Properties of the pillbox cavity

From last class, the solution to the z component of the electric field (and also magnetic field) in cylindrical coordinates can be expressed as

$$E_z = f(\rho)\, g(\phi)\, h(z)$$

$$h(z) = A_1 \cos(\beta_z z) + B_1 \sin(\beta_z z)$$

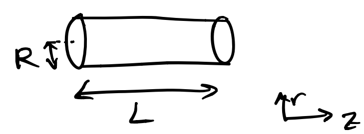
$$g(\phi) = A_2 \cos(m\phi) + B_2 \sin(m\phi)$$

$$f(\rho) = A_3 J_m(\beta_\rho \rho) + B_3 Y_m(\beta_\rho \rho), \qquad \beta_\rho = \beta^2 - \beta_z^{\,2}$$

B_z will have the same form as E_z. The other components of the electric and magnetic field can be derived in TE mode ($E_z = 0$) or TM mode ($B_z = 0$, used in accelerators) through $\bar{\nabla} \times \bar{E}$ & $\bar{\nabla} \times \bar{B}$ eqn.

TM mode in cylindrical cavity

We can use the boundary conditions of the pillbox cavity to simplify the fields inside:



① there is no ϕ dependence in the solution because the geometry of the problem is independant of ϕ, i.e. $m = 0$

$\therefore$ $g(\phi) = A_2 = \text{constant}$

$$f(\rho) = A_3 J_0(\beta_\rho \rho) + B_3 Y_0(\beta_\rho \rho)$$

additionally, because $Y_0 \to \infty$ as $\rho \to 0$, $B_3 = 0$ for any physical solution:

$$f(\rho) = A_3 J_0(\beta_\rho \rho)$$

② Boundary conditions:

for a metal surface, $\begin{cases} E_{\parallel} \text{ at the bondary} = 0 \\ B_{\perp} \text{ at the boundary} = 0 \end{cases}$

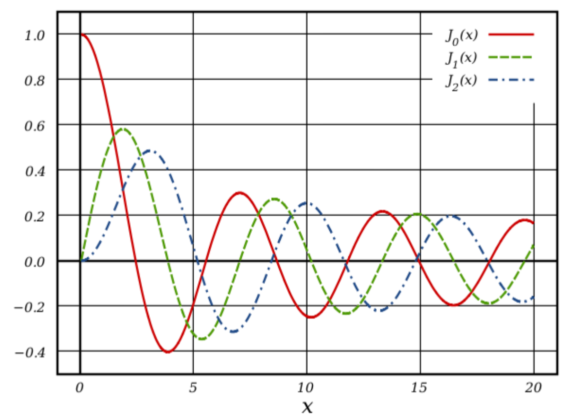
$\therefore$ at $\rho = R$, $E_z = 0$, $E_\phi = 0$, $B_r = 0$

$$\therefore \; f(R) = 0 \Rightarrow J_0(\beta_\rho R) = 0 \Rightarrow \boxed{\beta_\rho = \frac{2.405}{R}} \; \cdots (1)$$

Since $\beta_\rho^2 = \beta^2 - \beta_z^2$ & $\beta^2 = \frac{\omega^2}{c^2}$, if we have a condition for β_z, we can relate the frequency (ω) to the size of the cavity (R). We can use the bonady condition at $z=0$ & $z=h$

at $z=0$, $z=\ell$, $E_\rho = 0$

(3) $\nabla \cdot \bar{E} = 0$ can be used to determine E_ρ



Plot of Bessel function of the first kind, $J_\alpha(x)$, for integer orders $\alpha = 0, 1, 2$

$$\frac{1}{\rho}\frac{\partial(\rho E_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial \cancel{E_\phi}}{\partial \phi} + \frac{\partial E_z}{\partial z} = 0$$

