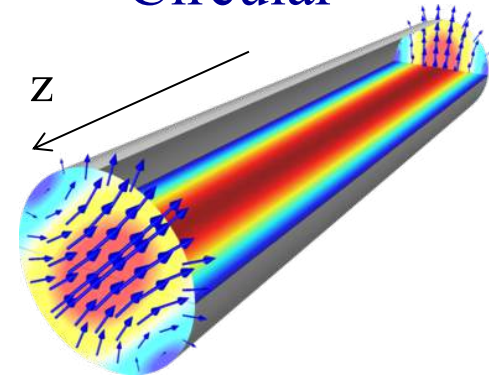
$$\vec{B}_\perp = \pm \frac{k_z}{k_o}\left[\hat{z} \times \vec{E}_\perp \right] \quad \text{for both TE and TM modes}$$

TM: $\vec{E}_\perp = \vec{\nabla}_\perp \psi_1(\vec{r}_\perp);$ TE: $\vec{B}_\perp = \vec{\nabla}_\perp \psi_2(\vec{r}_\perp);$

$$\vec{\nabla}_\perp^2 \psi + \left(k_o^2 - k_z^2\right)\psi = 0 + \textit{boundary conditions}$$

Different boundary conditions for TE and TM modes
In general case we need to find eigen function (modes)

$$TM : B_z \equiv 0; \; E_z\big|_s = 0;$$

$$TE : E_z \equiv 0; \; B_z\big|_s = 0;$$

$$TM : \psi\big|_s = 0; \; TE : \frac{\partial \psi}{\partial n}\bigg|_s = 0.$$

$$\boxed{\begin{array}{c} \vec{\nabla}_\perp^2 \psi_\lambda + \gamma_\lambda^2 \psi_\lambda = 0; \\ \lambda = 1, 2, 3, \ldots \end{array}}$$

Cut-off frequency

$$k_{z,\lambda}^2 = k_o^2 - \gamma_\lambda^2 > 0$$

$$k_{o\,\min} = \gamma_\lambda \rightarrow \omega_{cut-off} = c\gamma_\lambda$$

Below cut-off evanescent wave: Exp decay

$$\omega < \omega_{cut-off}$$

$$k_z = \pm i\sqrt{\omega^2_{cut-off} - \omega^2} = \pm i\kappa_z$$

$$\psi = \psi_0 e^{\pm \kappa_z z}$$

Cut-off frequency

Rectangular

z

b

a

Different boundary conditions for TE and TM modes

$$TM : \psi\big|_s = 0; \quad TE : \frac{\partial \psi}{\partial n}\bigg|_s = 0.$$

Circular

z
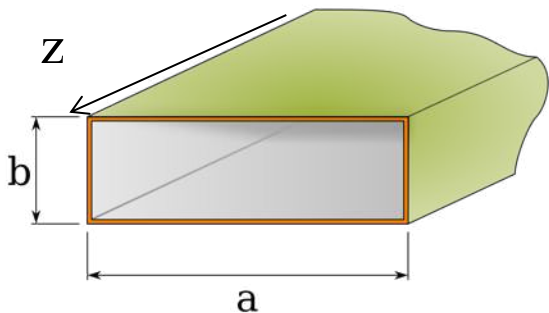
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\psi + \gamma_{mn}^2 \psi = 0$$

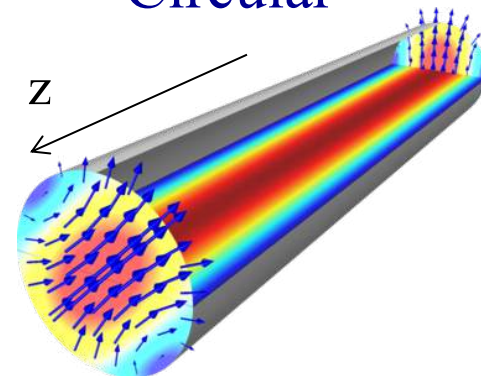$$TE : \psi^{TE}{}_{mn} = \psi_o \cos k_m x \cos k_n y; \quad m+n \geq 1;$$

$$TM : \psi^{TM}{}_{mn} = \psi_o \sin k_m x \sin k_n y; \quad m \geq 1; n \geq 1;$$

$$k_m = \pi \frac{m}{a}; k_m = \pi \frac{n}{b}; \gamma_{mn} = k_m{}^2 + k_n{}^2.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \gamma_{mn}^2 \psi = 0$$

$$\psi_{mn} = \varphi_{mn}(r) e^{in\theta} \Rightarrow r\frac{\partial}{\partial r}\left(r \frac{\partial \psi_m}{\partial r} \right) + \left(r^2 \gamma_{mn}^2 - n \right) = 0$$

$$\varphi_{mn} = J_n\left(\gamma_{mn} r \right)$$

Lowes cut-off frequency

Rectangular

$$TE : a > b; \ m = 1; \ n = 0; \ \omega_{cut-off} = \frac{\pi c}{a};$$
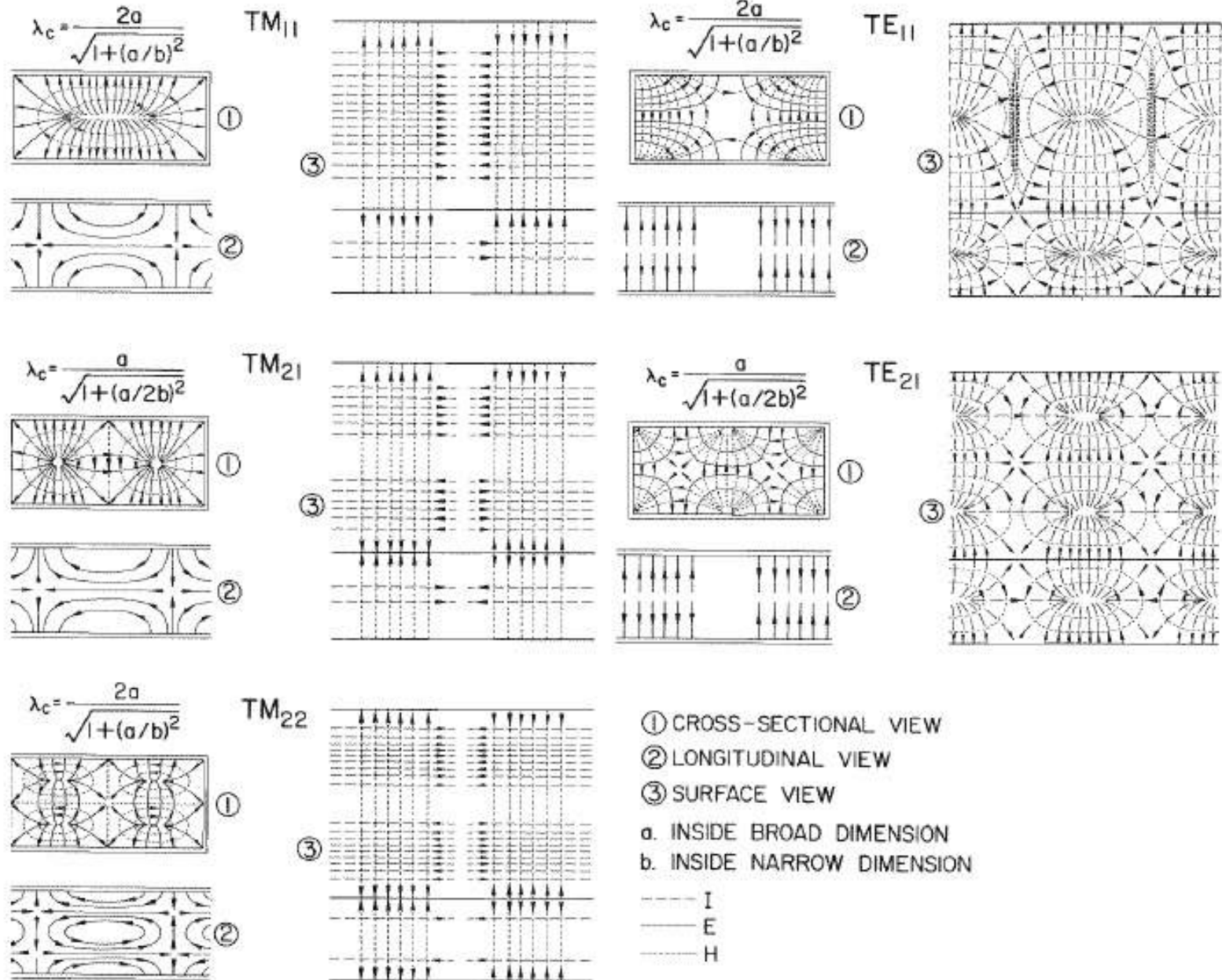
$$TM : \ m = 1; \ n = 1; \ \omega_{cut-off} = \frac{\pi c}{a}\sqrt{1 + \frac{a^2}{b^2}}.$$

Circular

$$TM : J_o\left(\gamma_{01} R \right) = 0 \rightarrow \gamma_{01} \cong \frac{2.40483..}{R}; \ \omega_{cut-off} \cong \frac{2.40\,c}{R};$$

$$TE : J_1\left(\gamma_{11} R \right) = 0 \rightarrow \gamma_{11} \cong \frac{1.84118....}{R}; \ \omega_{cut-off} \cong \frac{1.84\,c}{R}.$$

