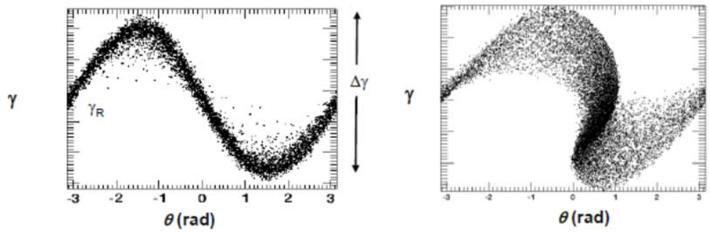
High Gain Regime: Concept

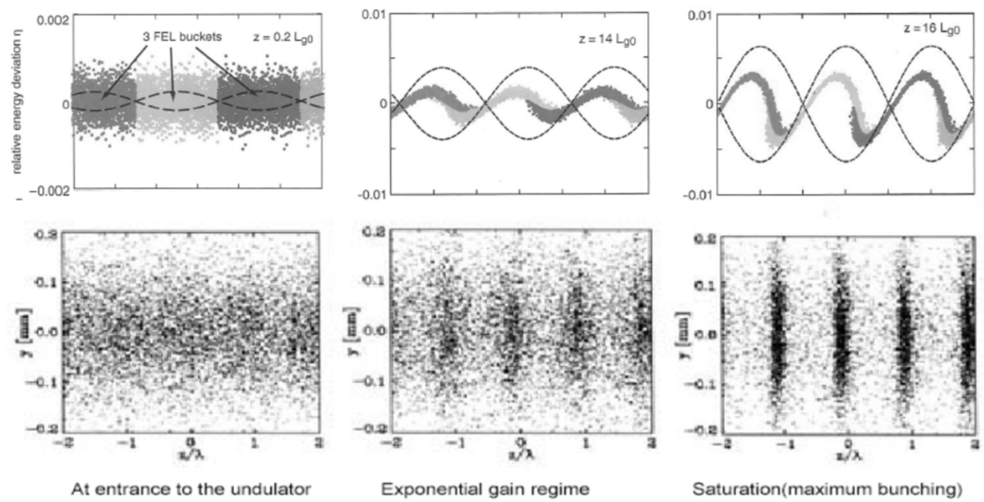
1. Energy kick from radiation field + dispersion/drift -> electron density bunching;

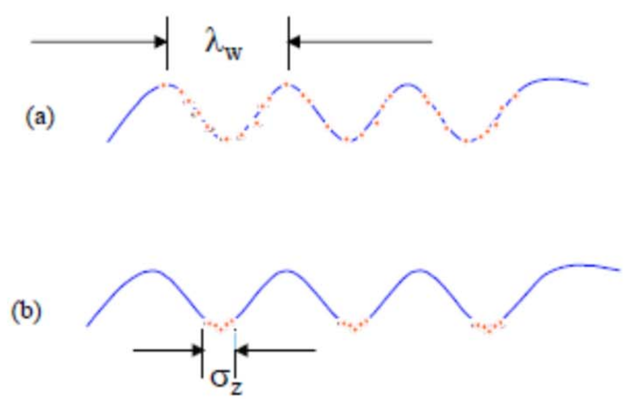


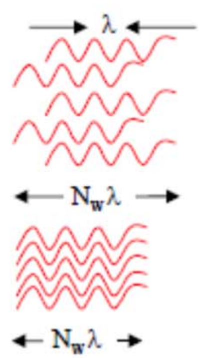
*The plots are for illustration only. The right plot actually shows somewhere close to saturation.

2. Electron density bunching makes more electrons radiates coherently -> higher radiation field;

3. Higher radiation fields leads to more density bunching through 1 and hence closes the positive feedback loop -> FEL instability.







$$|E| \propto \sqrt{N_e}$$

$$I_{incoherent} \propto N_e$$

$$|E| \propto N_e$$

$$I_{coherent} \propto N_e^2$$

The positive feedback loop between radiation field and electron density bunching is the underlying mechanism of high gain FEL regime.

1-D Model for cold beam without detuning

$$B(z)=\left\langle e^{-i\psi}\right\rangle=\frac{1}{N}\sum_{j=1}^{N}e^{-i\psi_j} \qquad D(z)=\left\langle Pe^{-i\psi}\right\rangle=\frac{1}{N}\sum_{j=1}^{N}P_j e^{-i\psi_j}$$

Assuming that $C=0$, it follows $\frac{d}{dz}\psi=C+\frac{\omega}{\gamma_z^2 c\mathcal{E}_0}P=\frac{\omega}{\gamma_z^2 c\mathcal{E}_0}P$

$$\boxed{\frac{d}{dz}B(z)=-i\left\langle e^{-i\psi}\frac{d}{dz}\psi\right\rangle=-i\frac{\omega}{c\gamma_z^2\mathcal{E}_0}\left\langle e^{-i\psi}P\right\rangle=-i\frac{\omega}{c\gamma_z^2\mathcal{E}_0}D(z)}$$

$$\frac{dP}{dz}=-e\theta_s E(z)\cos(\psi)$$

$$\boxed{\frac{d}{dz}D(z)=\left\langle e^{-i\psi}\frac{d}{dz}P\right\rangle-i\left\langle e^{-i\psi}P\frac{d}{dz}\psi\right\rangle\approx\left\langle e^{-i\psi}\frac{d}{dz}P\right\rangle=-\left\langle e^{-i\psi}eE\theta_s\cos(\psi)\right\rangle\approx-\frac{1}{2}e\theta_s E}$$

Wave Equation

$$\psi = k_w z + k\left(z - ct\right)$$

1-D theory and hence $\partial / \partial x = 0$ and $\partial / \partial y = 0$

Wave equation for transverse vector potential: $$\frac{\partial^2 \vec{A}_\perp}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \vec{A}_\perp}{\partial t^2} = -\mu_0 \vec{j}_\perp \qquad (1)$$

Transverse current perturbation: $$j_x + i j_y = \frac{1}{v_z}\left(v_x + i v_y\right) j_z = \theta_s e^{-ik_w z} j_z \qquad (2)$$

We seek the solution for vector potential of the form: $\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos\left(k_u z\right)\hat{x} - \sin\left(k_u z\right)\hat{y}\right]$

$$A_{x,y}(z,t) = \widetilde{A}_{x,y}(z) e^{i\omega(z/c - t)} + \widetilde{A}_{x,y}{}^*(z) e^{-i\omega(z/c-t)} \qquad (3)$$