$\partial / \partial \phi = 0$

$$\therefore \; \frac{1}{\rho}\frac{\partial(\rho E_\rho)}{\partial \rho} = -A_4 J_0(\beta_\rho \rho)\left[-A_1 \beta_z \sin(\beta_z z) + B_1 \beta_z \cos(\beta_z z)\right]$$

$$\therefore \; \frac{\partial}{\partial \rho}(\rho E_\rho) = \rho J_0(\beta_\rho \rho) A_4 \left[A_1 \beta_z \sin(\beta_z z) - \beta_z B_1 \cos(\beta_z z)\right]$$

$$\rho E_\rho = \left[\int \rho J_0(\beta_\rho \rho)\right]\left[\qquad \qquad \right]$$

use identity $\frac{d}{dx}(x J_1(x)) = x J_0(x)$

$$\cancel{\rho} E_\rho = \frac{\cancel{\rho}}{\beta_\rho} J_1(\beta_\rho \rho)\left[\qquad \qquad \right]$$

$$E_\rho = \frac{J_1(\beta_\rho \rho)}{\beta_\rho} A_4 \left[A_1 \beta_z \sin(\beta_z z) - \beta_z B_1 \cos(\beta_z z)\right]$$

$$E_\rho\big|_{z=0} = 0 \Rightarrow B_1 = 0$$

$$E_\rho\big|_{z=L} = 0 \Rightarrow \frac{J_1(\beta_\rho \rho)}{\beta_\rho} A_4 A_1 \beta_z \sin(\beta_z L) = 0$$

$$\therefore \beta_z L = n\pi \Rightarrow \boxed{\beta_z = \frac{n\pi}{L}}$$

$$\therefore \boxed{\begin{aligned} E_\rho &= C_\rho J_1(\beta_\rho \rho) \sin\left(\frac{n\pi}{L} z\right) \\ E_z &= C_z J_0(\beta_\rho \rho) \cos\left(\frac{n\pi}{L} z\right) \\ \beta_\rho^2 &= \frac{\omega^2}{c^2} - \left(\frac{n\pi}{L}\right)^2 = \left(\frac{2.405}{R}\right)^2 \end{aligned}}$$

(folding all the constants into C_ρ)

← Comple description of fields inside the pillbox cavity for $m=0$ & the first zero of the Bessel function

What about E_ϕ? For a transverse magnetic wave (TM wave), where $B_z = 0$, it can be shown that $E_\phi = 0$ in a cylindrically symmetric geometry.

<u>Properties of the fundamental mode</u>

In a fundamental mode, n=0.

$$E_\rho = E_\phi = 0, \quad E_z = E_0 J_0\left(\frac{\omega}{c}\rho\right)$$

$$\therefore \text{frequency of cavity: } \frac{\omega^2}{c^2} - \underbrace{\left(\frac{n\pi}{L}\right)^2}_{0} = \left(\frac{2.405}{R}\right)^2$$

$$\therefore \boxed{\frac{\omega}{c} R = 2.405}$$ ← frequency of the fundamental mode

Let $R \sim 30\,\text{cm}$ (some reasonable scale),

$$\frac{30\,\text{cm}}{3\times10^{10}\,\text{cm/s}}(2\pi f) = 2.405 \Rightarrow f \sim 400\,\text{MHz}$$

(independant of the length of the cavity)

Figures of merit for a cavity

There are three important figures of merit for an accelerator that we will explore here:

1. Transit time factor, which is a measure of efficiency of the accelerating particles in a time varying field as compared with a stationary field
2. Quality factor, which is a measure of efficiency of the cavity in storing energy or the fractional energy loss as compared with energy
3. Shunt impedance, which is a measure of energy gained by the particles per energy lost due to ohmic loss.

