

Complex Analysis Refresher

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Definition

$$i = \sqrt{-1}$$

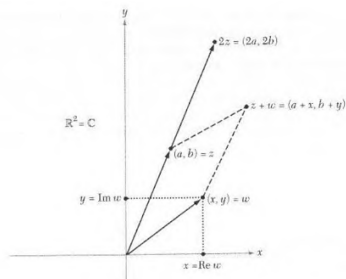
$$w = x + iy = (x, y), w \in \mathbb{C}, (x, y) \in \mathbb{R}^2$$

$$x = \mathbf{Re} w, y = \mathbf{Im} w$$

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

$$k(a + ib) = ka + ikb$$

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc)$$

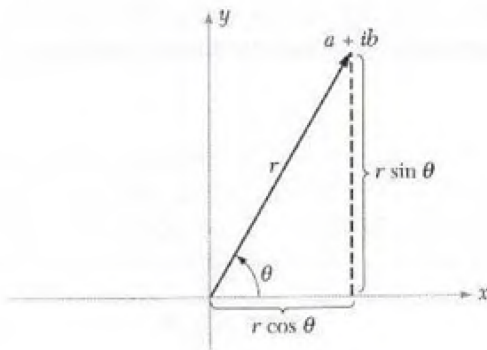


Polar Representation of Complex Numbers

$$z = a + ib = r \cos \theta + i(r \sin \theta) = r(\cos \theta + i \sin \theta)$$

$|z| = r = \sqrt{a^2 + b^2}$ is called norm, or modulus, or absolute value of z .

$\theta = \arg z$ is called the argument of z .



Multiplication of Complex Numbers in Polar Representation

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

$$|z_1 z_2| = |z_1| \cdot |z_2|, \arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}$$

$$z^n = r^n(\cos n\theta + i \sin n\theta), n \text{ is a positive integer}$$

$$w^n = z = r(\cos \theta + i \sin \theta),$$

$$w = \sqrt[n]{r} \left[\cos \left(\frac{\theta}{n} + \frac{k}{n} 2\pi \right) + i \sin \left(\frac{\theta}{n} + \frac{k}{n} 2\pi \right) \right]$$

$$k = 0, 1, 2, \dots, n-1$$

Complex Conjugation

$$\begin{aligned}z &= a + ib, \bar{z} = a - ib \\ \overline{z + z'} &= \bar{z} + \bar{z}' \\ \overline{zz'} &= \bar{z}\bar{z}' \\ \overline{z/z'} &= \bar{z}/\bar{z}' \text{ for } z' \neq 0 \\ z\bar{z} &= |z|^2 \\ z &= \bar{z} \text{ if and only if } z \text{ is real} \\ \operatorname{Re} z &= (z + \bar{z})/2, \operatorname{Im} z = (z - \bar{z})/2i \\ \overline{\bar{z}} &= z\end{aligned}$$

Some Elementary Functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{iy} = 1 + \frac{(iy)}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \dots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots \right) + i \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots \right)$$

$$= \cos y + i \sin y$$

Some Elementary Functions

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

$$e^{z+w} = e^z e^w$$

e^z is never zero

$$|e^{x+iy}| = e^x$$

$$e^{i\pi/2} = i, e^{i\pi} = -1, e^{i3\pi/2} = -i, e^{i2\pi} = 1$$

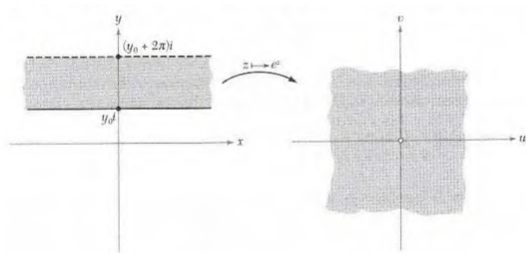
e^z is periodic, each period has the form $i2\pi n$, for some integer n

$e^z = 1$ if and only if $z = i2\pi n$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Logarithm Function

$A_{y_0} = \{x + iy \mid x \in \mathbb{R}, y_0 \leq y < y_0 + 2\pi\}$, e^z maps A_{y_0} in a one-to-one manner onto the set $\mathbb{C} \setminus \{0\}$