Modes in rectangular waveguide

$$\lambda_c = \frac{2a}{\sqrt{1+(a/b)^2}}$$

TM$_{11}$

$$\lambda_c = \frac{2a}{\sqrt{1+(a/b)^2}}$$

TE$_{11}$

$$\lambda_c = \frac{a}{\sqrt{1+(a/2b)^2}}$$

TM$_{21}$

$$\lambda_c = \frac{a}{\sqrt{1+(a/2b)^2}}$$

TE$_{21}$

$$\lambda_c = \frac{2a}{\sqrt{1+(a/b)^2}}$$

TM$_{22}$

① CROSS-SECTIONAL VIEW
② LONGITUDINAL VIEW
③ SURFACE VIEW

a. INSIDE BROAD DIMENSION
b. INSIDE NARROW DIMENSION

- - - - - I
———— E
········· H

RF cavities

are designed to confine the EM field inside: It means that they operate at frequency below cut-off of the beam-pipes attached to them
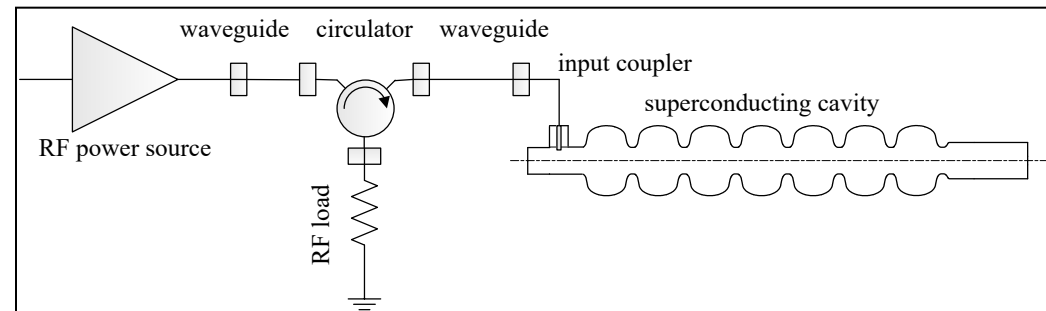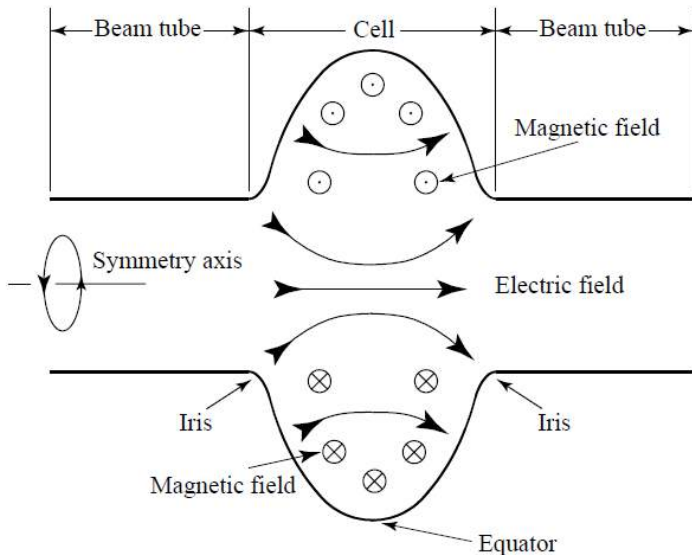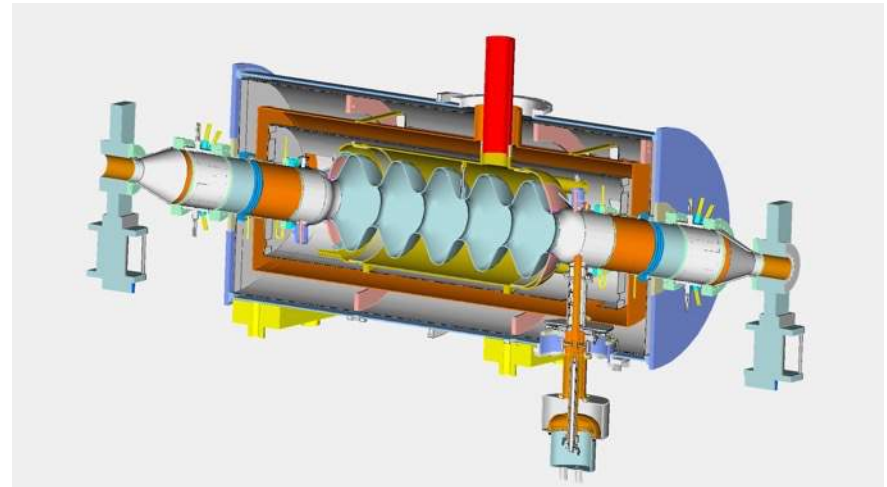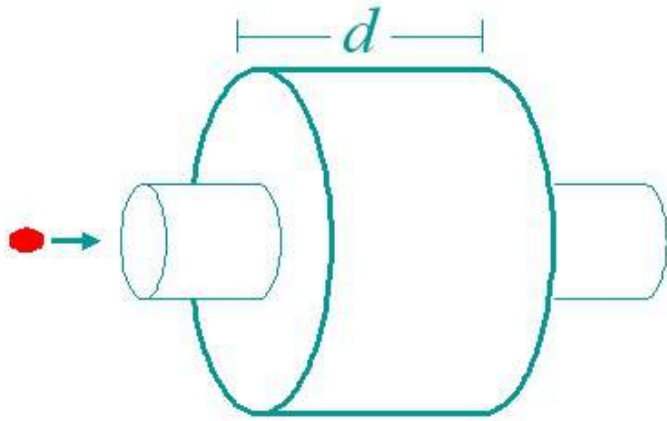
RF Cavity Modes:
the lowest accelerating is TM$_{010}$ mode

- Fields in the cavity are solutions of the equation
- Subject to the boundary conditions $\hat{n} \times \mathbf{E} = 0, \ \hat{n} \cdot \mathbf{H} = 0$

$$\left(\nabla^2 - \frac{1}{c}\frac{\partial^2}{\partial t} \right)\left\{ \begin{matrix} \mathbf{E} \\ \mathbf{H} \end{matrix} \right\} = 0$$

- Two extra surfaces (z=0 and z=d): but this is no problem for TM mode
- An infinite number of solutions (eigen modes) belong to two families of modes with different field structure and eigen frequencies: TE modes have only transverse electric fields, TM modes have only transverse magnetic fields.
- One needs longitudinal electric field for acceleration, hence the lowest frequency TM$_{010}$ mode is used.
- For the pillbox cavity w/o beam tubes
- Note that frequency does not depend of the cavity length! But only its radius.
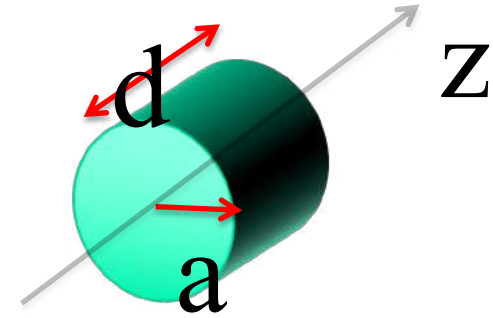
$$E_z = E_0 J_0 \left(\frac{2.405r}{R} \right) e^{i\omega t}$$

$$H_\phi = -iE_0 J_1 \left(\frac{2.405r}{R} \right) e^{i\omega t}$$

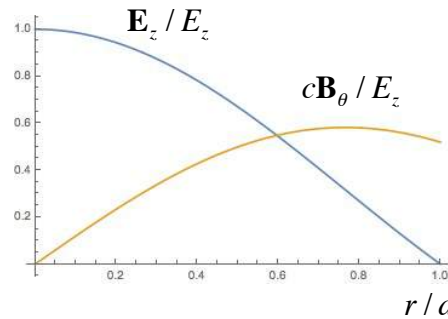$$\omega_{010} = \frac{2.405c}{R}, \ \lambda_{010} = 2.61R$$
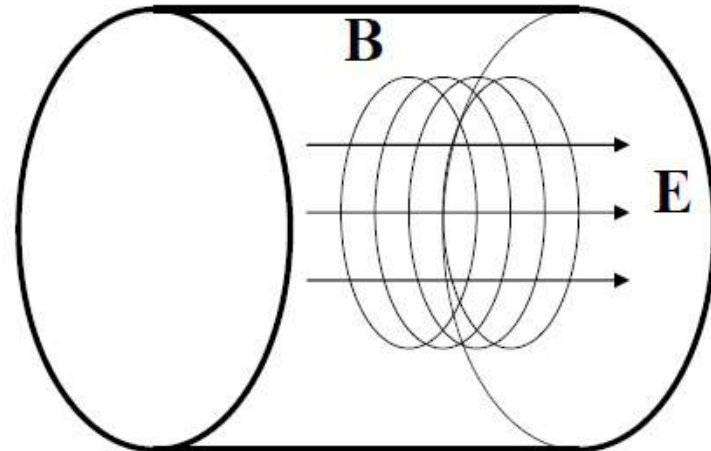
Pillbox Cavity

- Similarly to a previous exercise, we need to pick TM mode to have non-zero E_z component
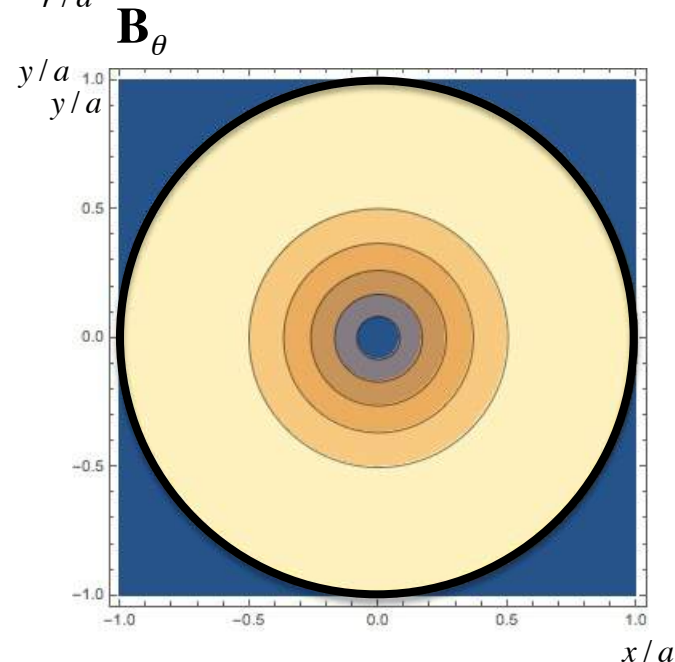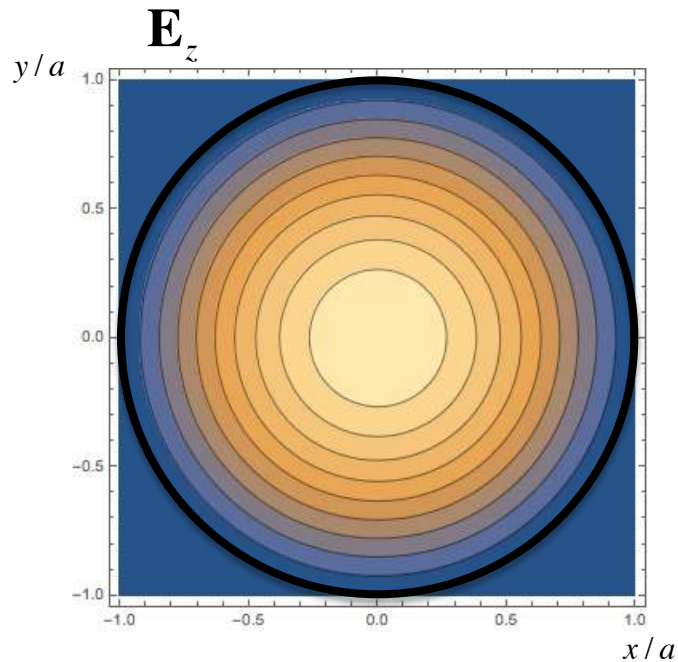- We also select TEM01 waveguide mode and $k_z=0$

$$\mathbf{E}_z = E_z \cdot J_o\left(2.405\frac{x}{a}\right)\sin(\omega t);$$

$$\mathbf{B}_\theta = \frac{1}{c}E_z \cdot J_1\left(2.405\frac{x}{a}\right)\cos(\omega t)$$

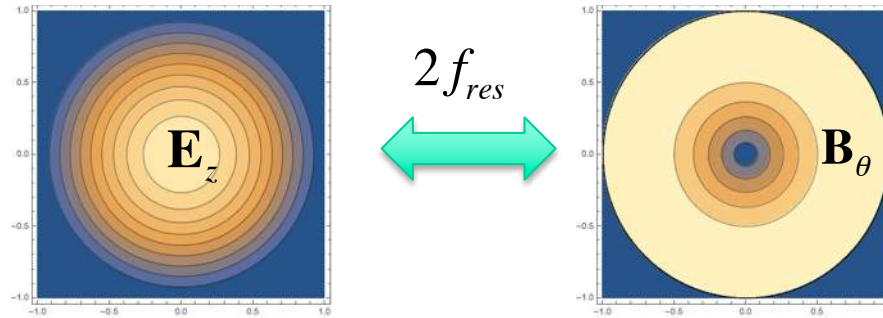EM energy oscillates between electric field magnetic field: Peaking at the same value: total energy

EM Cavity

$$\vec{E} = \vec{E}_o(\vec{r})\cos(\omega t + \varphi(\vec{r}))$$
$$\vec{B} = \vec{B}_o(\vec{r})\sin(\omega t + \psi(\vec{r}))$$
$$\int \vec{E}_o^2 \, dV = \int \vec{B}_o^2 \, dV \quad (\cdot c^2 \text{ for } SI)$$