Inserting eq. (2) and (3) into eq. (1) yields

$$e^{i\omega(z/c-t)}\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\widetilde{A}_x\\ \widetilde{A}_y\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\widetilde{A}_x\\ \widetilde{A}_y\end{pmatrix}\right\} + C.C. = -\mu_0\theta_s\begin{pmatrix}\cos(k_w z)\\ -\sin(k_w z)\end{pmatrix} j_z$$

Multiplying both sides by $e^{ik_w z}$ and neglecting terms proportional to $e^{ik_w z - ik(z-ct)}$ since they will change fast over the FEL (same as the helicity argument).

$$\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\widetilde{A}_{tot,x}\\ \widetilde{A}_{tot,y}\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\widetilde{A}_{tot,x}\\ \widetilde{A}_{tot,y}\end{pmatrix}\right\} = -\frac{\mu_0 N\theta_s}{2}\begin{pmatrix}e^{ik_w z} + e^{-ik_w z}\\ ie^{ik_w z} - ie^{-ik_w z}\end{pmatrix}\left\langle j_z e^{-i\psi}\right\rangle e^{ik_w z}$$

1. Ignoring fast oscillating term $\sim e^{2ik_w z}$

2. Ignoring second derivative by assuming that the variation of $\widetilde{A}_x'$ is negligible over the optical wave length.

Wave Equation

After neglecting the fast oscillation terms, we get the following relation between the current perturbation and the vector potential of the radiation field:

$$\frac{\partial}{\partial z}\widetilde{A}_{tot,x} = -\frac{c\mu_0 N\theta_s}{4i\omega}\left\langle j_z e^{-i\psi}\right\rangle \qquad \frac{\partial}{\partial z}\widetilde{A}_{tot,y} = \frac{\mu_0 Nc\theta_s}{4\omega}\left\langle j_z e^{-i\psi}\right\rangle$$

In order to relate the vector potential to the electric field, we use the Maxwell equation:

$$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \Rightarrow \nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0 \Rightarrow \left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=\vec{\nabla}\varphi \Rightarrow E_{x,y}=-\frac{\partial A_{x,y}}{\partial t}$$

1D: $\frac{\partial}{\partial x}=0\ ,\quad \frac{\partial}{\partial y}=0$

$$\Rightarrow E_x + iE_y = Ee^{i\omega(z/c-t)} = -\frac{\partial}{\partial t}\left[\left(\tilde{A}_{tot,x}+i\tilde{A}_{tot,y}\right)e^{i\omega(z/c-t)}\right]$$

$$\Rightarrow E = i\omega\left(\tilde{A}_{tot,x}+i\tilde{A}_{tot,y}\right)$$

$$\vec{E}_\perp(z,t) = E\left[\cos\left(k(z-ct)\right)\hat{x}+\sin\left(k(z-ct)\right)\hat{y}\right]$$

Finally, the relation between the radiation field and the current modulation is obtained:

$$\boxed{\frac{d}{dz}E = i\omega\left(\frac{\partial}{\partial z}\tilde{A}_{tot,x}+i\frac{\partial}{\partial z}\tilde{A}_{tot,y}\right) = -\frac{c\mu_0 N\theta_s}{2}\left\langle j_z e^{-i\psi}\right\rangle = \frac{ec^2 N\mu_0\theta_s}{2V}B = \frac{ec^2 n\mu_0\theta_s}{2}B}$$

$$\left\langle j_z e^{-i\psi}\right\rangle = -\frac{ec}{NV}\sum_{k=1}^{N}e^{-i\psi_k} = -\frac{ecB}{V} \qquad n = N/V$$

1-D High Gain FEL Equation for Cold Beam and Zero Detuning

$$\frac{d}{dz}B(z) = -i\frac{\omega}{c\gamma_z^2\mathcal{E}_0}D(z)$$

$$\frac{d}{dz}D = -\frac{1}{2}e\theta_s E$$

$$\frac{d}{dz}E = \frac{ec^2 n\mu_0\theta_s}{2}B$$

$$\Rightarrow \quad \frac{d^3}{d\hat{z}^3}E = iE$$

$\hat{z} \equiv \Gamma z$ is normalized longitudinal location along wiggler,

$$\Gamma \equiv \left[\frac{\pi j_0\theta_s^2\omega}{c\gamma_z^2\gamma I_A}\right]^{1/3}$$ is the 1-D Gain rate parameter

$$I_A = \frac{4\pi\varepsilon_0 mc^3}{e}$$ is called Alfven current

$$E(\hat{z}) = \sum_{k=1}^{3} B_k e^{\lambda_k \hat{z}}$$

$$\lambda^3 = i \Rightarrow$$

$\lambda_1 = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ ← Growing mode

$\lambda_2 = e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$ ← Damping mode

$\lambda_3 = e^{-i\frac{\pi}{2}} = -i$ ← Oscillating mode

1D Gain Length

- At high gain limit, i.e. $\hat{z} >> 1$, the radiation field is given by

$$E(\hat{z}) \approx B_1 e^{\lambda_k \hat{z}} = B_1 \exp\left[\frac{\sqrt{3}}{2}\Gamma z\right] \exp\left[i\frac{1}{2}\Gamma z\right]$$

and the radiation power is

A: cross section of the radiation field

$$P(\hat{z}) = \varepsilon_0 c \left| E(\hat{z})^2 \right| A = \varepsilon_0 c \left| B_1 \right|^2 \exp\left(\sqrt{3}\Gamma z\right) = \varepsilon_0 c \left| B_1 \right|^2 A \exp\left(\frac{z}{L_G}\right)$$

and the 1-D power gain length is

$$\boxed{L_G \equiv \frac{1}{\sqrt{3}\Gamma} = \frac{\lambda_w}{4\pi\sqrt{3}\rho}}$$

Pierce Parameter

$$\boxed{\rho \equiv \frac{\gamma_z^2 \Gamma c}{\omega} = \frac{\Gamma}{2k_w}}$$

1-D amplitude gain length is $L_{GA} = 2L_G \equiv \frac{2}{\sqrt{3}\Gamma} = \frac{\lambda_w}{2\pi\sqrt{3}\rho}$