Transit Time Factor (TTF) and the length of the cavity

Define a time factor T (called the transit time factor,TTF) as the ratio of energy given to a particle that passes the cavity center (peak of the Bessel function) to the energy that would be received if the field were constant with time at its peak value. This figure of merit characterizes the efficiency of accelerating an electron beam in a time varying field as compared to a field of constant amplitude. For cavity length L, this ratio is

$$T = \frac{2E_0 \int_0^{L/2} \cos(\omega z/v)\, dz}{E_0 L} = \frac{\sin u}{u}, \quad u = \frac{\omega L}{2v}$$

if $T = 0.9$, $v \simeq c$ & $u = 0.8$ & $L/R = 2/3$

(we have implicitly assumed that velocity change through one accelerator section is small, which is a usually true)

When we start talking about longitudinal dynamics of the accelerator, we will ignore the transit time across a cavity, and consider the energy gain across the cavity to be given by

$$W = qV \sin(\omega_{rf} t)$$

Where *t* will indicate the time the particle reaches the accelerating section and the TTF is folded into the amplitude V.

<u>Figure of merit: Quality factor</u>
The quality factor, Q, is a measure of the efficiency with which a resonator stores energy. For superconducting cavity, it is especially important because the dissipated power is exhausted into the liquid helium!

Q is defined as the ratio of the stored energy to the energy lost in one radian of (time) oscillation. In other words, it's a measure of how fast you need to replenish the energy inside the resonator.

How to calculate the energy inside the cavity?
When the electric field is maximum, the magnetic field is zero (because of the Ampere equation.) So we can calculate the stored energy from electric field alone.

Volume

$$\text{Stored energy} = U = \frac{1}{2}\epsilon_0 \int E^2 dV$$

$$= \frac{1}{2}\epsilon_0 E_0^2 V J_1^2(2.405)$$

using $\int_0^1 J_0^2(\alpha x)\, x\, dx = \frac{1}{2}[J_0'(\alpha)]$

& $J_0'(\alpha) = -J_1$

$J_1(2.405) = 0.52 \Rightarrow$ Energy stored in cavity is a little over $\frac{1}{4}$ of the energy that would be associated w/ field E_0 filling the entire volume (e.g. capacitor)

So, we were looking at Q calculation, which was stored energy divided by the loss in one radian divided. So what is the source of loss? It is ohmic resistance at the walls of the cavity.

Ohmic loss rate per unit area averaged over one cycle is

$$\frac{1}{2}\rho_s J^2$$

ρ_s: surface resistivity $= \frac{\rho}{\delta} = \frac{1}{\sigma\delta}$ (σ: conductivity, δ: skin depth)

J: current density

$$\therefore P = \int_A \frac{1}{2}\rho_s J^2 \, dA$$

Find J

$$\bar{\nabla}\times\bar{B} = \mu_0 \bar{J} + \mu_0\epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

Find $\bar{B}$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t} = -j\omega\bar{B}$$

$$\bar{E} = E_z(\rho)e^{j\omega t}$$

$$= E_0 J_0\left(\frac{\omega}{c}\rho\right)e^{j\omega t}\hat{z}$$

$\Rightarrow$

$$B_\rho = -\frac{1}{j\omega}\left(\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right) = 0$$

$$B_z = -\frac{1}{j\omega}\cdot\frac{1}{\rho}\left(\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi}\right) = 0$$

$$B_\phi = -\frac{1}{j\omega}\left(\underbrace{\frac{\partial E_\rho}{\partial z}}_{0} - \frac{\partial E_z}{\partial \rho}\right)$$

$$= -\frac{j}{\omega}E_0\frac{d}{d\rho}\left\{J_0\left(\frac{\omega}{c}\rho\right)\right\}e^{j\omega t}$$

$$= -\frac{j}{\omega}E_0\left\{-J_1\left(\frac{\omega}{c}\rho\right)\cdot\frac{\omega}{c}\right\}e^{j\omega t}$$

$$\boxed{B_\phi = \frac{E_0}{c}J_1\left(\frac{\omega}{c}\rho\right)e^{j(\omega t + \pi/2)}}$$

$\therefore$ Through the integration of J^2 over the surface, it can be shown that

$$P = \frac{1}{2} \rho_s \frac{E_0^2}{Z_0^2} 2\pi R L \left(1 + \frac{R}{L}\right) J^2(2.405)$$