$2f_{res}$

E_z $\longleftrightarrow$ B_θ

- Each mode has full analogy with a resonant LC circuit or a mechanical oscillator: energy stored in electric field can be compared to potential energy, and energy stored in magnetic filed – to kinetic energy
- Typical energy stored in 5 cell, 700 MHz SRF cavity operating at 20 MV/m is ~ 500 J
- What much more impressive is the intra-cavity power of about 2,000 GW!

$$E_M = \frac{LI^2}{2}; E_E = \frac{Q^2}{2C}; \omega_o = \frac{1}{\sqrt{LC}}$$

V_c L R C
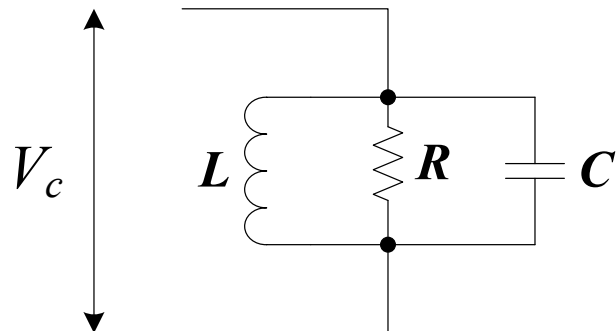
$$E_K = \frac{Mv^2}{2}; E_P = \frac{Kx^2}{2}; \omega_o = \sqrt{\frac{K}{M}}$$
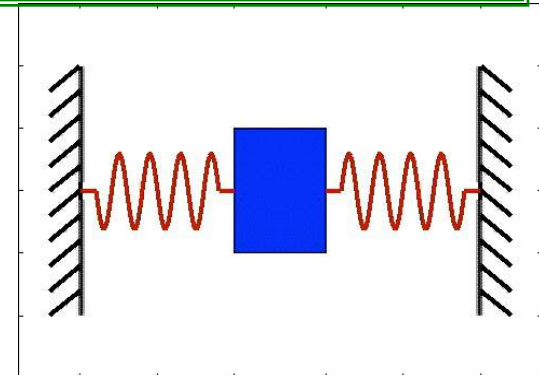
Quality factor - definition

- Let's consider a stand-alone cavity without any external couplers

- Energy stored in the cavity
$$W = \int \left(\varepsilon_o \frac{\vec{\mathbf{E}}^2}{2} + \mu_o \frac{\vec{\mathbf{H}}^2}{2} \right) dV = \mu_o \int \vec{\mathbf{H}}_o^{\ 2}\, dV$$
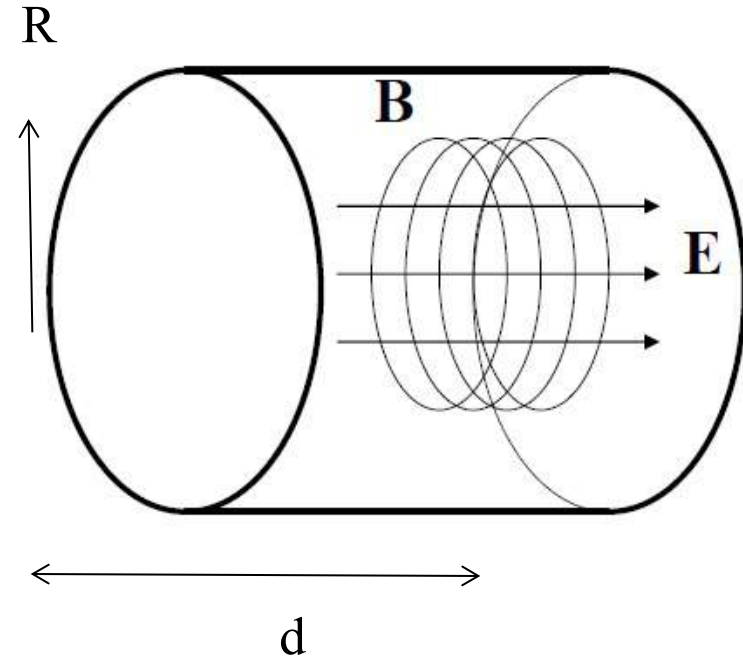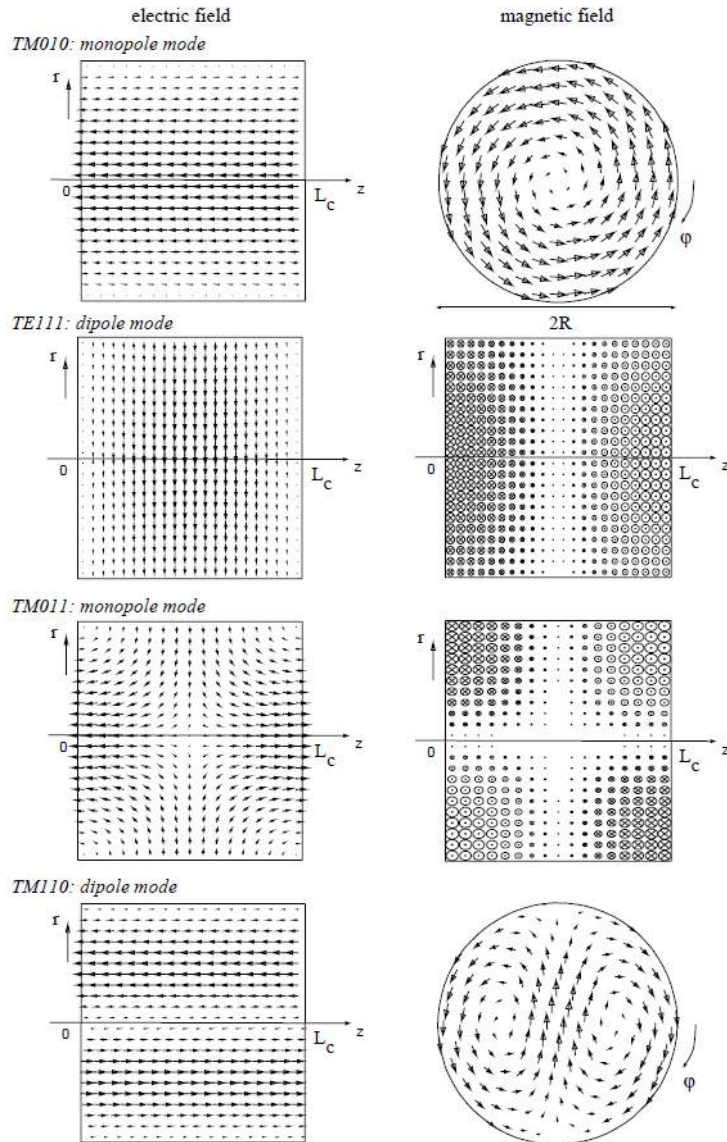
- Losses in the power
$$P_{loss} = \frac{W\omega}{Q}$$

- Quality factor - Q - it is the number of RF oscillation times 2π required for energy inside the cavity to reduce e-fold.

Fundamental and high order modes (HOMs)
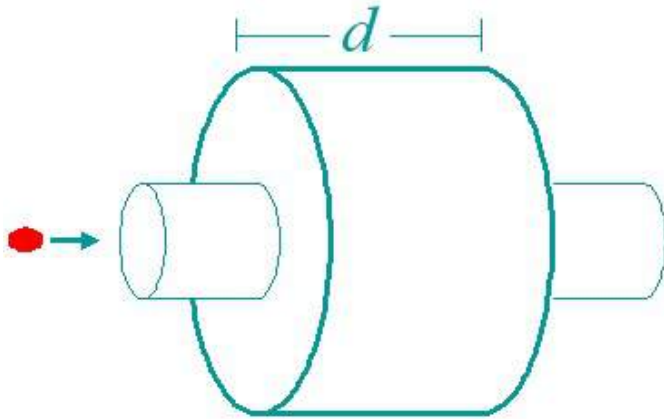
Eigenmodes in a Pill-box cavity

electric field | magnetic field

TM010: monopole mode

TE111: dipole mode

TM011: monopole mode

TM110: dipole mode

$$TM : \varphi_{mnl} = J_n\left(\gamma_{mn}r\right)\cos k_{z,l}z; \ \ k_{z,l} = l\frac{\pi}{d}; \ \ J_n\left(\gamma_{mn}R\right) = 0;$$

$$\omega_{res} = c\sqrt{\gamma_{mn}^2 + l^2\frac{\pi^2}{d^2}}; l = 0,1,2....$$

$$TE : \varphi_{mnl} = J_n\left(\kappa_{mn}r\right)\sin k_{z,l}z; \ \ k_{z,l} = l\frac{\pi}{d}; \ \ J_n'\left(\kappa_{mn}R\right) = 0;$$

$$\omega_{res} = c\sqrt{\kappa_{mn}^2 + l^2\frac{\pi^2}{d^2}}; l = 1,2....$$

Accelerating voltage & transit time

- Assuming charged particles moving along the cavity axis, one can calculate accelerating voltage as

$$V_C = \left| \int_{-\infty}^{\infty} E_z(\rho = 0, z) e^{i\omega_0 z/\beta c} dz \right|$$

For the pillbox cavity one can integrate this analytically:

$$V_C = E_0 \left| \int_0^d e^{i\omega_0 z/\beta c} dz \right| = E_0 d \frac{\sin\left(\dfrac{\omega_0 d}{2\beta c}\right)}{\dfrac{\omega_0 d}{2\beta c}} = E_0 d \cdot T$$


where T is the transit time factor.

- To get maximum acceleration:

$$T_{transit} = t_{exit} - t_{enter} = \frac{T_0}{2} \Rightarrow d = \beta\lambda/2 \Rightarrow V_c = \frac{2}{\pi} E_0 d$$

Thus for the pillbox cavity $T = 2/\pi$.