Solution for Cold Beam with Nonzero Detuning

$$C \equiv k_w + k - \frac{\omega}{v_z(\mathcal{E}_0)}$$

$$\hat{C} = C / \Gamma$$

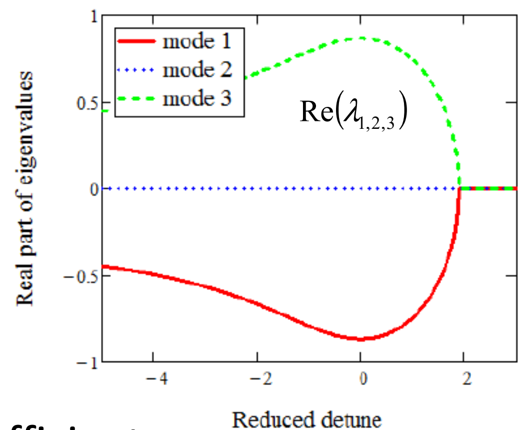
For non-vanishing detuning, the differential equation becomes

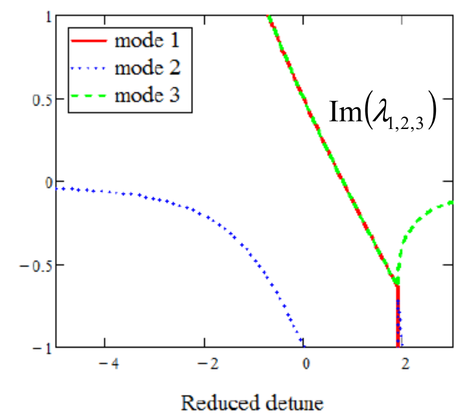
$$\frac{d^3}{d\hat{z}^3}E(\hat{z}) + 2i\hat{C}\frac{d^2}{d\hat{z}^2}E(\hat{z}) - \hat{C}^2\frac{d}{d\hat{z}}E(\hat{z}) = iE(\hat{z})$$

The general solution of the ODE reads:

$$E(\hat{z}) = \sum_{k=1}^{3} B_k e^{\lambda_k \hat{z}}$$

$$\lambda^3 + 2i\hat{C}\lambda^2 - \hat{C}^2\lambda = i$$





Applying initial condition to get the coefficients

$$\begin{pmatrix} E(0) \\ E'(0) \\ E''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \Rightarrow \qquad \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}^{-1}\begin{pmatrix} E(0) \\ E'(0) \\ E''(0) \end{pmatrix}$$

For $E(0) = E_{ext}$ and $E'(0) = E''(0) = 0$, the solution can be explicitly written as

$$E(\hat{z}) = E_{ext}\left[\frac{\lambda_2\lambda_3 e^{\lambda_1\hat{z}}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2\hat{z}}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3\hat{z}}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right]$$

Low Gain Limit of High Gain Solution

Can we reproduce the previously obtained low gain solution by taking the proper limit of the high gain solution?

$$\tau \equiv \frac{2\pi j_0 \theta_s^2 \omega}{c\gamma_z^2\gamma}\frac{l_w^3}{I_A} = 2\Gamma^3 l_w^3$$

$$g_l = \frac{\left(E_{ext}+\Delta E\right)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau \cdot f\left(\hat{C}_l\right) = 2\Gamma^3 l_w^3 f_l\left(\hat{C}_l\right)$$

$$\boxed{f_l\left(\hat{C}_l\right) = \frac{2}{\hat{C}_l^{\,3}}\left(1-\cos\hat{C}_l - \frac{\hat{C}_l}{2}\sin\hat{C}_l\right)}$$

$$\hat{C}_l = Cl_w$$

$$g_h\left(\hat{C}_l\right) = \frac{\tilde{E}^2 - E_{ext}^2}{E_{ext}^2} = \left|\frac{\lambda_2\lambda_3 e^{\lambda_1\hat{l}_w}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2\hat{l}_w}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3\hat{l}_w}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right|^2 - 1$$

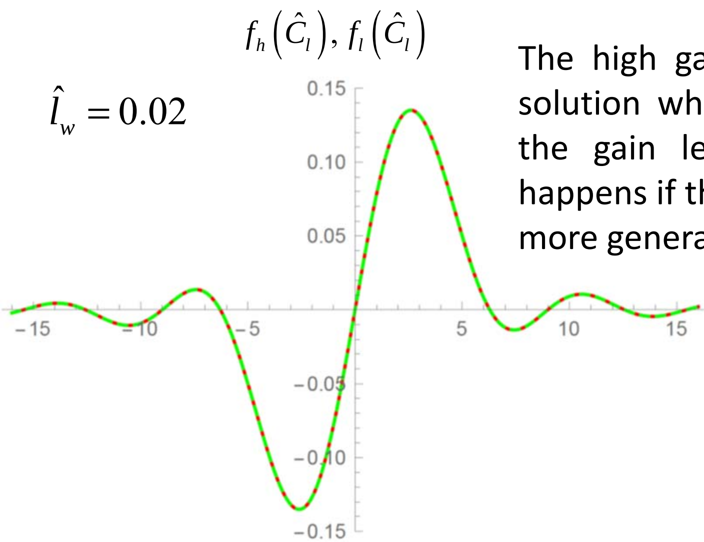
$$= 2\Gamma^3 l_w^3 f_h\left(\hat{C}_l\right) \qquad \hat{l}_w = l_w\Gamma$$

$$\boxed{f_h\left(\hat{C}_l\right) = \frac{1}{2\hat{l}_w^3}\left\{\left|\frac{\lambda_2\lambda_3 e^{\lambda_1\hat{l}_w}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2\hat{l}_w}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3\hat{l}_w}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right|^2 - 1\right\}}$$

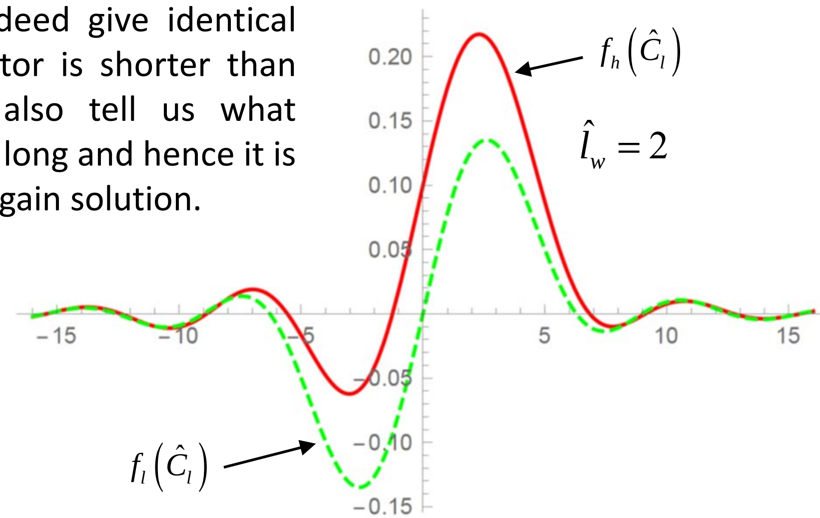
The normalization factor for high gain is different from that of low gain:

$$\hat{C}_h = C/\Gamma = Cl_w/\hat{l}_w = \hat{C}_l/\hat{l}_w$$

$$\lambda^3 + 2i\frac{\hat{C}_l}{\hat{l}_w}\lambda^2 - \left(\frac{\hat{C}_l}{\hat{l}_w}\right)^2\lambda = i$$