$Z_0^2 = \frac{\mu_0}{\epsilon_0}$ = impedence of free space

$$Q = \frac{\overset{\text{energy stored}}{U}}{\underset{\text{energy dissipated per radian of oscillation}}{P/\omega}} = \frac{2.405\, \mu_0 c}{2 \rho_s \left(1 + (R/L)\right)}$$

For copper, in the 400 MHz region, this expression gives Q of the order of 1e4 for the geometry that we are using.
For reference, a typical super conducting cavity has a Q~5e9
For a refresher on the relation between Q, the bandwidth, and the damping of a harmonic oscillator, see the Appendix 1 on the course website.

Figure of merit: shunt impedance
Shunt impedance is a measure of how much energy is given to the accelerating particle compared to the energy lost through the surface current on the surface of the resonator. It is given by

$$R_s = \frac{(\text{energy gain per unit charge})^2}{P}$$

$$T = \frac{2E_0 \int_0^{L/2} \cos(\omega z / v)\, dz}{E_0 (L/2)}$$

$\Rightarrow$ energy gain per unit charge through the cavity $= T E_0 L$

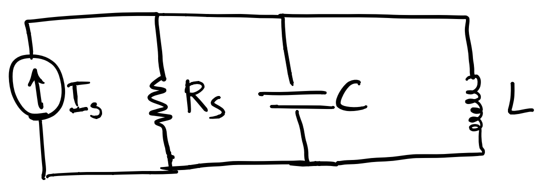
$$R_S = \frac{T^2 \cancel{E_0^2} L^{\cancel{2}}}{\frac{1}{2}\rho_s \frac{\cancel{E_0^2}}{Z_0^2} 2\pi R \cancel{L} \left(1+\frac{R}{L}\right) J_1^2(2.405)}$$

$$\boxed{R_S = \frac{Z_0^2}{\pi \rho_S} \frac{L}{R} \frac{T^2}{(1+R/L)\, J_1^2(2.405)}}$$

For numbers we have been using, $R_S \sim 7M\Omega$

The term shunt impedance comes from lumped circuit model, where a resonant circuit with a capacitor and an inductor are used to model the conversion of electric and magnetic field to each other, and the shunt impedance is used to model energy loss.

I_S is the current source with ∞ internal impedence. It feeds energy into LC resonant circuit



Resistor Rs shunts the circuit w/ Loss $\frac{V^2}{R}$

When the generator is turned ohm the voltage in the circuit builds up to an asymptotic limit (how fast it builds up depends on Q). At this limit, the energy supplied to the circuit equals the dissipation in the resistor.

$$\text{shunt impedence} = \frac{V_{peak}^2}{2P_{rms}} = R_S$$

$$U = \frac{1}{2}CV^2 = \frac{1}{2}LI^2$$ (Larger than the energy delivered in a single RF cycle)

Reactance of L and C :

$$X_L = \omega L, \qquad X_C = \frac{1}{\omega C}$$

at resonance, $X_L = X_C$

$$X_L = X_C \Rightarrow \omega L = \frac{1}{\omega C} \Rightarrow \omega = \frac{1}{\sqrt{LC}}$$ resonant frequency

$$X_L = \omega L = \frac{L}{\sqrt{LC}} = \sqrt{\frac{L}{C}} = X_C$$

$$Q = \frac{\omega U}{P}$$

also comes from averaging over a cycle

$$= \omega \cdot \frac{1}{2} C \cancel{V^2} \cdot \frac{2 R_s}{\cancel{V^2}}$$

$$= \frac{R_s}{1/\omega C}$$

$$\boxed{Q = \frac{R_s}{X_C} = \frac{R_s}{X_L}}$$

Note that $\frac{R_s}{Q} = X_C = X_L$ results in a quantity that only depends on "geometry"; i.e. L & C, which are determined by physical properties. By the same token, R_s/Q for the resonator cavity results in a quantity that is independant of resistivity