- The accelerating field E_{acc} is defined as $E_{acc} = V_c/d$. Unfortunately the cavity length is not easy to specify for shapes other than pillbox so usually it is assumed to be $d = \beta\lambda/2$. This works OK for multi-cell cavities, but poorly for single-cell ones.
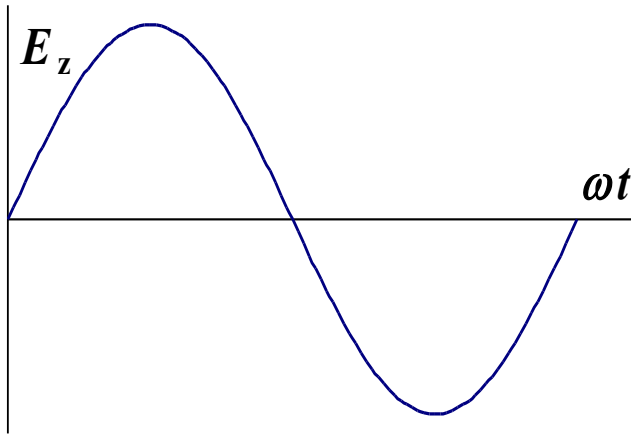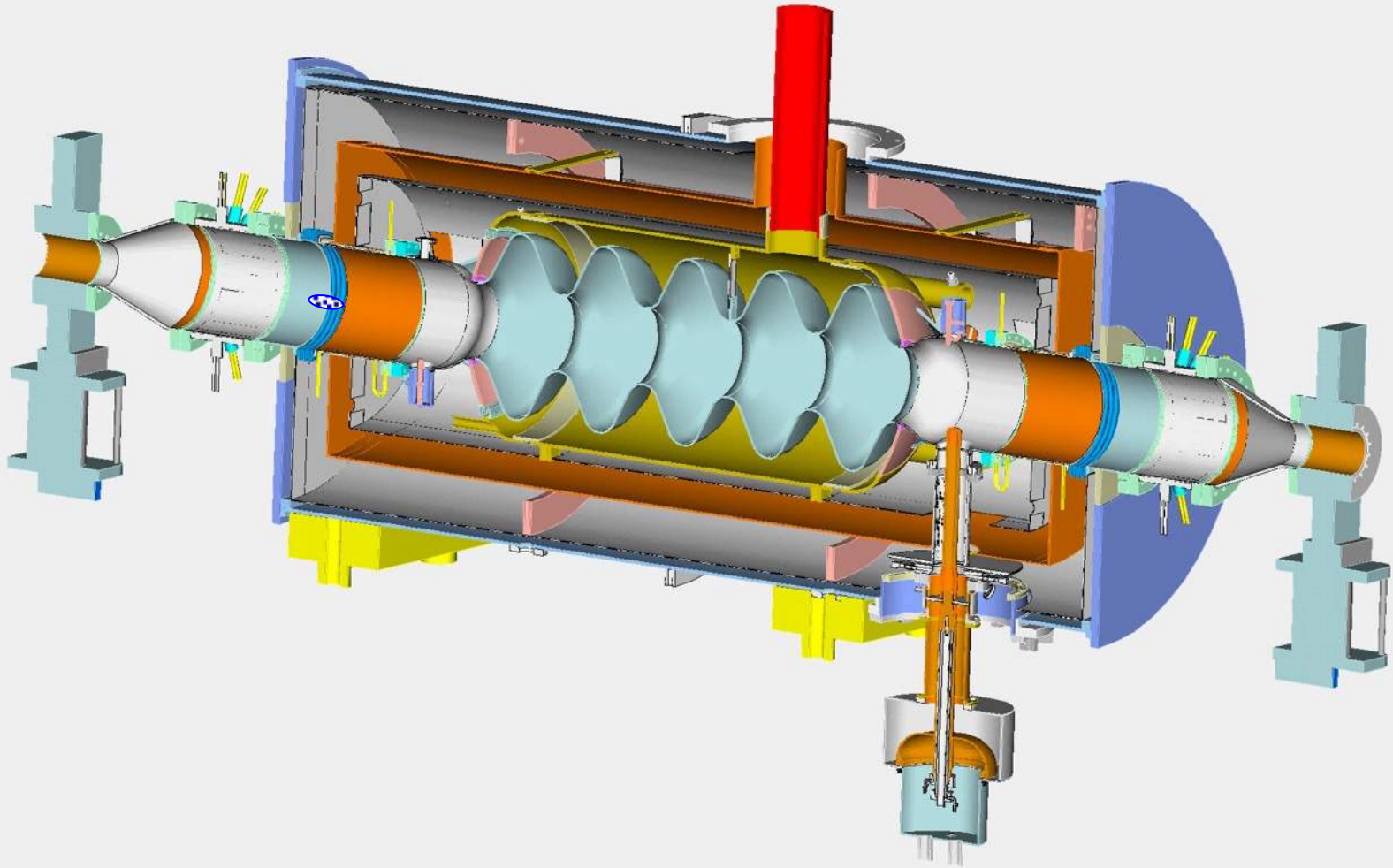
Multicell cavities: coupled oscillators

- Several cells can be connected together to form a multicell cavity.