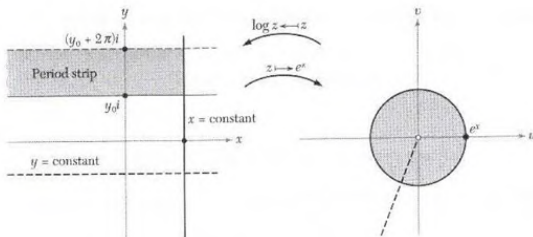
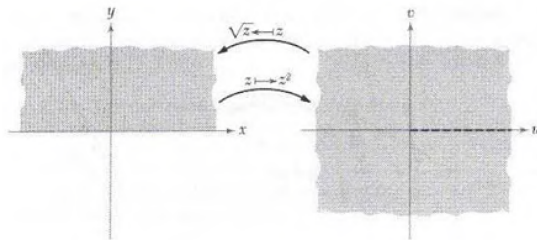
The function $\log : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$, with range $y_0 \leq \text{Im } \log z < y_0 + 2\pi$, is defined by

$$\log z = \log |z| + i \arg z$$

where $\arg z$ takes values in the interval $[y_0, y_0 + 2\pi)$.

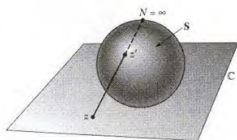
$$\sqrt[n]{z} = z^{1/n} = e^{(\log z)/n}$$

Geometry of some functions



Functions

- Open sets, closed sets, connected sets, compact sets, point at infinity, Riemann sphere \mathbf{S} , $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$



- Functions on $\mathbb{C} \rightarrow \mathbb{C}$ can be written as $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, where $u(x, y) = \mathbf{Re} f(z)$ and $v(x, y) = \mathbf{Im} f(z)$.
- Limit, continuity, uniform continuity
- Extreme value theorem: Continuous function on a compact set attains finite maximum and minimum values.

Analytic Functions

Function f is said to be differentiable (in the complex sense) at $z_0 \in A (A \subset \mathbb{C})$ if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. f is said to be analytic (or holomorphic) on A if f is complex differentiable at each $z_0 \in A$.

$$(af + bg)'(z) = af'(z) + bg'(z)$$

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{[g(z)]^2}$$

$$\frac{d}{dz}(g \circ f)(z) = g'(f(z)) \cdot f'(z) \text{ (Chain Rule)}$$

Cauchy-Riemann Theorem: $f'(z_0)$ exists if and only if f is real differentiable

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Inverse Function Theorem:

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)} \text{ where } w = f(z)$$

Harmonic:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

If $f = u + iv$ is analytic, then u and v are harmonic, u and v are harmonic conjugates. For example,

$$u(x, y) = x^2 - y^2, v(x, y) = 2xy$$

Differentiation of Elementary Functions

$$\frac{de^z}{dz} = e^z$$

$$\frac{d}{dz} \log z = \frac{1}{z}$$

$$\frac{d}{dz} \sin z = \cos z$$

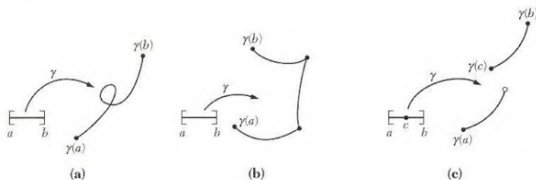
$$\frac{d}{dz} \cos z = -\sin z$$

$$\frac{d}{dz} a^z = (\log a)a^z$$

$$\frac{d}{dz} z^b = bz^{b-1}$$

Contour Integrals

A curve or contour $\gamma : [a, b] \rightarrow \mathbb{C}$, smooth, piecewise C^1 , discontinuous.



$\int_{\gamma} f = \int_{\gamma} f(z) dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t)) \gamma'(t) dt$ is called the integral of f along γ .

$$f(z) = u(x, y) + iv(x, y)$$

$$f(\gamma(t))\gamma'(t) = [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)] \\ + i[v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]$$

$$f(z)dz = (u + iv)(dx + idy) = udx - vdy + i(vdx + udy)$$

$$\left| \int_{\gamma} f \right| \leq \int_{\gamma} |f| |dz|$$

Fundamental Theorem of Calculus

$\gamma : [0, 1] \rightarrow \mathbb{C}$ is a piecewise smooth curve.

F is a function defined and analytic on an open set G containing γ .

$$\int_{\gamma} F'(z) dz = F(\gamma(1)) - F(\gamma(0))$$

$$\int_{\gamma} F'(z) dz = 0 \text{ if } \gamma(0) = \gamma(1)$$

Integrals are path-independent.

