Solutions of HW# 21

Problem a)

At resonance, $\hat{C}=0$ and the dispersion relation reads

$$\lambda^3 = i. \tag{1}$$

Solving eq. (1) yields

$$\lambda_1 = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + i\frac{1}{2} , \tag{2}$$

$$\lambda_2 = e^{i\frac{1}{3}\left(\frac{\pi}{2}-2\pi\right)} = -i \ , \tag{3}$$

and

$$\lambda_3 = e^{i\frac{1}{3}\left(\frac{\pi}{2}+2\pi\right)} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}. \tag{4}$$

Inserting eq. (2)-(4) into eq. (1) leads to

$$\begin{aligned}\tilde{E}(\hat{z}) &= E_{ext}\left[\frac{\lambda_2\lambda_3 e^{\lambda_1\hat{z}}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2\hat{z}}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3\hat{z}}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right] \\ &= \frac{E_{ext}}{3}\left[e^{\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)\hat{z}} + e^{-i\hat{z}} + e^{\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)\hat{z}}\right]\end{aligned}. \tag{5}$$

Problem b)

Approach 1: A general proof for arbitrary $\hat{C}$

The Taylor expansion of the electric field reads

$$\tilde{E}(\hat{z}) = \tilde{E}(0) + \tilde{E}'(0)\hat{z} + \frac{1}{2}\tilde{E}''(0)\hat{z}^2 + \frac{1}{6}\tilde{E}'''(0)\hat{z}^3 + \frac{1}{24}\tilde{E}''''(0)\hat{z}^4 + \dots . \tag{6}$$

Applying the initial condition

$$\tilde{E}(0) = E_{ext} \ , \tag{7}$$

$$\tilde{E}'(0) = 0 \ , \tag{8}$$

and

$$\tilde{E}''(0) = 0 \tag{9}$$

to eq. (6) yields

$$\tilde{E}(\hat{z}) = E_{ext} + \frac{1}{6}\tilde{E}'''(0)\hat{z}^3 + \frac{1}{24}\tilde{E}''''(0)\hat{z}^4 + \dots \quad (10)$$

The differential equation for the electric field is

$$\frac{d^3}{d\hat{z}^3}\tilde{E}(\hat{z}) + 2i\hat{C}\frac{d^2}{d\hat{z}^2}\tilde{E}(\hat{z}) - \hat{C}^2\frac{d}{d\hat{z}}\tilde{E}(\hat{z}) = i\tilde{E}(\hat{z}) \ , \quad (11)$$

Applying eq. (11) at $\hat{z} = 0$ produces

$$\tilde{E}'''(0) = iE_{ext} \ . \quad (12)$$

Taking the derivative of eq. (11) with respect to $\hat{z}$ and applying it at $\hat{z} = 0$ lead to

$$\tilde{E}''''(0) = -2i\hat{C}\tilde{E}'''(0) = 2\hat{C}E_{ext} \ , \quad (13)$$

where we used eq. (12) for the last step. Inserting eq. (12) and (13) into eq. (10) generates

$$\tilde{E}(\hat{z}) = E_{ext} + \frac{1}{6}iE_{ext}\hat{z}^3 + \frac{\hat{C}}{12}E_{ext}\hat{z}^4 + \dots \quad (14)$$

The gain can then be calculated as

$$g_h = \frac{\tilde{E}^2 - E_{ext}^2}{E_{ext}^2} \approx \left(1 + \frac{\hat{C}}{12}\hat{z}^4\right)^2 - 1 \approx \frac{\hat{C}}{6}\hat{z}^4 \ . \quad (15)$$

Approach 2: Demonstrate the $\hat{z}^4$ dependence for a specific value of $\hat{C}$

The dispersion relation can be written as

$$\left(\lambda + 2i\hat{C}\right)\lambda^2 - \hat{C}^2\left(\lambda + \frac{i}{\hat{C}^2}\right) = 0 \ , \quad (16)$$

For simplicity, we can solve the dispersion relation at $\hat{C} = 2^{-\frac{1}{3}}$, which leads to

$$\left(\lambda + i2^{2/3}\right)\left(\lambda^2 - 2^{-2/3}\right) = 0. \quad (17)$$

The solutions of eq. (17) are

$$\lambda_1 = 2^{-1/3} \ , \quad (18)$$

$$\lambda_2 = -i2^{2/3} \ , \quad (19)$$

and

$$\lambda_3 = -2^{-1/3} \quad . \tag{20}$$

$$\tilde{E}(\hat{z}) = E_{ext}\left[\frac{i2^{1/3}e^{\lambda_1\hat{z}}}{\left(2^{-1/3}+i2^{2/3}\right)\left(2^{-1/3}+2^{-1/3}\right)} + \frac{-2^{-1/3}2^{-1/3}e^{\lambda_2\hat{z}}}{\left(-i2^{2/3}+2^{-1/3}\right)\left(-i2^{2/3}-2^{-1/3}\right)} + \frac{-i2^{2/3}\lambda_1 e^{\lambda_3\hat{z}}}{\left(-2^{-1/3}-2^{-1/3}\right)\left(-2^{-1/3}+i2^{2/3}\right)}\right]. \tag{21}$$

$$= \frac{E_{ext}}{5}\left[\left(2+i\right)e^{\lambda_1\hat{z}} + e^{\lambda_2\hat{z}} + \left(2-i\right)e^{\lambda_3\hat{z}}\right]$$

For $\hat{z} << 1$, the exponential function can be expanded into power series:

$$e^{\lambda\hat{z}} = 1 + \lambda\hat{z} + \frac{\lambda^2}{2}\hat{z}^2 + \frac{\lambda^3}{6}\hat{z}^3 + \frac{\lambda^4}{24}\hat{z}^4 + \ldots \quad . \tag{22}$$

Inserting eq. (22) into eq. (21) leads to

$$\begin{aligned}
\tilde{E}(\hat{z}) &= \frac{E_{ext}}{5}\left\{5 + \left(2i\lambda_1 + \lambda_2\right)\hat{z} + \frac{1}{2}\left(4\lambda_1^2 + \lambda_2^2\right)\hat{z}^2 + \frac{1}{6}\left(2i\lambda_1^3 + \lambda_2^3\right)\hat{z}^3 + \frac{1}{24}\left(4\lambda_1^4 + \lambda_2^4\right)\hat{z}^4 + \ldots\right\} \\
&= \frac{E_{ext}}{5}\left\{5 + i\frac{5}{6}\hat{z}^3 + \frac{2^{2/3}}{24}5\hat{z}^4 + \ldots\right\} \quad . \\
&= E_{ext}\left\{1 + i\frac{1}{6}\hat{z}^3 + \frac{2^{2/3}}{24}\hat{z}^4 + \ldots\right\}
\end{aligned} \tag{23}$$

Thus the gain is given by

$$g_h = \frac{\tilde{E}^2 - E_{ext}^2}{E_{ext}^2} \approx \left(1 + \frac{2^{2/3}}{24}\hat{z}^4\right)^2 - 1 \approx \frac{2^{2/3}}{12}\hat{z}^4 = 0.132\hat{z}^4 \, . \tag{24}$$

Note: it is fine to assume other values of $\hat{C}$ as long as it is not zero. In most values of $\hat{C}$, the dispersion relation has to be solved numerically.