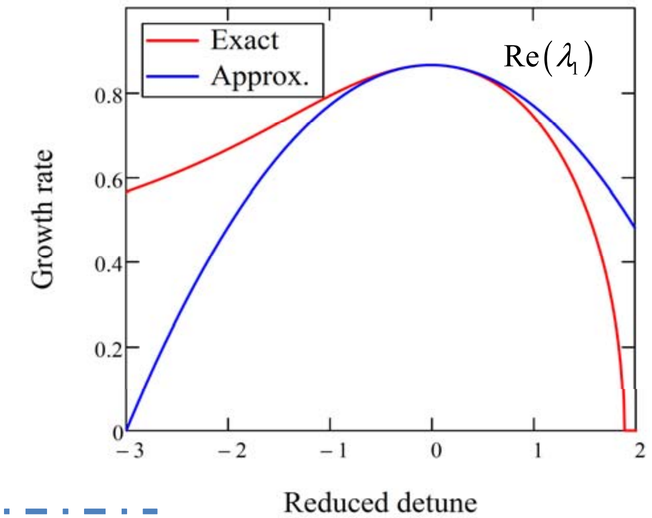
The high gain solution indeed give identical solution when the undulator is shorter than the gain length. But it also tell us what happens if the undulator is long and hence it is more general than the low gain solution.



Bandwidth at High Gain Limit I

It is sometimes hard to extract insights from the exact solution of the 3rd order polynomial equation for the eigenvalue. Therefore, it is useful to get the approximate solution which is simpler but gives accurate results for the region that we are interested in.

$$\lambda^3 + 2i\hat{C}\lambda^2 - \hat{C}^2\lambda = i \qquad \boxed{\lambda = a_0 + a_1\hat{C} + a_2\hat{C}^2}$$



$$f\left(\hat{C}\right) = \left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right)^3 + 2i\hat{C}\left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right)^2 - \hat{C}^2\left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right) - i = 0$$

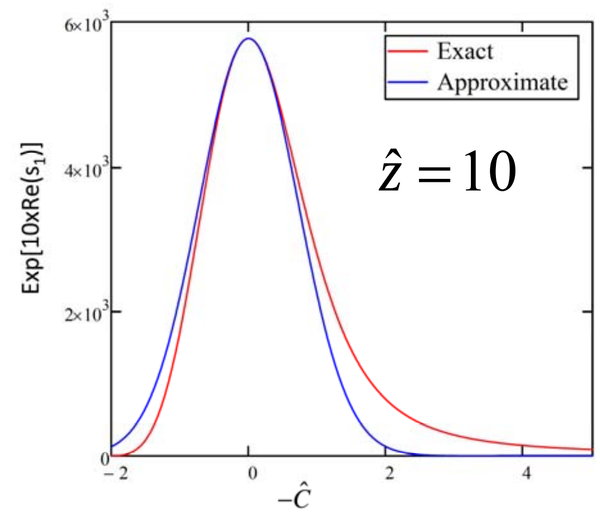
$$f\left(\hat{C}\right) = f_0\left(a_0, a_1, a_2\right) + f_1\left(a_0, a_1, a_2\right)\hat{C} + f_2\left(a_0, a_1, a_2\right)\hat{C}^2 = 0$$

(Homework)

Zeroth order equation: $f(0) = 0 \Rightarrow \quad \boxed{a_0 = \frac{\sqrt{3}}{2} + i\frac{1}{2}}$

First order equation: $\left.\frac{d}{d\hat{C}} f\left(\hat{C}\right)\right|_{\hat{C}=0} = 0 \Rightarrow \quad \boxed{a_1 = -i\frac{2}{3}}$

Second order equation: $\left.\frac{d^2}{d\hat{C}^2} f\left(\hat{C}\right)\right|_{\hat{C}=0} = 0 \Rightarrow \quad \boxed{a_2 = -\frac{1}{9}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)}$



Bandwidth at High Gain Limit II

After taking the approximate eigenvalue, the radiation field in frequency domain is

$$\boxed{E\left(\hat{C}\right) \sim \exp\left[a_0\hat{z} + a_1\hat{C}\hat{z} + a_2\hat{C}^2\hat{z}\right] \sim \exp\left[-\frac{\hat{C}^2}{2\sigma_{\hat{C}}^{\ 2}}\right] \Rightarrow \sigma_{\hat{C}} = \sqrt{-\frac{1}{2\,\mathrm{Re}\left(a_2\right)\hat{z}}}}$$

$$\mathrm{Re}\left(a_2\right) = -\frac{\sqrt{3}}{18} \qquad \sigma_{\hat{C}} = 3\sqrt{\frac{1}{\sqrt{3}\Gamma z}} \qquad \hat{C} \equiv \frac{1}{\Gamma}\left(k_w - \frac{\omega}{2c\gamma_z^2}\right) \qquad \Gamma = \rho\frac{\omega}{\gamma_z^2 c}$$

1D FEL bandwidth for radiation field:

$$\boxed{\sigma_\omega = \Gamma 2c\gamma_z^2\sigma_{\hat{C}} = 6c\gamma_z^2\sqrt{\frac{\Gamma}{\sqrt{3}z}} = 3\omega_0\sqrt{\frac{2\rho}{\sqrt{3}k_w z}}}$$