Integrals around closed curves are 0.

An example: let γ be the circle of radius r around $a \in \mathbb{C}$, n is an integer,

$$\int_{\gamma} (z - a)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

Cauchy's Theorem

γ is a simple closed curve (γ intersects itself only at its endpoint). f is analytic on and inside γ

$$\int_{\gamma} f = 0$$

Green's Theorem

$$\int_{\gamma} P(x, y)dx + Q(x, y)dy = \int \int_A \left[\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y) \right] dx dy$$

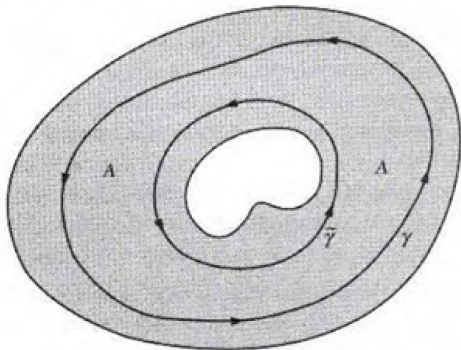
Suppose f is analytic on and inside a simple closed curve γ

$$\begin{aligned} \int_{\gamma} f &= \int_{\gamma} f(z)dz = \int_{\gamma} (u + iv)(dx + idy) \\ &= \int_{\gamma} (udx - vdy) + i \int_{\gamma} (udy + vdx) \\ &= \int \int_A \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] dx dy + i \int \int_A \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] dx dy \\ &= 0 \text{ (Cauchy-Riemann equations)} \end{aligned}$$

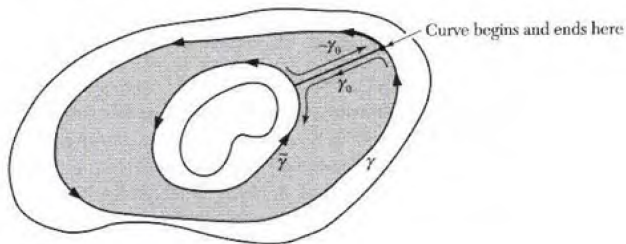
Deformation Theorem

f is analytic on a region A , γ is a simple closed curve in A . If γ can be continuously deformed to another simple closed curve $\tilde{\gamma}$ without passing outside the region A (γ is homotopic to $\tilde{\gamma}$ in A), then

$$\int_{\gamma} f = \int_{\tilde{\gamma}} f$$



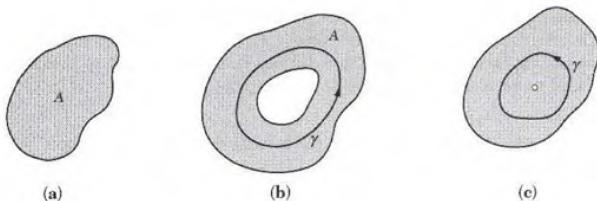
Deformation Theorem



$$0 = \int_{\gamma + \gamma_0 - \tilde{\gamma} - \gamma_0} f = \int_{\gamma} f + \int_{\gamma_0} f - \int_{\tilde{\gamma}} f - \int_{\gamma_0} f = \int_{\gamma} f - \int_{\tilde{\gamma}} f$$

Simply Connected Regions

A is called simply connected if A is connected and every closed curve γ in A can be deformed in A to some constant curve $\tilde{\gamma} = z_0$.

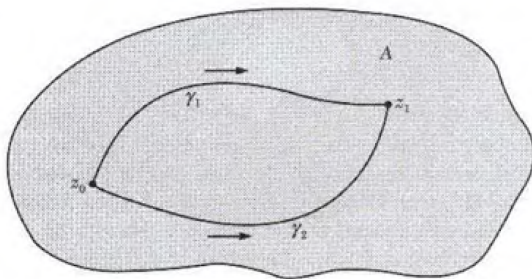


$\int_{\gamma} f = 0$, if f is analytic on a simply connected region G and γ is a closed curve in G .