- Coupling of TM_{010} modes of the individual cells via the iris (primarily electric field) causes them to split:
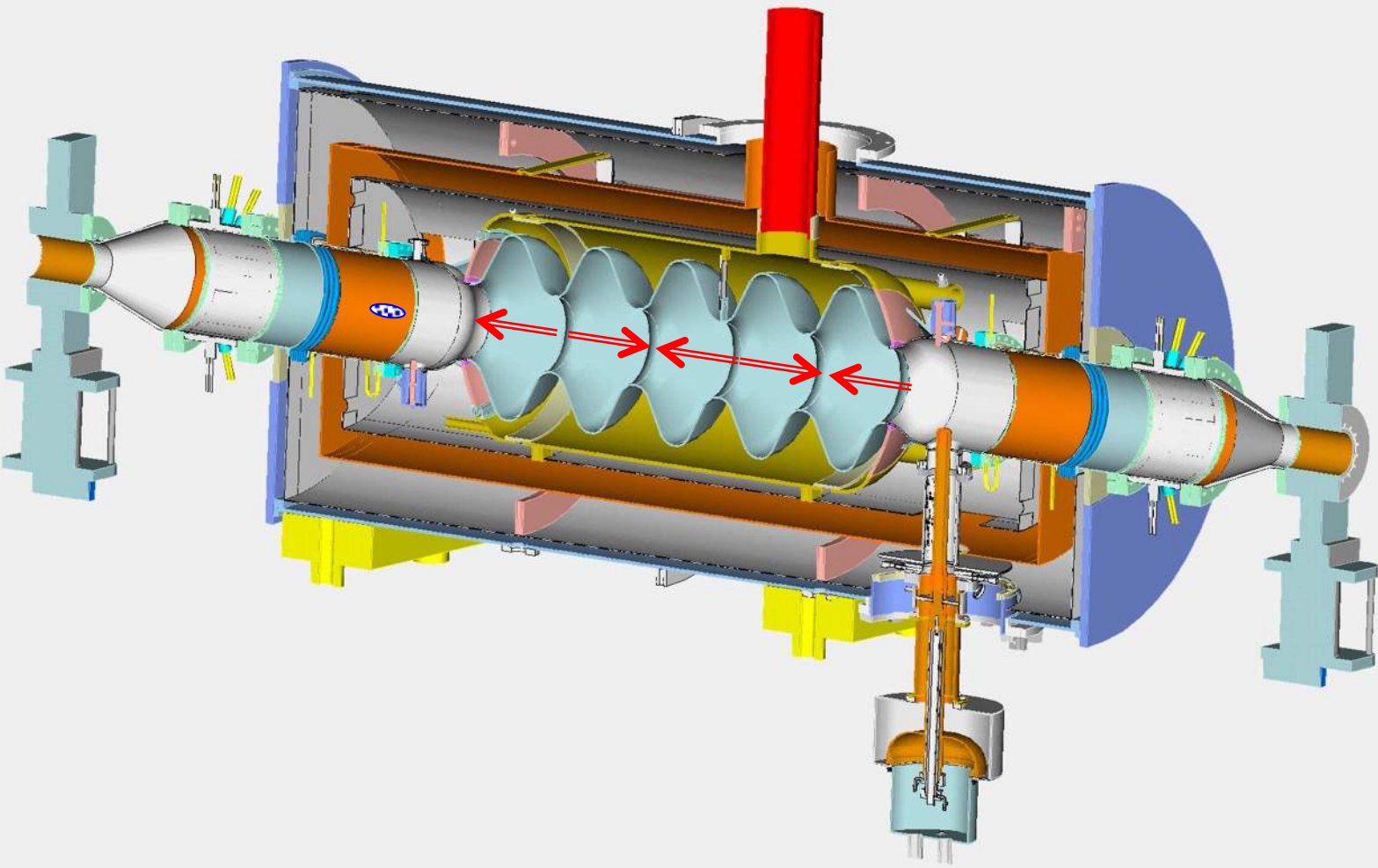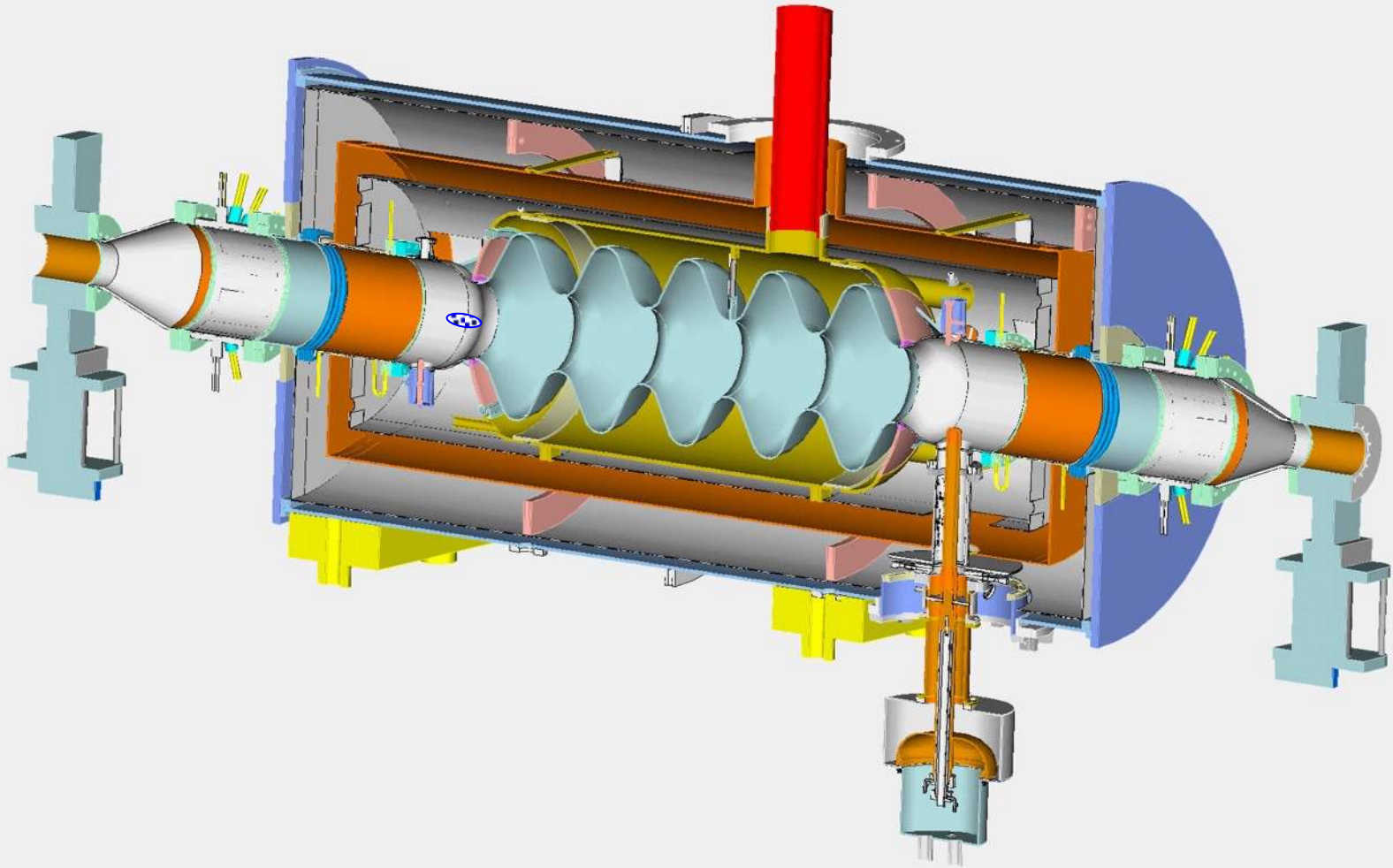
How β=1 RF accelerator works?
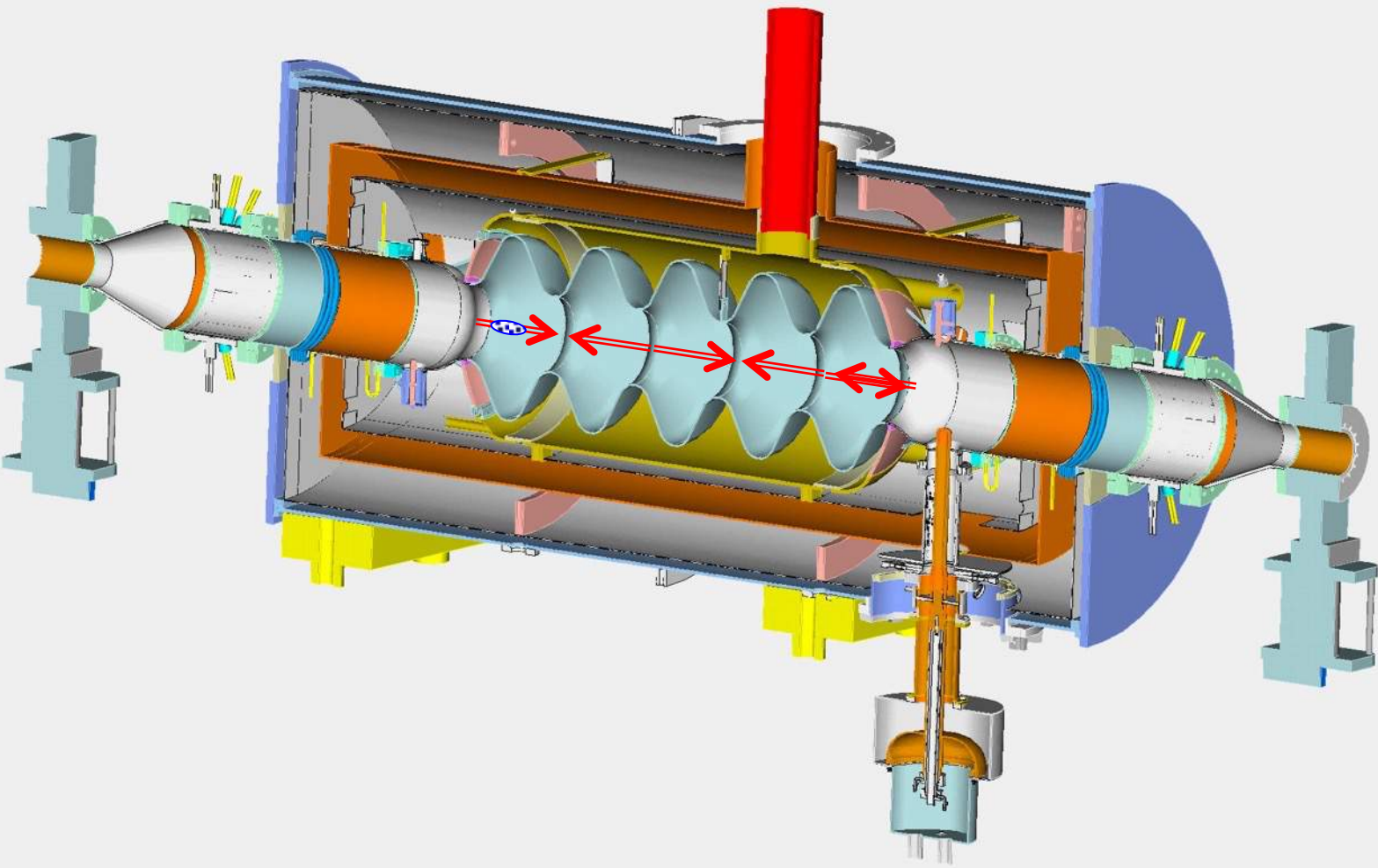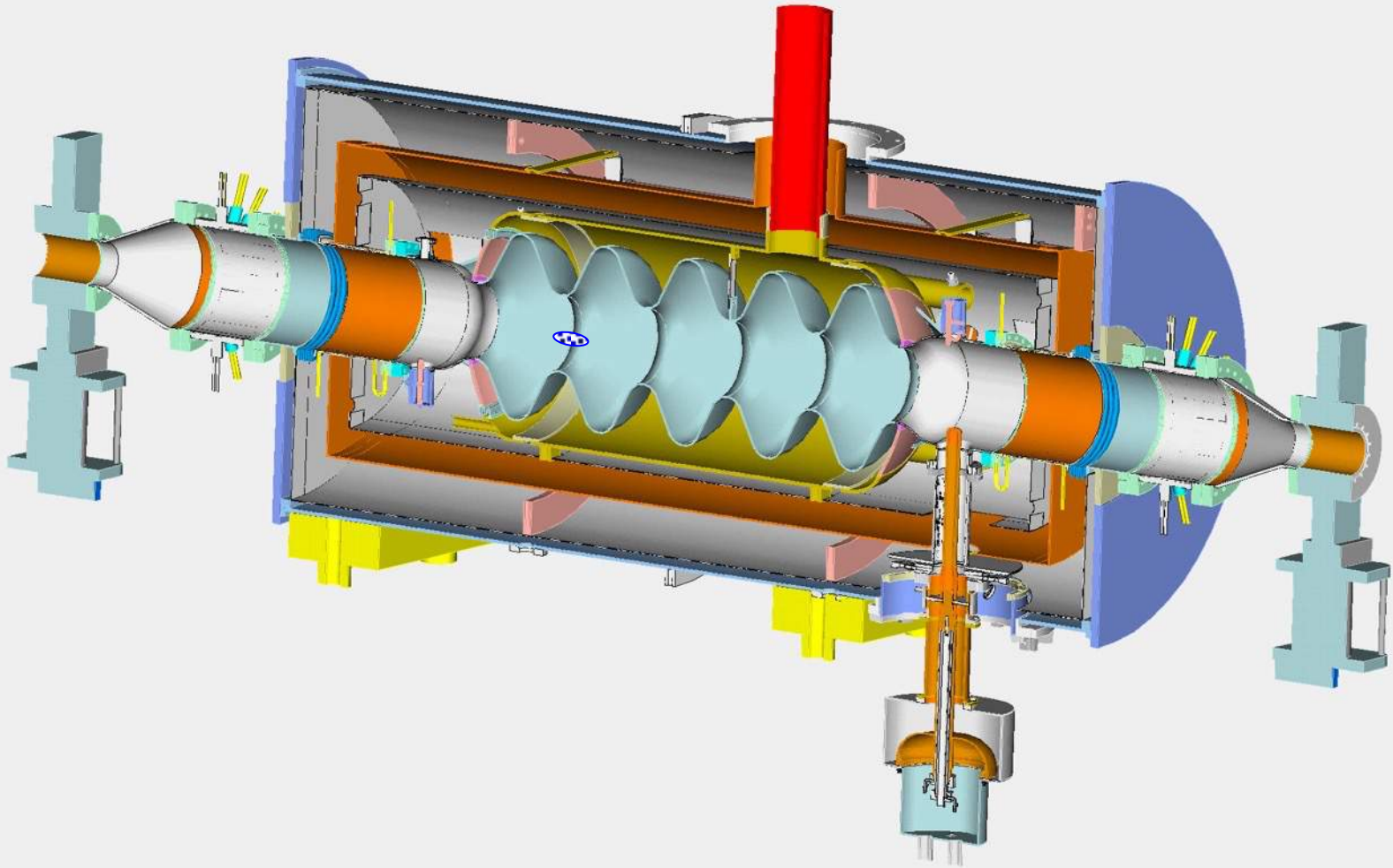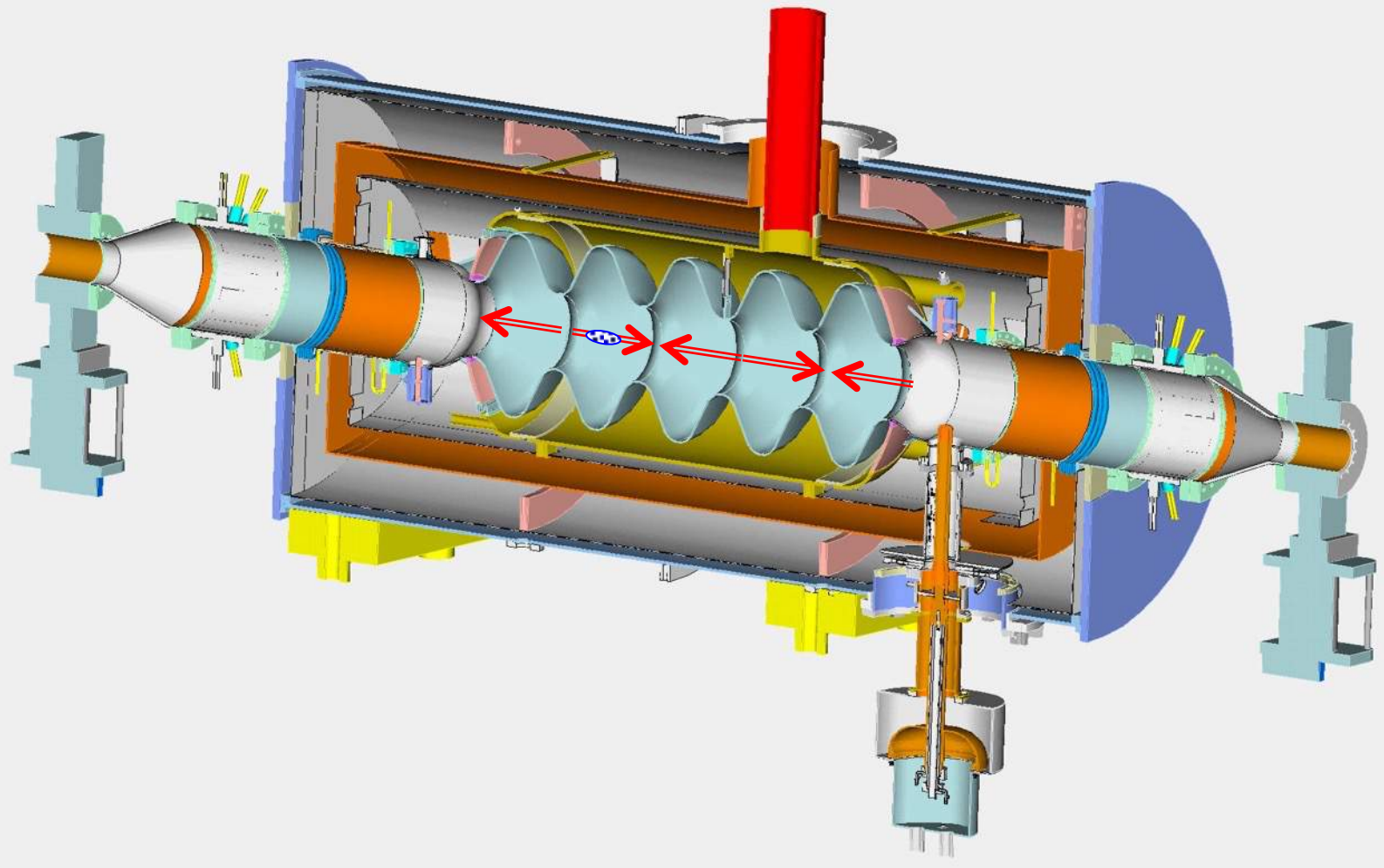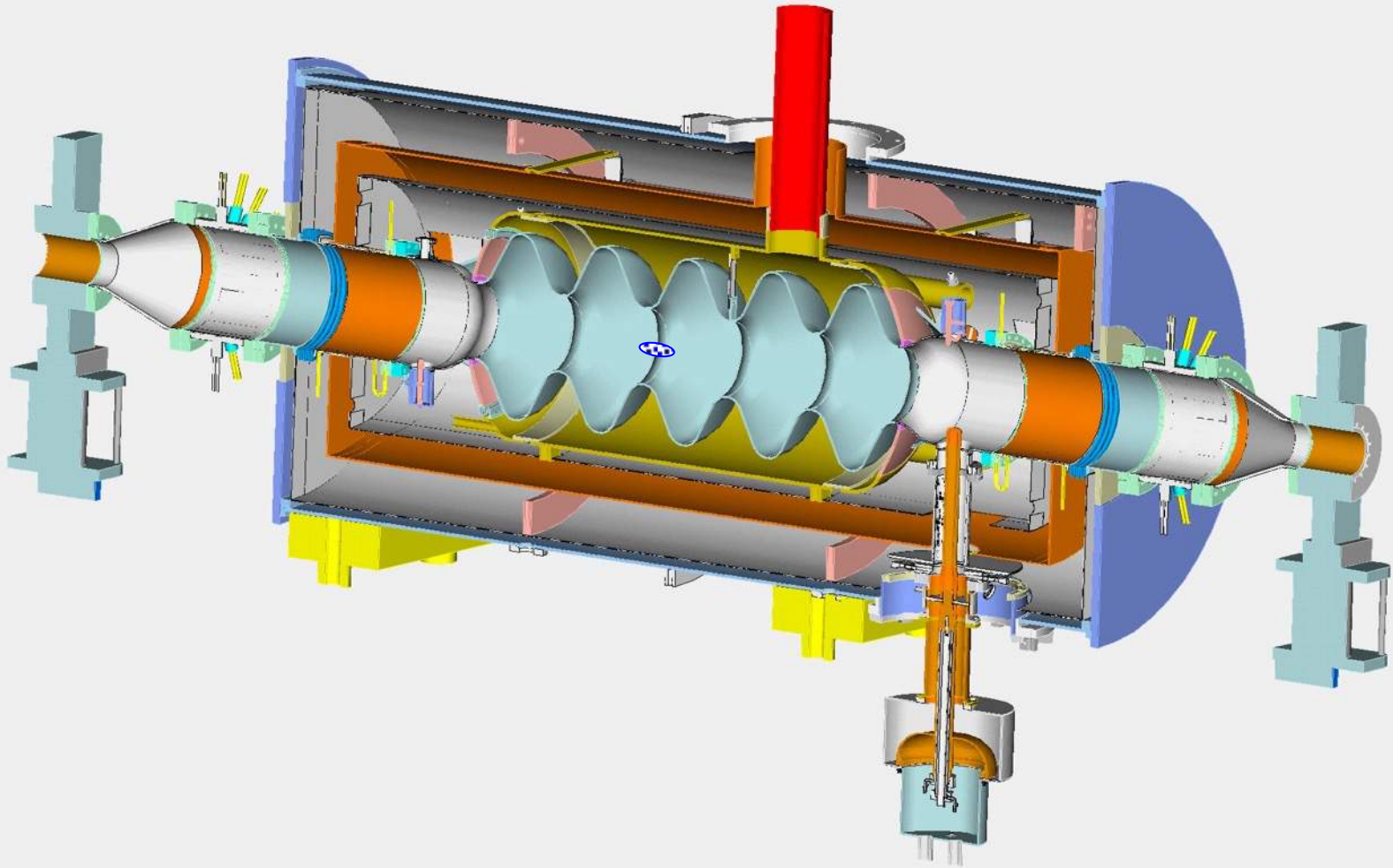## In pictures