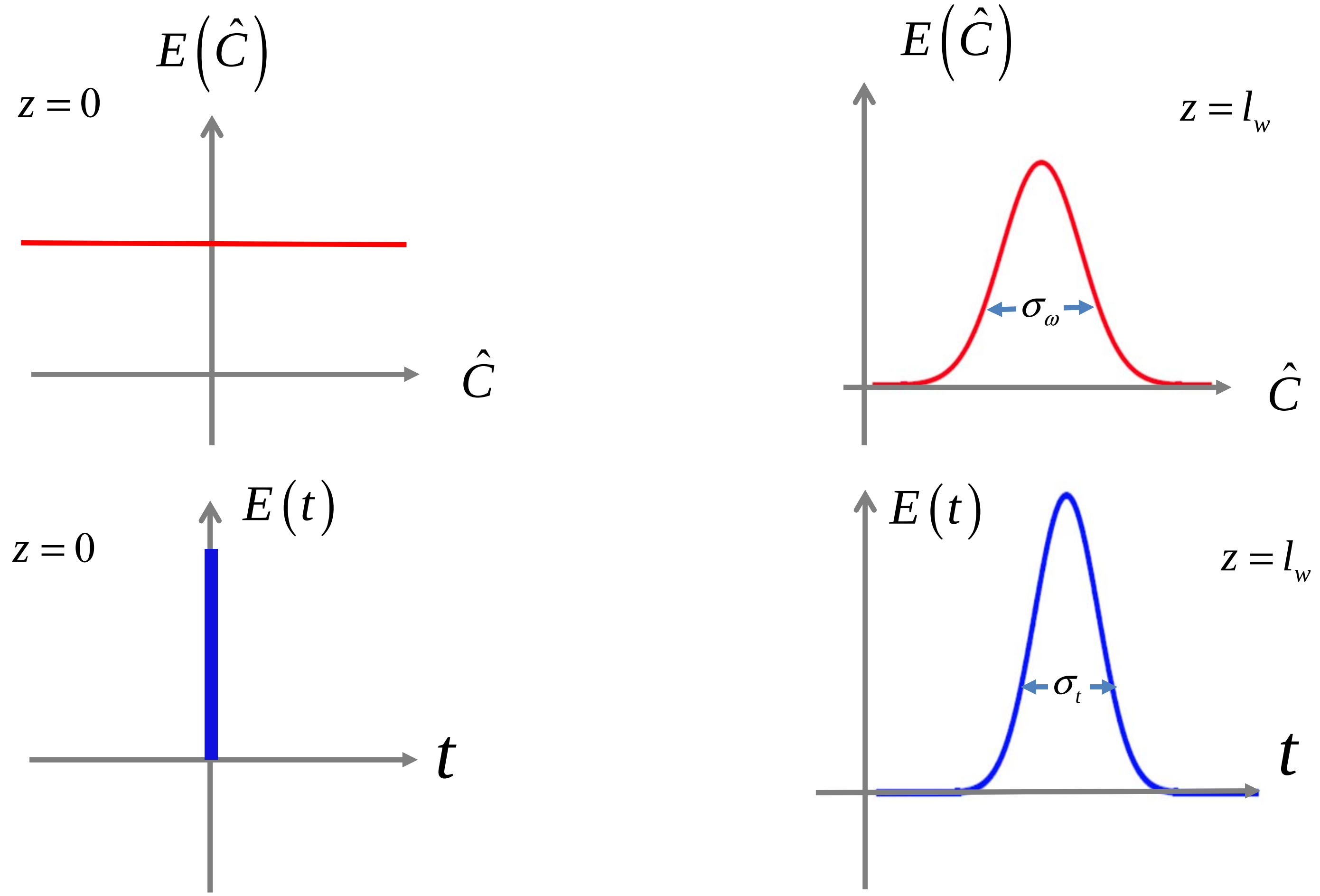
1D FEL bandwidth for radiation power:

$$\boxed{\sigma_A = \frac{\sigma_\omega}{\sqrt{2}} = \omega_0\sqrt{\frac{3\sqrt{3}\rho}{k_w z}}}$$

Pierce Parameter

$$\boxed{\rho = \frac{\gamma_z^2\Gamma c}{\omega}}$$

Coherent Length

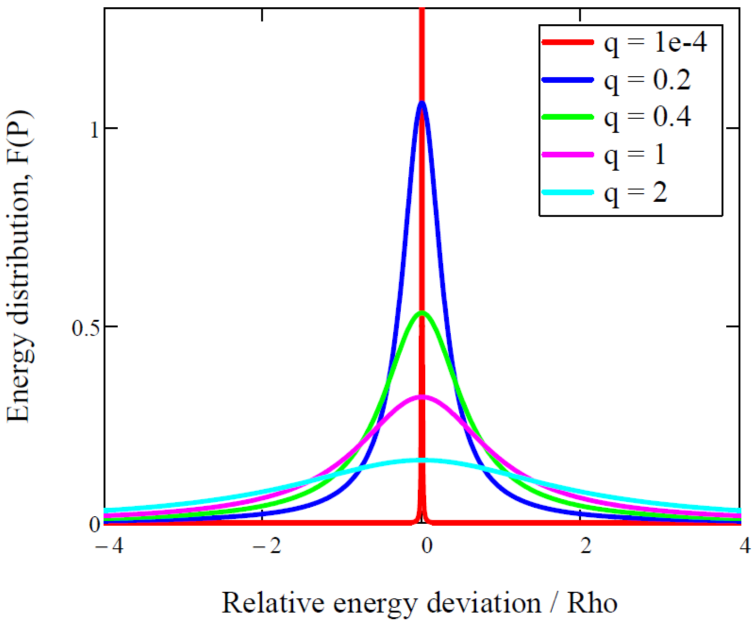


Coherent length is the width of the radiation wave-packet generated by a delta-like excitation.

$$E(\omega) \sim \exp\left[-\frac{\omega^2}{2\sigma_\omega^2}\right] \Rightarrow E(t) \sim \exp\left[-\frac{t^2}{2\sigma_t^2}\right]$$

$$\boxed{\sigma_t = \frac{|a_2|}{k_0 c}\sqrt{\frac{-k_w z}{\rho \operatorname{Re}(a_2)}} = \frac{1}{3k_0 c}\sqrt{\frac{2k_w z}{\rho\sqrt{3}}} = \frac{2}{\sqrt{3}\sigma_\omega}}$$