Independence of Path

f is analytic on simply connected region A , γ_1 and γ_2 are two curves joining two points z_0 and z_1 in A ,

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$



Antiderivative Theorem

f is analytic on simply connected region A . Then there is an analytic function F defined on A that is unique up to an additive constant, such that $F'(z) = f(z)$, F is the antiderivative of f on A .

$$\begin{aligned}F_0(z) &= F(z) + C \\(F_0 - F)'(z) &= F_0'(z) - F'(z) = f(z) - f(z) = 0\end{aligned}$$

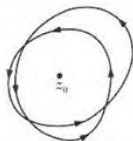
A is a simply connected region and $0 \notin A$. Then there is an analytic function $F(z)$, unique up to the addition of multiples of $2\pi i$, such that $e^{F(z)} = z$. We write $F(z) = \log z$ and call F a branch of the logarithm function.

$$\begin{aligned}e^{F(z)} &= z, e^{G(z)} = z \\e^{F(z)-G(z)} &= 1 \\F(z_0) - G(z_0) &= 2\pi ni \text{ for some integer } n \text{ at fixed } z_0\end{aligned}$$

Index of a Closed Path



Index = -1



Index = 2



Index = 1

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}$$

$I(\gamma; z_0)$ is the index of γ with respect to z_0 . We say that γ winds around z_0 , $I(\gamma; z_0)$ times.

$\gamma(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi n$ has index n with respect to z_0 ,

$-\gamma(t) = z_0 + re^{-it}$, $0 \leq t \leq 2\pi n$ has index $-n$ with respect to z_0 .

γ and $\tilde{\gamma}$ are homotopic in $\mathbb{C} \setminus \{z_0\}$, z_0 does not lie on either γ or $\tilde{\gamma}$, then

$$I(\gamma; z_0) = I(\tilde{\gamma}; z_0)$$

Cauchy's Integral Formula

$$f(z_0) \cdot I(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz$$

When γ is a simple closed curve and $I(\gamma; z_0) = 1$, then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz$$

Example: $\int_{\gamma} \frac{e^z}{z} dz = 2\pi i \cdot e^0 = 2\pi i$ by choosing $f(z) = e^z$ and $z_0 = 0$.

Cauchy's Integral Formula for Derivatives:

$$f^{(k)}(z_0) \cdot I(\gamma; z_0) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{k+1}} dz, k = 1, 2, 3, \dots$$

Cauchy's Inequalities and Liouville's Theorem

Cauchy's Inequalities: f is analytic on a region A , γ is a circle with radius R and center z_0 . Suppose $|f(z)| \leq M$ for all z on γ , then for $k = 0, 1, 2, \dots$

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$$

Liouville's Theorem: f is entire and $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is constant.

Cauchy Inequalities with $k = 1$, $|f'(z_0)| \leq M/R$. Hold z_0 and let $R \rightarrow \infty$, we have $|f'(z_0)| = 0$ and therefore $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$.

Mean Value Property

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

$$\gamma(\theta) = z_0 + re^{i\theta}, 0 \leq \theta \leq 2\pi$$

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \end{aligned}$$

Poisson's Formula

Mean Value Property for Harmonic Functions:

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

Poisson's Formula real form:

$$u(\rho e^{i\phi}) = \frac{r^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{u(re^{i\theta})}{r^2 - 2r\rho \cos(\phi - \theta) + \rho^2} d\theta$$

Poisson's Formula complex form:

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta$$

Series Representation

Taylor's series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) (\zeta - z_0)^{n+1} d\zeta, \quad n = 1, 2, \dots$$

Isolated Singularities

$$f(z) = \dots + \frac{b_n}{(z - z_0)^n} + \dots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

If f is analytic on $\{z \mid 0 < |z - z_0| < r\}$, z_0 is an isolated singularity of f

- A finite number of b_n are zero, z_0 is called a pole of f . k is the highest integer such that $b_k \neq 0$, z_0 is a pole of order k . A first-order pole is a simple pole.
- An infinite number of b_k are nonzero, z_0 is an essential singularity.
- b_1 is the residue of f at z_0 .
- All the b_k are zero, z_0 is a removable singularity.

$$\int_{\gamma} f(z) dz = b_1 \cdot 2\pi i$$

Isolated Singularities

z_0 is a removable singularity if and only if one of the following conditions holds:

- f is bounded in a deleted neighborhood of z_0
- $\lim_{z \rightarrow z_0} f(z)$ exists
- $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$

z_0 is a simple hole if and only if $\lim_{z \rightarrow z_0} (z - z_0)f(z)$ exists and is unequal to zero. This limit equals the residue of f at z_0 .

f and g have zeros of order n and k respectively at z_0 . $h(z) = f(z)/g(z)$:

- if $k > n$, h has a pole of order $k - n$ at z_0 .
- if $k = n$, h has a removable singularity with nonzero limit at z_0 .
- if $k < n$, h has a zero of order $n - k$ at z_0 .

$$f(z) = \dots + \frac{b_2}{(z - z_0)^2} + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

$$b_1 = \operatorname{Res}(f; z_0)$$

$f(z) = g(z)/h(z)$: If g and h have zeros at z_0 of the same order, f has a removable singularity at z_0 , $\operatorname{Res}(f; z_0) = 0$.