How β=1 RF accelerator works?
In pictures

How β=1 RF accelerator works?
In pictures

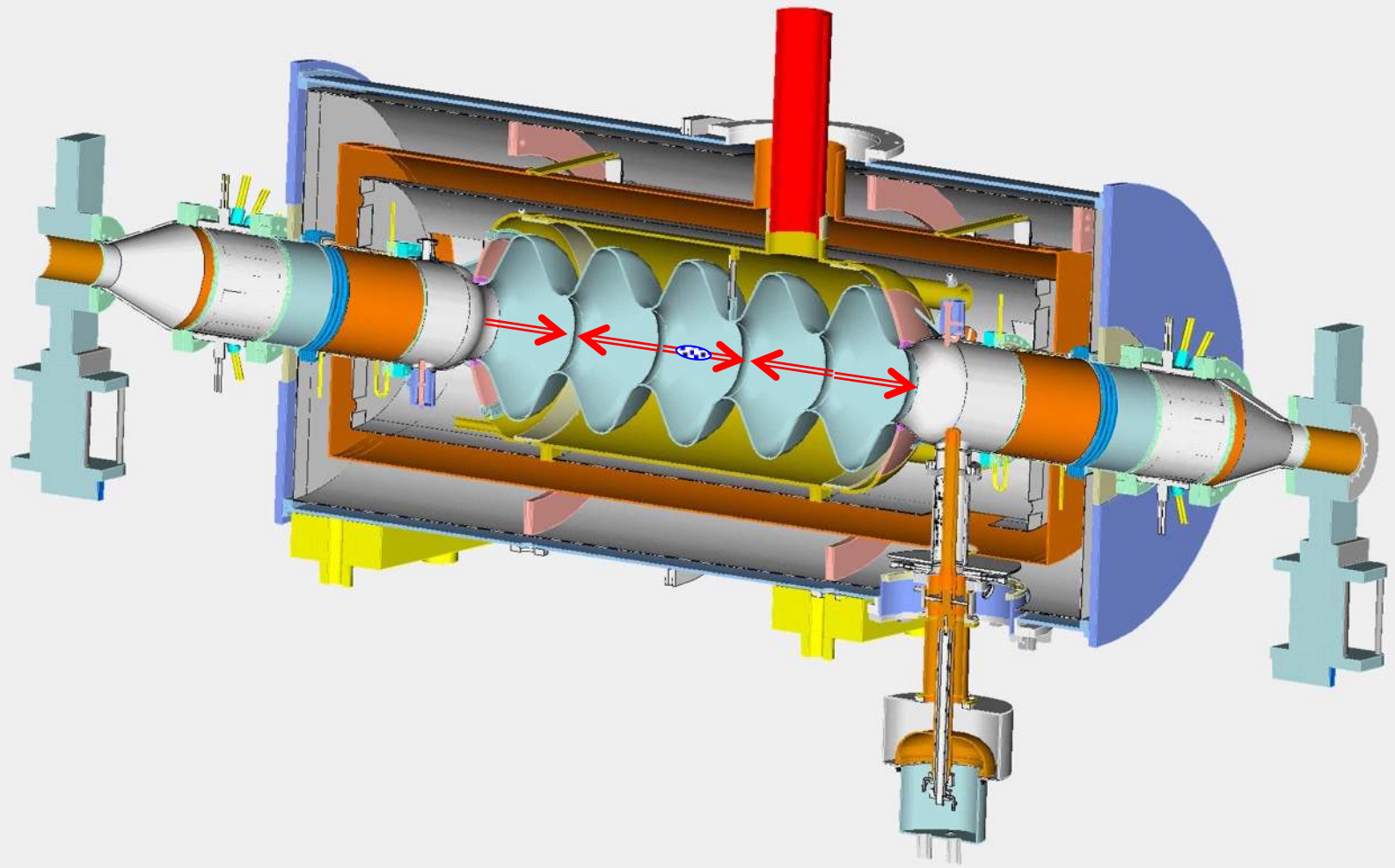# How β=1 RF accelerator works?
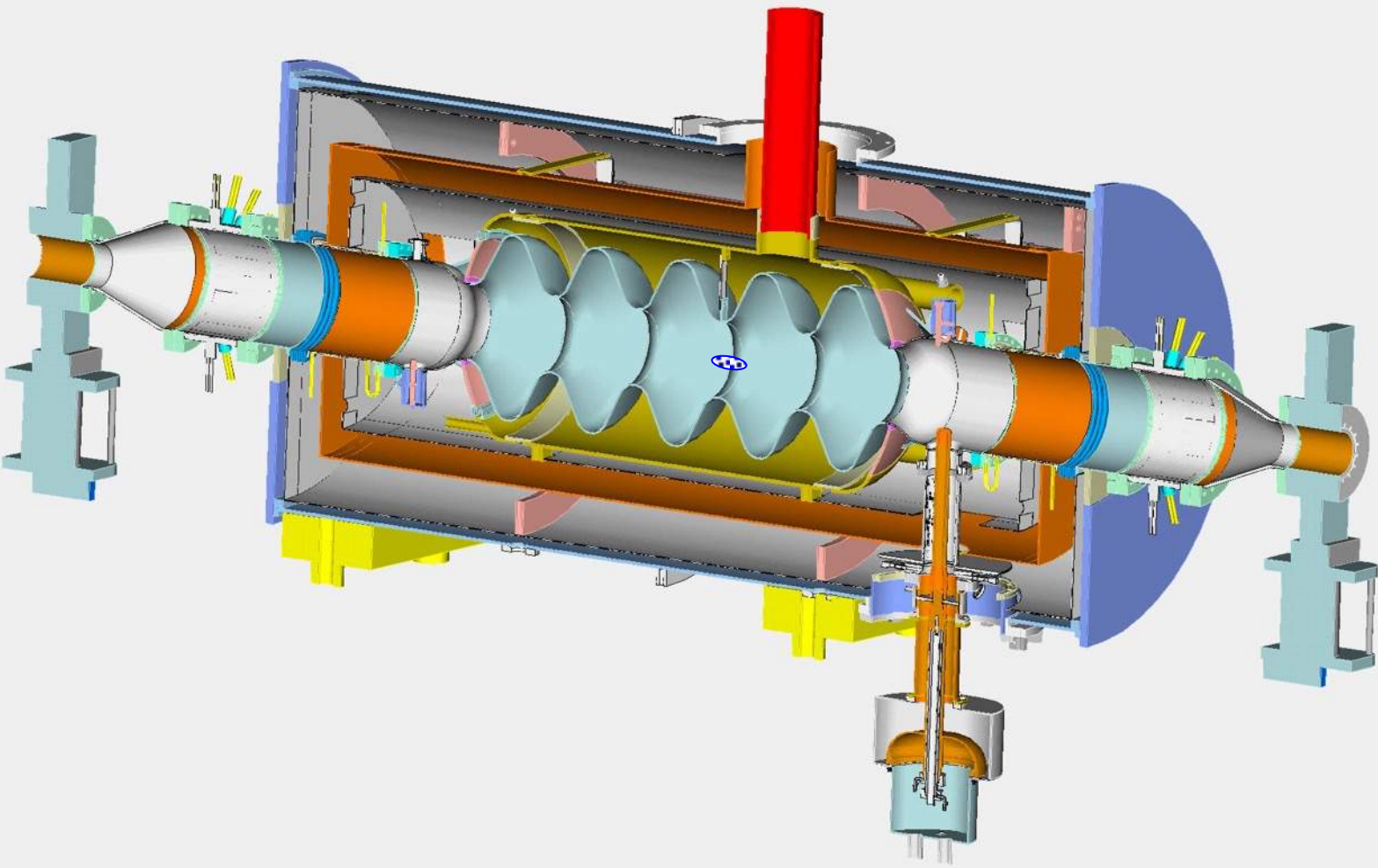## In pictures