FEL Gain for warm Beam with Lorentzian Energy Distribution

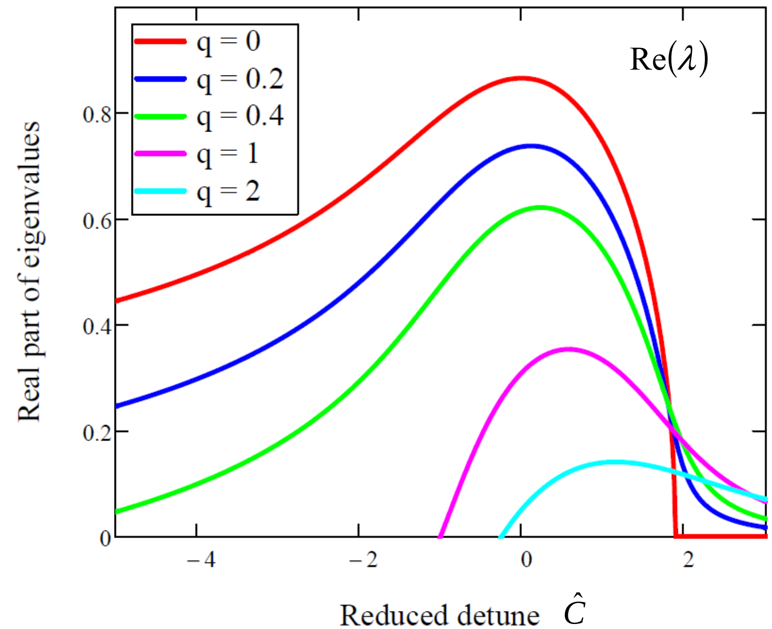


$$F(\hat{P}) = \frac{1}{\pi \hat{q}} \frac{1}{1 + \frac{\hat{P}^2}{\hat{q}^2}}$$

$$\hat{P} = \frac{\mathcal{E} - \mathcal{E}_0}{\mathcal{E}_0 \rho}$$

Pierce Parameter

$$\boxed{\rho = \frac{\gamma_z^2 \Gamma c}{\omega}}$$



If there is no initial modulation in the electron beam:

$$\frac{d^3}{d\hat{z}^3} E(\hat{z}) + 2\left(i\hat{C} + \hat{q}\right) \frac{d^2}{d\hat{z}^2} E(\hat{z}) + \left(i\hat{C} + \hat{q}\right)^2 \frac{d}{d\hat{z}} E(\hat{z}) = iE(\hat{z})$$

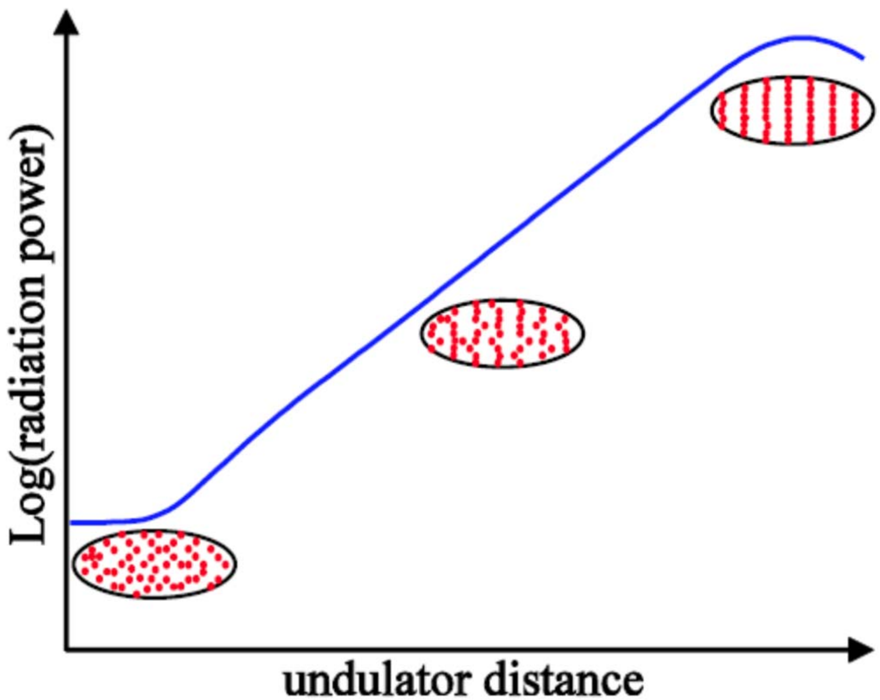
The eigenvalues are determined by : $\lambda\left(\lambda + \hat{q} + i\hat{C}\right)^2 = i$

- FEL gain reduced substantially when the relative energy spread become comparable or larger than the Pierce parameter.

FEL Saturation I

Like any other amplification mechanism, the exponential growth of FEL radiation can not continue forever. One of the criteria to determine the onset of saturation is when there is no electrons to be bunched further, i.e. $\delta n / n_0 \sim 1$, which happens to be the point where nonlinear effects starts to take over.

$$n(\psi) = n_0 + \delta n(\psi)$$



For FEL process starts from shot noise, i.e. SASE, the maximal gain can be derived as

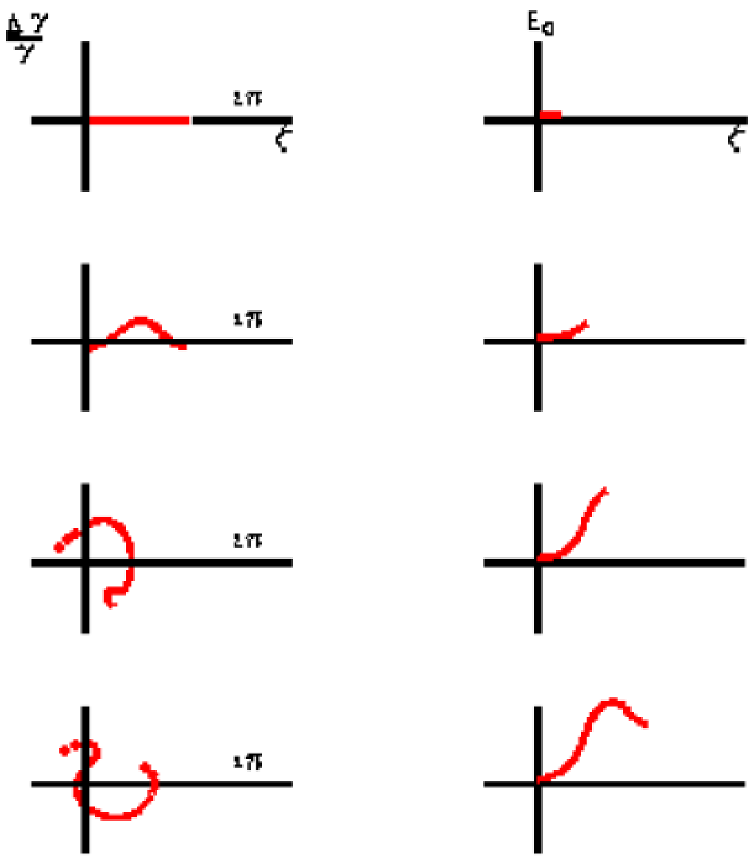
$$\delta n / n_0 \sim 1 \Rightarrow g_{\max} \le \sqrt{\frac{M_e}{N_c}}$$