If $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$, f has a simple pole at z_0 ,

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}.$$

If g has a zero of order k at z_0 and h has a zero of order $k + 1$, then f has a simple pole, $\operatorname{Res}(f; z_0) = (k + 1) \frac{g^{(k)}(z_0)}{h^{k+1}(z_0)}$.

If $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) = 0$, $h''(z_0) \neq 0$, f has a second-order pole at z_0 , $\operatorname{Res}(f; z_0) = 2 \frac{g'(z_0)}{h''(z_0)} - \frac{2}{3} \frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$.

High-Order Poles

k is the smallest integer ≥ 0 such that $\lim_{z \rightarrow z_0} (z - z_0)^k f(z)$ exists. Then f has a pole of order k . Let $\phi(z) = (z - z_0)^k f(z)$

$$\operatorname{Res}(f; z_0) = \frac{\phi^{(k-1)}(z_0)}{(k-1)!}$$

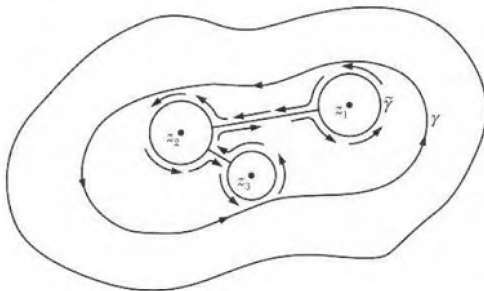
$g(z_0) \neq 0, h(z_0) = 0 = \dots = h^{(k-1)}(z_0), h^{(k)}(z_0) \neq 0$, then g/h has a pole of order k

$$\operatorname{Res}(g/h; z_0) = \left[\frac{k!}{h^{(k)}(z_0)} \right]^k \times$$

$\frac{h^{(k)}(z_0)}{h^{(k+1)}(z_0)}$	0	0	...	0	$g(z_0)$
$\frac{k!}{(k+1)!}$	$\frac{h^{(k)}(z_0)}{h^{(k+1)}(z_0)}$	0	...	0	$g^{(1)}(z_0)$
$\frac{h^{(k+2)}(z_0)}{(k+2)!}$	$\frac{h^{(k+1)}(z_0)}{(k+1)!}$	$\frac{h^{(k)}(z_0)}{k!}$...	0	$\frac{g^{(2)}(z_0)}{2!}$
\vdots	\vdots	\vdots			\vdots
$\frac{h^{(2k-1)}(z_0)}{(2k-1)!}$	$\frac{h^{(2k-2)}(z_0)}{(2k-2)!}$	$\frac{h^{(2k-3)}(z_0)}{(2k-3)!}$...	$\frac{h^{(k+1)}(z_0)}{(k+1)!}$	$\frac{g^{(k-1)}(z_0)}{(k-1)!}$

Residue Theorem

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{i=1}^n [\text{Res}(f; z_i)] I(\gamma; z_i)$$



$$F(z) = f(1/z)$$

f has a pole of order k at ∞ if F has a pole of order k at 0 ;

f has a zero of order k at ∞ if F has a zero of order k at 0 .

Define $\text{Res}(f; \infty) = -\text{Res}((1/z^2)F(z); 0)$.

$$\int_{\gamma} f = -2\pi i \sum \{\text{residues of } f \text{ outside } \gamma \text{ including at } \infty\}$$