μ_0/ε_0

$$R_s/Q = \frac{Z_0^2}{\pi \cancel{R_s}} \frac{L}{R} \frac{T^2}{\cancel{(1+(R/L))} J_1^2(2.405)} \times \frac{2\cancel{R_s}\,\cancel{(1+(R/L))}}{2.405\, \underline{\underline{\mu_0 C}}}$$

$$\frac{\mu_0}{\epsilon_0 \mu_0 c} = \frac{\mu_0 c^2}{c} = \mu_0 c$$

$$\boxed{\frac{R_s}{Q} = \frac{2}{\pi} \frac{L}{R} \frac{\mu_0 c\, T^2}{2.405\, J_1^2(2.405)}}$$

Purely geometric factor (material independant)

Note that $\frac{R_s}{Q} = \frac{(\text{energy gain per charge})^2}{\omega \cdot \text{stored energy}}$ is a very useful tool in describing the accelerating efficiency of a cavity.

Another material independant Quantity is $R_s \cdot Q$

$$R_s \cdot Q = \frac{\omega U}{P/R_s} = \frac{\omega_0 \mu_0 L R^2}{2(R^2 + RL)} = Z_0 \frac{2.405\, L}{2(R+L)} = \frac{453\, L/R}{1 + L/R}\,\Omega$$

Choose length such that particle time to transit cavity correspont to the time it takes for field to switch

$$\text{TTF} \rightarrow \sin\left(\frac{\omega_0 L}{2v}\right)$$

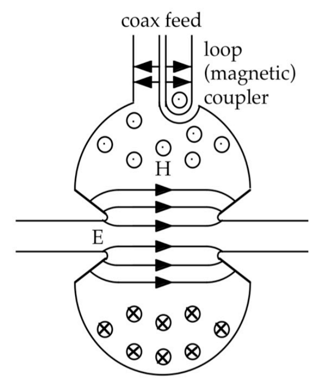
$$\rightarrow \left.\begin{array}{l} \dfrac{\omega_0 L}{2\beta c} = \dfrac{\pi}{2} \\[2ex] \dfrac{\omega_0 R}{c} = 2.405 \end{array}\right\} \Rightarrow \frac{2.405\, \cancel{c}\, L}{2\beta \cancel{c}\, R} = \pi \Rightarrow \frac{L}{R} = \frac{\beta \pi}{2.405}$$

$\therefore\ G = 257\beta\ [\Omega]$ for all optimized pillbox cavity resonators!

Note that β scaling means that the resonator is more efficient at higher velocities.

The pillbox is easy to analyze and could in fact be used in the cases where variable frequency is not necessary. However, because of the cost of RF power, a lot of work has gone into optimizing the shape:

- Nose: energy gain could be increased by the nose
- The losses occur predominantly on the cylindrical surfaces and so higher Q can be achieved by spreading the current out on a sphere like surface



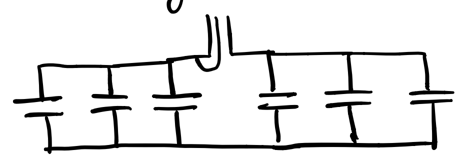
"nose-cone" copper cavity

(Recall : $P \sim \int_A \rho_s J^2/2 \, da$)

Accelerating Structures

Multiple accelerating cavities are required in an actual accelerator. We want to power multiple oscillators with a single source to save RF power. This means that the frequency "degeneracy" is split, and there will be five fundamental frequencies different in phase relationship of the fields from cell to cell. We will explore this topic later as part of the discussion on coupled oscillators

Cells are operated in π mode, i.e. at a given instant, electric fields are in opposite direction at adjacent cells



An electron linac is a different matter because an electron with an energy of 3MeV is already traveling at 99% speed of light. In this case, you can think of a modified uniform waveguide for accelerating structure. The modification involves the insertion of irises to slow down the phase velocity, which for a hollow cylinder is greater than c. The radar S-band near 3 GHz was the choice for SLAC.