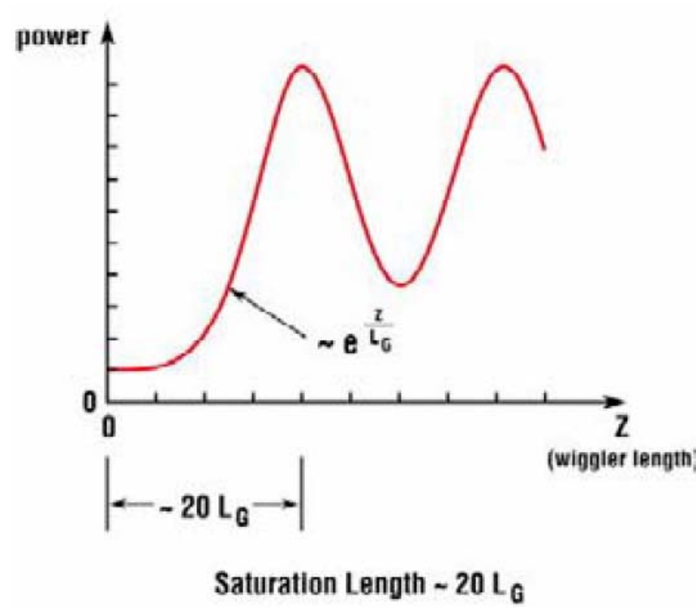
$N_c = L_c / \lambda_{opt}$ is the ratio between coherent length and the radiation wavelength.

M_e is the number of electrons in a radiation wavelength.

FEL Saturation II

There are other criteria which give similar results for the maximal Gain in SASE:





$$\frac{d^2}{d\hat{z}^2}\left(\psi+\frac{\pi}{2}\right)+\hat{u}\sin\left(\psi+\frac{\pi}{2}\right)=0$$

$$\Omega_p=\frac{\sqrt{\hat{u}}}{l_w}=\sqrt{\frac{eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}}\approx\frac{1}{L_G}=\sqrt{3}\Gamma$$

$$\boxed{P_{sat}=\varepsilon_0 cE_{sat}{}^2 A=\chi\cdot\rho\cdot\frac{\mathcal{E}_0}{e}I_e}$$

A: cross section of the beam (and the radiation field)

χ: a numerical factor in the order of one. (homework)

Hence the Pierce parameter is also called efficiency parameter.

FEL Saturation III

- If we use the result that FEL typically saturates at 20 power gain length, the FEL bandwidth at saturation is given by

$$\sigma_{\omega,sat} = 3\omega_0 \sqrt{\frac{2\rho}{\sqrt{3}k_w z_{sat}}} \approx 3\omega_0 \sqrt{\frac{2\rho}{\sqrt{3}k_w 20 L_G}} \qquad L_G \equiv \frac{1}{\sqrt{3}\Gamma} = \frac{\lambda_w}{4\pi\sqrt{3}\rho}$$

FEL bandwidth for radiation amplitude at saturation:

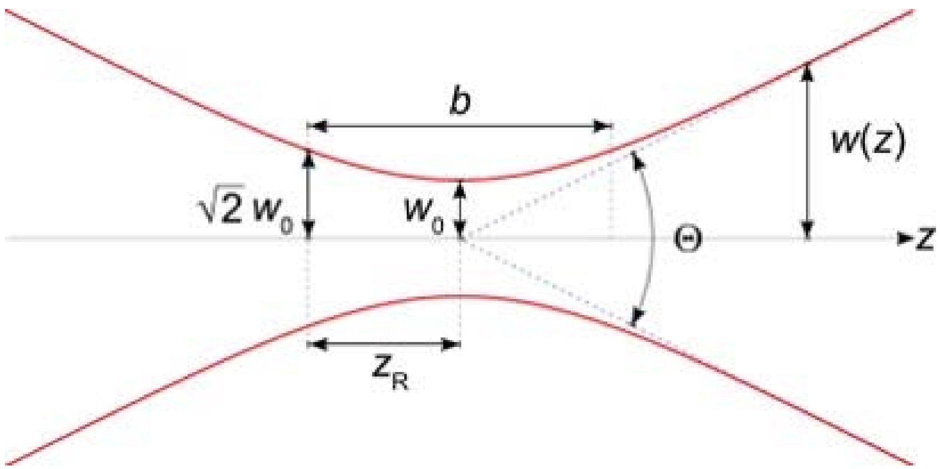
$$\frac{\sigma_{\omega,sat}}{\omega_0} = 3\omega_0 \sqrt{\frac{2\rho}{\sqrt{3}k_w z_{sat}}} \approx \rho\sqrt{1.8}$$

FEL bandwidth for radiation power at saturation:

$$\boxed{\frac{\sigma_{A,sat}}{\omega_0} = \frac{\sigma_{\omega,sat}}{\sqrt{2}\omega_0} = \sqrt{0.9}\rho \approx \rho}$$

Pierce parameter is roughly equal to the bandwidth of the FEL at saturation.

3D Effects: Diffraction



The radius of the radiation at a given distance is given by $w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$

The Rayleigh length or Rayleigh range is the distance along the propagation direction of a beam from the waist to the place where the area of the cross section is doubled.

For a Gaussian radiation beam: $$z_R = \frac{\pi w_0^2}{\lambda_{opt}}$$

The size of the electron beam and the seeding radiation field optics have to be properly chosen so that the interaction efficiency between radiation fields and electrons can be optimized.

Three Dimensional Effects: 3D Gain

- In reality, the gain length will be longer than the 1D gain length due to diffraction, electron emittance, and electron beam energy spread. It is difficult to obtain a general analytical expression for the gain length with all these effects taken into account.
- The analytical approach typically involves expansion over a series of transverse modes.
- For the dominant transverse mode, there is a fitting formula derived by Ming Xie, which is typically of the accuracy of 10% compared with simulation results.