Evaluation of Definite Integrals

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i}$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2}$$

$$dz = ie^{i\theta} d\theta = izd\theta$$

$$\int_0^{2\pi} R(\cos \theta, \sin \theta) = 2\pi i \sum \{\text{residues of } f(z) \text{ inside the uni circle}\}$$

$$f(z) = \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)$$

Integrals on the Whole Real Line

$$\int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{H}\}$$

$$\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im}(z) \geq 0\}$$

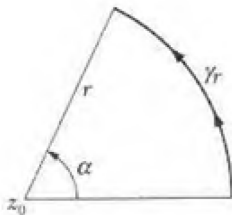
$$\int_{-\infty}^{\infty} f(x) dx = -2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{L}\}$$

$$\mathcal{L} = \{z \in \mathbb{C} \mid \text{Im}(z) \leq 0\}$$

Cauchy Principal Value

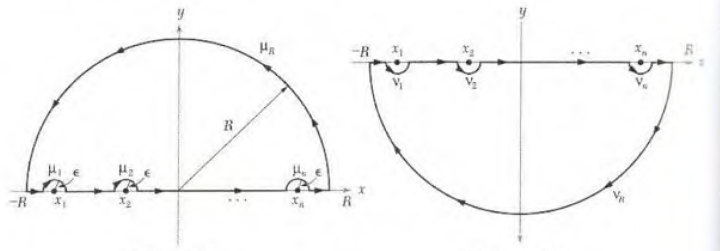
The regular integral $\int_{-\infty}^{\infty} f(x)dx = \lim_{A,B \rightarrow \infty} \int_{-A}^B f(x)dx$ may not be convergent.

The Cauchy principal value may exist, with $A, B \rightarrow \infty$ using the symmetric way: $P.V. \int_{-\infty}^{\infty} f(x)dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x)dx$



$$\lim_{r \rightarrow 0} \int_{\gamma_r} f = \alpha i \text{Res}(f; z_0)$$

Singularities on Real Axis



$$(H)_0 = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$$

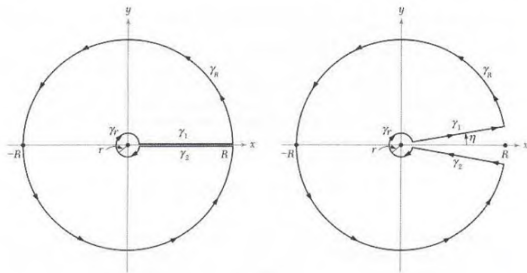
$$P.V. \int_{-\infty}^{\infty} f(z) dz = 2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{H}_0\} + \pi i \sum_{j=1}^n \text{Res}(f; x_j)$$

$$(L)_0 = \{z \in \mathbb{C} \mid \text{Im}(z) < 0\}$$

$$P.V. \int_{-\infty}^{\infty} f(z) dz = -2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{L}_0\} - \pi i \sum_{j=1}^n \text{Res}(f; x_j)$$

An example

Evaluate $\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}$



$$g(z) = \sqrt[3]{z}/(1+z^2), 0 < r < 1, R > 1, r \rightarrow 0, R \rightarrow \infty$$

$$\Sigma = \text{Res}(g; i) + \text{Res}(g; -i) = \frac{\sqrt[3]{i}}{2i} + \frac{\sqrt[3]{-i}}{-2i} = -e^{\pi i/3}/2$$

$$2\pi i \Sigma = \int_{\gamma_1} g + \int_{\gamma_R} g + \int_{\gamma_2} g + \int_{\gamma_r} g$$

An example

$$\left| \int_{\gamma_R} g \right| \leq \frac{R^{1/3} 2\pi R}{R^2 - 1} \rightarrow 0$$

$$\left| \int_{\gamma_r} g \right| \leq \frac{r^{1/3} 2\pi r}{1 - r^2} \rightarrow 0$$

$$\int_{\gamma_1} g = \int_r^R \frac{x^{1/3}}{1+x^2} dx \rightarrow \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx$$

$$\int_{\gamma_2} g = \int_R^r \frac{x^{1/3} e^{2\pi i/3}}{1+x^2 e^{4\pi i}} e^{2\pi i} dx \rightarrow -e^{2\pi i/3} \int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx$$

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2} dx = \frac{\pi}{\sqrt{3}}$$

Laplace Transforms

$f : [0, \infty) \rightarrow \mathbb{C}$ or \mathbb{R} is of exponential order: $|f(t)| \leq Ae^{tB}$ for all $t \geq 0$ with constants $A > 0, B \in \mathbb{R}$.

$$\begin{aligned}\tilde{f}(z) &= \int_0^{\infty} e^{-zt} f(t) dt \\ \frac{d}{dz} \tilde{f}(z) &= - \int_0^{\infty} t e^{-zt} f(t) dt\end{aligned}$$

There exists a unique number $\sigma, -\infty \leq \sigma < \infty$, such that the integral converges if $\operatorname{Re} z > \sigma$ and diverges if $\operatorname{Re} z < \sigma$. σ is called the abscissa of convergence.