How β=1 RF accelerator works?
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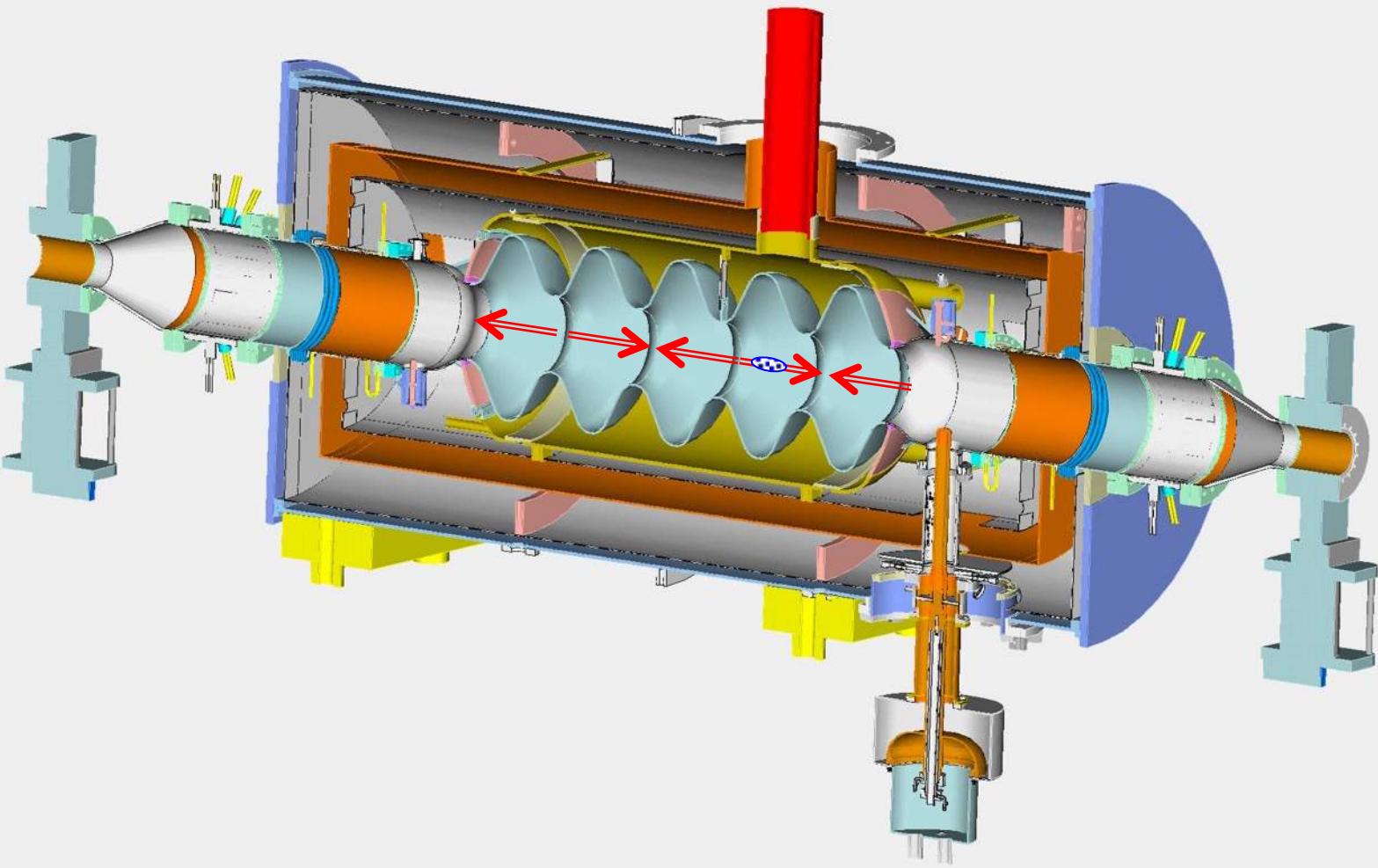# How β=1 RF accelerator works?
In pictures