Ming Xie's fitting formula for 3D gain length $\boxed{L_{3D} = L_{1D}\left(1+\Lambda\right)}$

$$\Lambda = 0.45\eta_d^{0.57} + 0.55\eta_\varepsilon^{1.6} + 3\eta_\gamma^2 + 0.35\eta_\varepsilon^{2.9}\eta_\gamma^{2.4} + 51\eta_d^{0.95}\eta_\gamma^3 + 0.62\eta_d^{0.99}\eta_\varepsilon^{1.1}$$
$$+5.3\eta_d^{0.76}\eta_\varepsilon^{2.3}\eta_\gamma^{2.7} + 120\eta_d^{2.1}\eta_\varepsilon^{2.9}\eta_\gamma^{2.8} + 3.7\eta_d^{0.43}\eta_\varepsilon\eta_\gamma$$

Energy spread effects ↓

$$\eta_\gamma = \left(\frac{L_{1D}4\pi}{\lambda_w}\right)\frac{\delta\gamma}{\gamma}$$

Electron emittance effects ↓

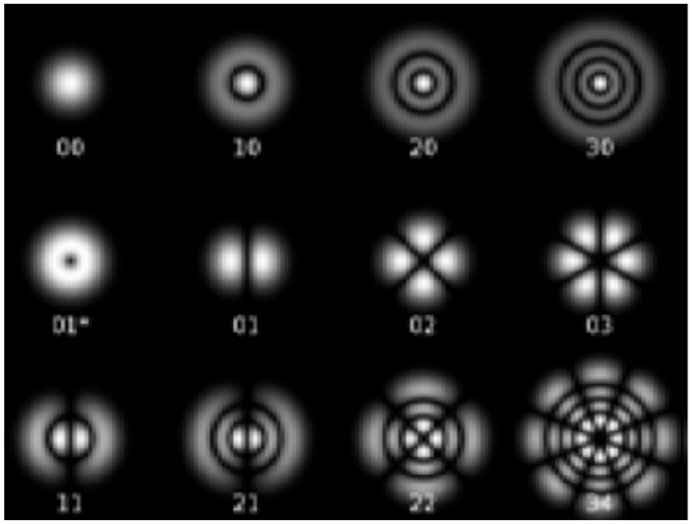
$$\eta_\varepsilon = \left(\frac{L_{1D}4\pi}{\beta_b\gamma\lambda}\right)\varepsilon_n$$

Diffraction effects ↓

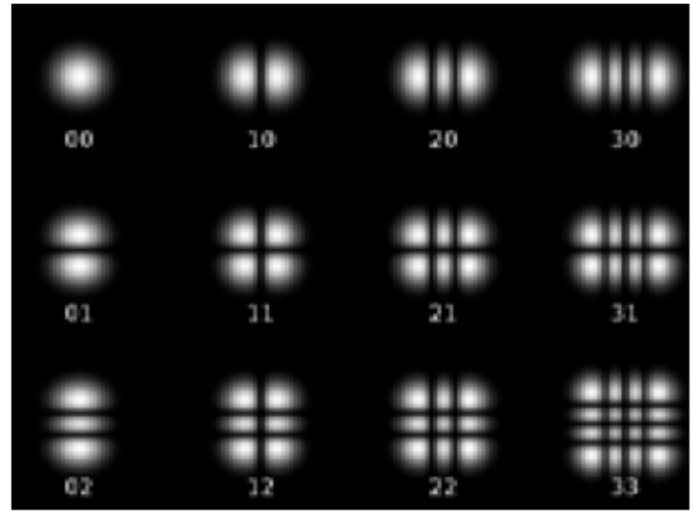
$$\eta_d = \frac{L_{1D}}{Z_R}$$

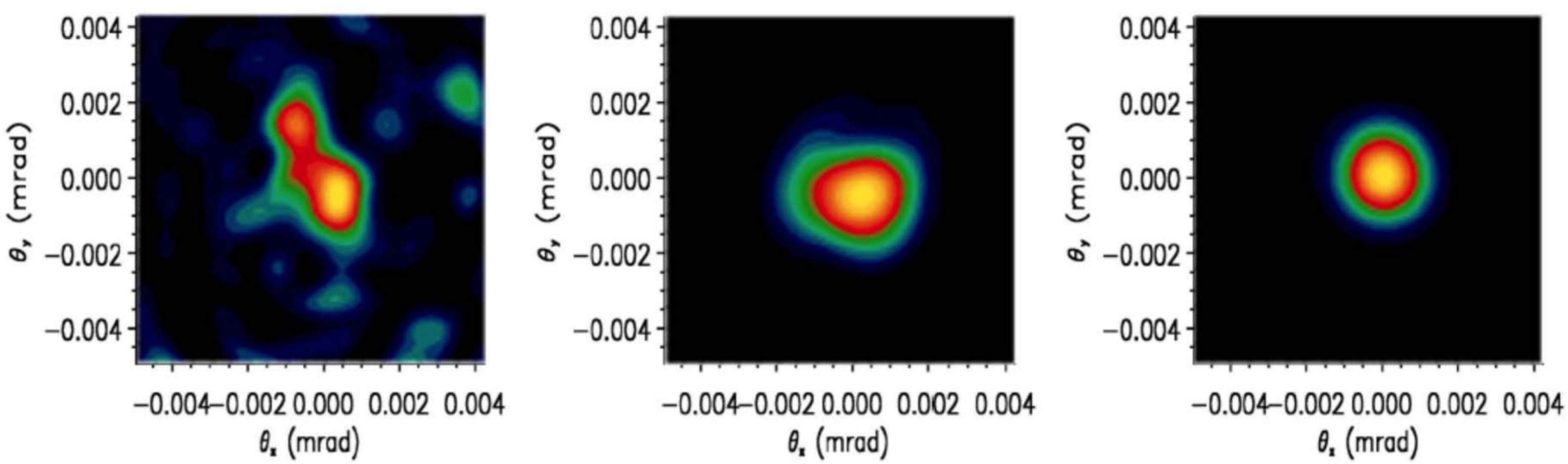
Three-Dimensional Effects: transverse modes

Cylindrical coordinates, Laguerre-Gaussian modes



Cartesian coordinates, Hermite-Gaussian modes





(a) $z = 25$ m (b) $z = 50$ m (c) $z = 75$ m

FIG. 9. (Color) Evolution of the LCLS transverse profiles at different z locations (courtesy of Sven Reiche, UCLA).

Backup Slides

Homeworks

- Show that for $\hat{C} << 1$, the eigenvalue of the growing mode for the 1-D FEL (cold beam) can be approximated as (slides 27)

$$\lambda = a_0 + a_1 \hat{C} + a_2 \hat{C}^2$$

with $a_0 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$, $a_1 = -i\frac{2}{3}$, and $a_2 = -\frac{1}{9}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$.

Homework II

- Assuming the saturation of a FEL takes place at the condition (slides 32)

$$\Omega_{p,sat} = \sqrt{\frac{eE_{sat}\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}} \approx \frac{1}{L_G} = \sqrt{3}\Gamma$$

show that the radiation power at saturation is given by

$$P_{sat} = \varepsilon_0 c E_{sat}{}^2 A = \chi \cdot \rho \cdot \frac{\mathcal{E}_0}{e} I_e$$

and find the numerical coefficient χ.