Define $\rho = \inf\{B \in \mathbb{R} \mid \text{there exists an } A > 0 \text{ such that } |f(t)| \leq Ae^{Bt}\}$, then $\sigma \leq \rho$.

$\tilde{f}(z) = \tilde{h}(z)$ for $\operatorname{Re} z > \gamma_0$ for some γ_0 . Then $f(t) = h(t)$ for all $t \in [0, \infty)$.

$$\widetilde{\left(\frac{df}{dt}\right)}(z) = z\tilde{f} - f(0)$$

$$\widetilde{\left(\frac{d^2f}{dt^2}\right)}(z) = z^2\tilde{f} - zf(0) - \frac{df}{dt}(0)$$

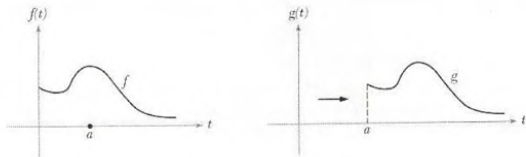
$$\tilde{g}(z) = d\tilde{f}(z)/dz, \text{ where } g(t) = -tf(t)$$

$$g(t) = \int_0^t f(\tau)d\tau, \text{ then } \tilde{g}(z) = \frac{\tilde{f}(z)}{z}$$

$$g(t) = e^{-at}f(t), \text{ then } \tilde{g}(z) = \tilde{f}(z+a) \text{ (First Shifting Theorem)}$$

Second Shifting Theorem

$H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 0$, $a \geq 0$, $g(t) = f(t - a)H(t - a)$.
 $g(t) = 0$ if $t < a$ and $g(t) = f(t - a)$ if $t \geq a$,



$$\tilde{g}(z) = e^{-az} \tilde{f}(z)$$

If $g(t) = f(t)H(t - a)$, then $\tilde{g}(z) = e^{-az} \tilde{F}(z)$ where $F(t) = f(t + a)$

$$(f * g)(t) = \int_0^{\infty} f(t - \tau) \cdot g(\tau) d\tau$$

$$\widetilde{(f * g)}(z) = \tilde{f}(z) \cdot \tilde{g}(z)$$

$$\begin{aligned}\widetilde{(f * g)}(z) &= \int_0^{\infty} e^{-zt} \left[\int_0^{\infty} f(t - \tau) \cdot g(\tau) d\tau \right] dt \\ &= \int_0^{\infty} \left[\int_0^{\infty} e^{-z\tau} e^{-z(t-\tau)} f(t - \tau) g(\tau) d\tau \right] dt \\ &= \int_0^{\infty} e^{-z\tau} \left[\int_0^{\infty} e^{-z(t-\tau)} f(t - \tau) dt \right] g(\tau) d\tau \\ &= \int_0^{\infty} e^{-z\tau} \tilde{f}(z) g(\tau) d\tau \\ &= \tilde{f}(z) \cdot \tilde{g}(z)\end{aligned}$$

Some Common Laplace Transforms

If $f(t) = e^{-at}$, then $\tilde{f}(z) = \frac{1}{z+a}$ and $\sigma(f) = -\operatorname{Re} a$

If $f(t) = \cos at$, then $\tilde{f}(z) = \frac{z}{z^2+a^2}$ and $\sigma(f) = |\operatorname{Im} a|$

If $f(t) = \sin at$, then $\tilde{f}(z) = \frac{a}{z^2+a^2}$ and $\sigma(f) = |\operatorname{Im} a|$

If $f(t) = t^a, a > -1$, then $\tilde{f}(z) = \frac{\Gamma(a+1)}{z^{a+1}}$ and $\sigma(f) = 0$

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \Gamma(n) = (n-1)!$

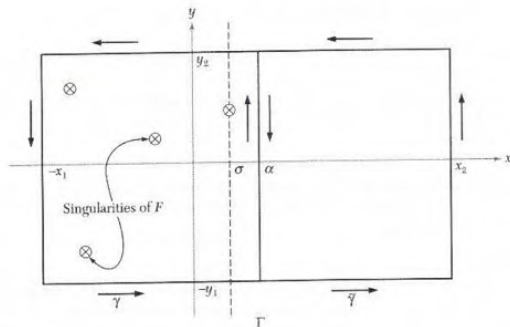
If $f(t) = 1$, then $\tilde{f}(z) = \frac{1}{z}$ and $\sigma(f) = 0$

Complex Inversion Formula

$$f(t) = \sum \{ \text{residues of } e^{zt} F(z) \text{ at each of its singularities in } \mathbb{C} \}$$

Then $\tilde{f}(z) = F(z)$ for $\text{Re } z > \sigma$.