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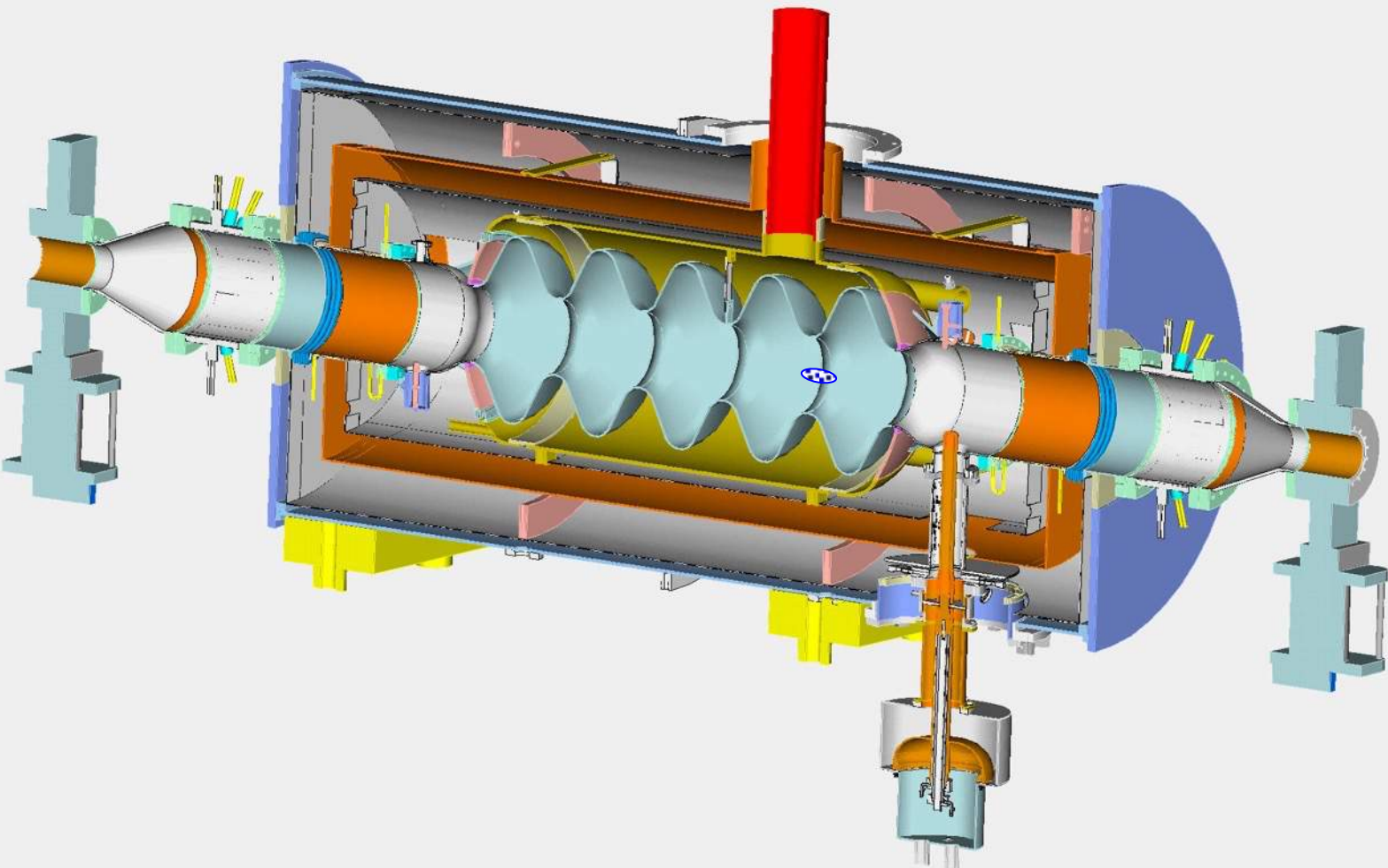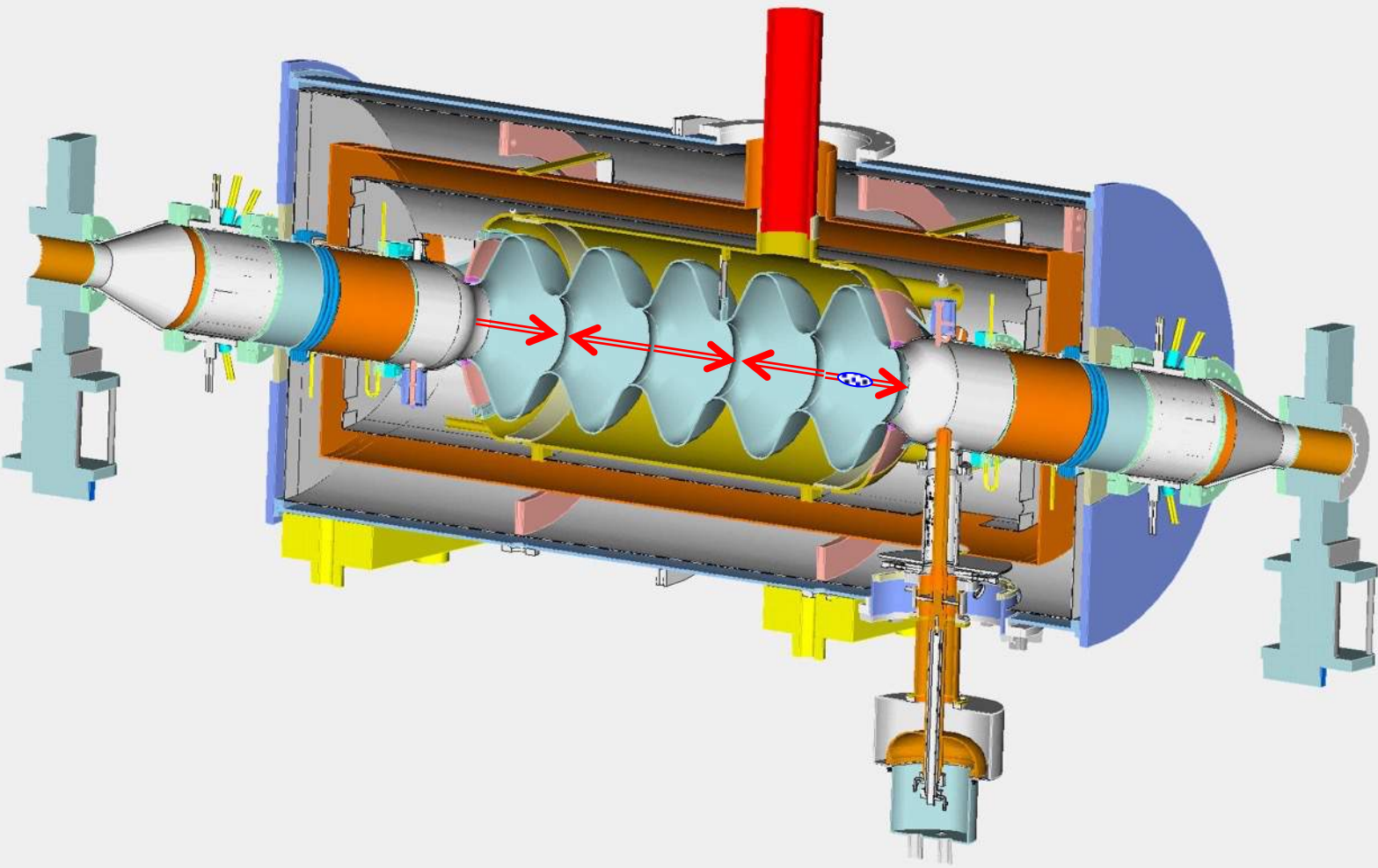# How β=1 RF accelerator works?
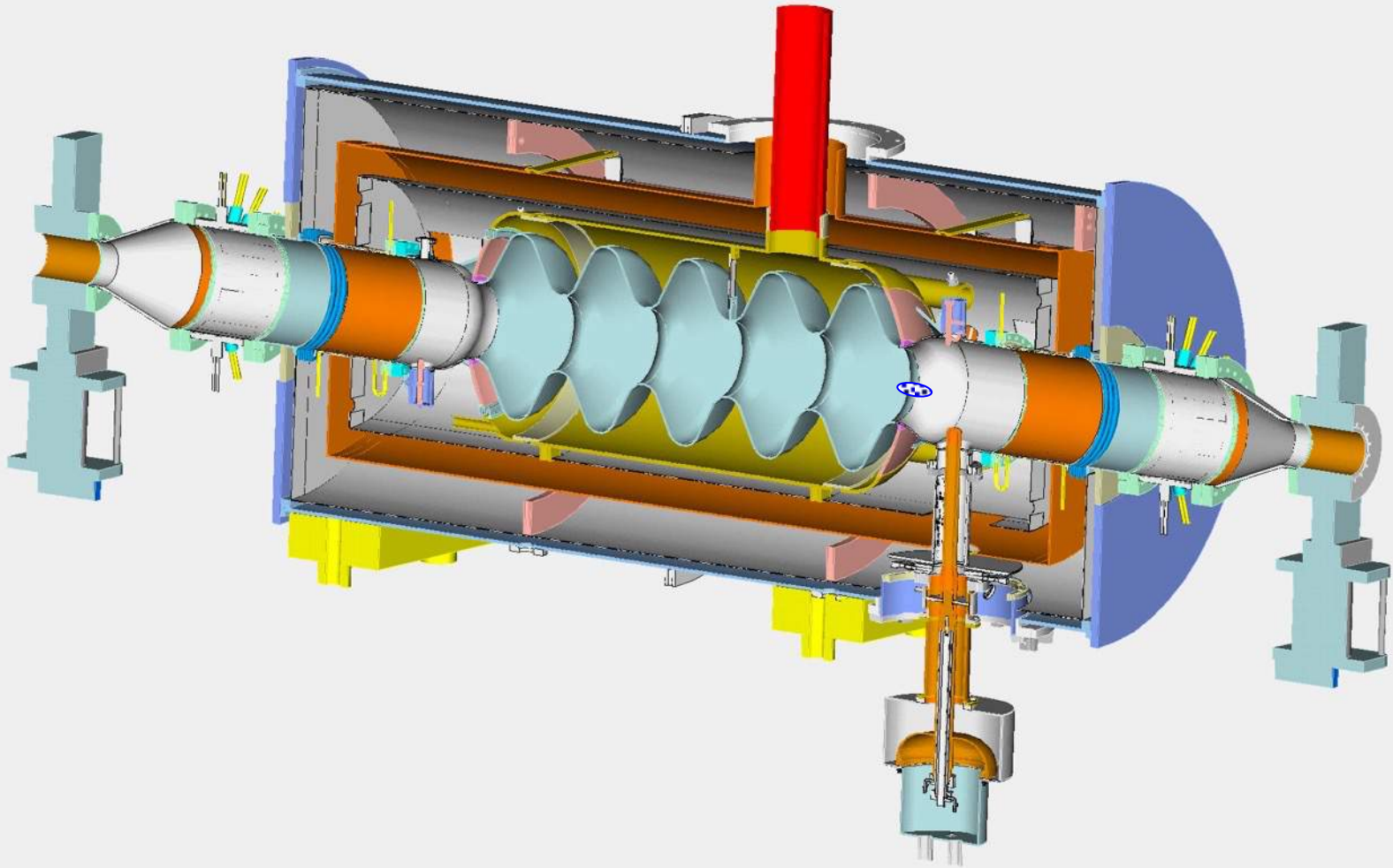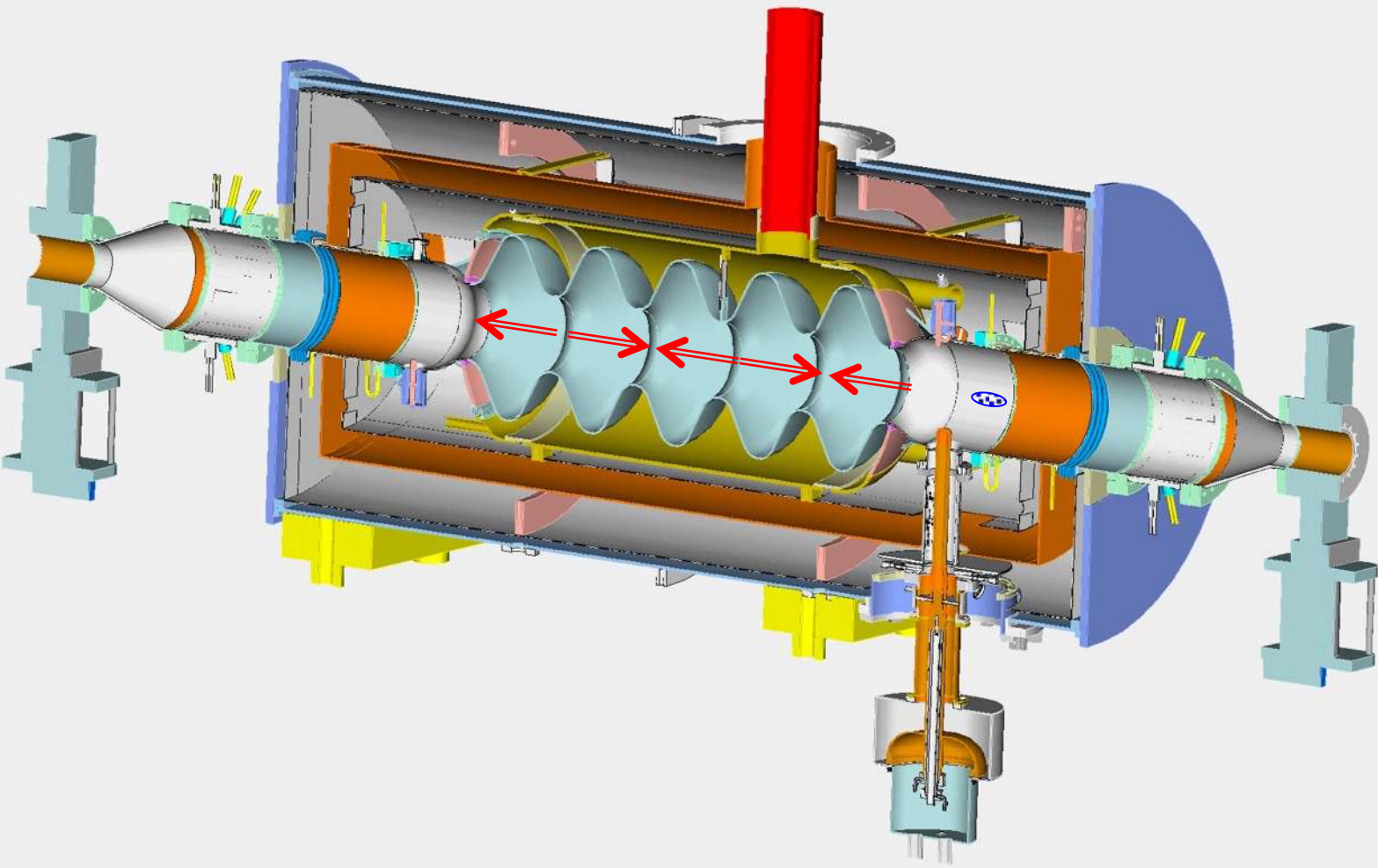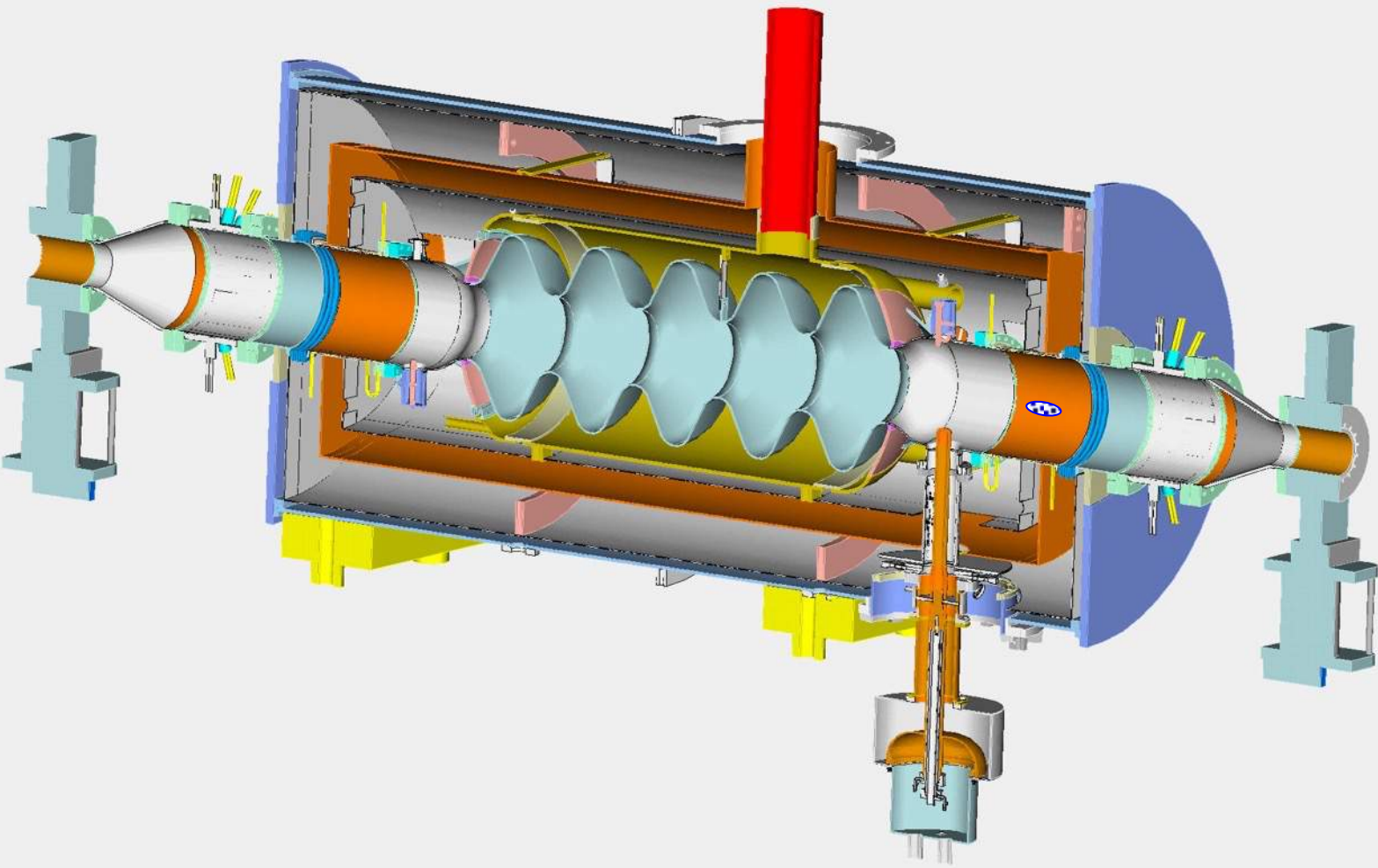# In pictures

How β=1 RF accelerator works?
In pictures

How β=1 RF accelerator works?
In pictures

Multi-cell cavities

- We learned so far that single cell RF cavity has limited accelerating voltage

$$Max\left(V_{RF}\right) = \frac{E_o \lambda_{RF}}{\pi}$$

- To gain more energy we can either use more individual cells or use multi-cell cavities
- The first path, while feasible, is expensive (each cavity would need individual transmitter, waveguide, controls, etc.) and less effective – the average accelerating gradient (energy gain per meter of real estate) would be low
- Thus, where the acceleration gradient is important, the accelerator community uses multi-cell cavities

9-cell Tesla design

7-cell

5-cell

What we learned

- Resonant modes in a cavity resonator belong to two families: TE and TM.

- There is an infinite number of resonant modes.

- The lowest frequency TM mode is usually used for acceleration.

- All other modes (HOMs) are considered parasitic as they can harm the beam.

- Several figures of merits are used to characterize accelerating cavities: main are accelerating voltage, transit time and Q-factor.

- In a multi-cell cavity every mode splits into a pass-band.

- The number of modes in each pass-band is equal to the number of cavity cells.

- The width of the pass-band is determined by the cell-to-cell coupling.

- Accelerating cavities operate at frequency below the cut-off frequency of vacuum pipes connected to them. The RF field decay exponentially along the pipes and reduces to a negligible level at length ~ few beam-pipe radii (assuming $R \ll \lambda_{RF}$)

- Coaxial lines and rectangular waveguides are commonly used in RF systems for power delivery to cavities

- Homework is posted on the website this evening: due in one week, March 2