Split Γ into a sum of two rectangular paths γ and $\tilde{\gamma}$ by a vertical line through $\text{Re } z = \alpha > \sigma$.



Complex Inversion Formula

$$\int_{\gamma} e^{zt} F(z) dz = 2\pi i \sum \{\text{residues of } e^{zt} F(z)\} = 2\pi i f(t)$$

$$\begin{aligned} 2\pi i \tilde{f}(z) &= \lim_{r \rightarrow \infty} \int_0^r e^{-zt} \left[\int_{\gamma} e^{\zeta t} F(\zeta) d\zeta \right] dt \\ &= \lim_{r \rightarrow \infty} \int_{\gamma} \int_0^r e^{(\zeta-z)t} F(\zeta) dt d\zeta \\ &= \lim_{r \rightarrow \infty} \int_{\gamma} \left(e^{(\zeta-z)r} - 1 \right) \frac{F(\zeta)}{\zeta - z} d\zeta \\ &= - \int_{\gamma} \frac{F(\zeta)}{\zeta - z} d\zeta = \int_{\tilde{\gamma}} \frac{F(\zeta)}{\zeta - z} d\zeta - \int_{\Gamma} \frac{F(\zeta)}{\zeta - z} d\zeta \\ &= 2\pi i F(z) \end{aligned}$$

Examples

If $\tilde{f}(z) = 1/(z - 3)$, $f(t) = e^{3t}$.

If $\tilde{f}(z) = \log(z^2 + z)$,

$$\tilde{g}(z) = -\frac{d}{dz}\tilde{f}(z) = -\frac{2z+1}{z^2+z} = -\frac{1}{z} - \frac{1}{z+1}$$

$$g(t) = -1 - e^{-t}$$

$$f(t) = -\frac{1}{t}(1 + e^{-t})$$

$$\tilde{f}(z) = F(z) = \frac{z}{(z+1)^2(z^2+3z-10)}$$

$$f(t) = \sum \left\{ \text{residues of } \frac{e^{zt}z}{(z+1)^2(z^2+3z-10)} = \frac{e^{zt}z}{(z+1)^2(z+5)(z-2)} \right\}$$

$$\begin{aligned} f(t) &= \text{Res}(e^{zt}F(z); -1) + \text{Res}(e^{zt}F(z); -5) + \text{Res}(e^{zt}F(z); 2) \\ &= \frac{1}{12} \left(te^{-t} - e^{-t} + \frac{e^{-t}}{12} \right) + \frac{5e^{-5t}}{112} + \frac{2e^{2t}}{63} \end{aligned}$$

Fourier Transforms

$f : \mathbb{R} \rightarrow \mathbb{R}$, the Fourier transform is defined as

$$\hat{f} := (\mathcal{F} \cdot f)(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

The factor of $1/\sqrt{2\pi}$ may be missing and the exponent may be $-2\pi i\omega x$.

$$\int_{-\infty}^{\infty} f(x) \cos(\omega x) dx = \operatorname{Re}(\hat{f}(\omega))$$

$$\int_{-\infty}^{\infty} f(x) \sin(\omega x) dx = -\operatorname{Im}(\hat{f}(\omega))$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dx f(x) e^{-i2\pi\omega x}$$

$$f(x) = \int_{-\infty}^{\infty} d\omega \hat{f}(\omega) e^{-i2\pi\omega x}$$

$\omega < 0$, $|e^{-i\omega z}| = e^{\omega y} \leq 1$ in the upper half plane \mathcal{H}

$$\int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx = 2\pi i \sum \{\text{residues of } f(z)e^{-i\omega z} \text{ in } \mathcal{H}\}$$

$\omega > 0$, $|e^{-i\omega z}| = e^{\omega y} \leq 1$ in the lower half plane \mathcal{L}

$$\int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx = -2\pi i \sum \{\text{residues of } f(z)e^{-i\omega z} \text{ in } \mathcal{L}\}$$