PHY 564
Advanced Accelerator Physics
Lecture 14

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Lecture 13 - finishing

Lecture 13. Linear Hamiltonian system and their transport matrices

Compact form

We went through following discussion during previous lectures:

1. A linear Hamiltonian *n*-dimensional system with *s* as independent variable and *n* canonical pairs of variables

$$X^T = \{q_1, P_1, ..., q_n, P_n\} \Leftrightarrow \{x_1, ..., x_{2n}\}$$
$$x_{2k-1} = q_k; x_{2k} = P_k; k = 1,..,n \tag{1}$$

is fully described by its Hamiltonian

$$H(X,s) = \frac{1}{2} X^T \mathbf{H}(s) X;\ \mathbf{H}^T(s) = \mathbf{H}(s) \tag{2}$$

where $\mathbf{H}(s)$ is *2n x* 2n symmetric matrix with coefficients, in general, depending on (e.g. being functions of) *s*:

$$[\mathbf{H}]_{ij} = h_{ij}(s). \tag{3}$$

Equations of motions can be written in a compact matrix form

$$\frac{dq_i}{ds} = \frac{\partial H}{\partial P_i} = \sum_{j=1}^{2n} h_{ij}(s) x_j; \frac{dP_i}{ds} = -\frac{\partial H}{\partial q_i} = -\sum_{j=1}^{2n} h_{ij}(s) x_j$$

$$\frac{dX}{ds} = \mathbf{S} \cdot \mathbf{H} \cdot X;\quad \mathbf{S} = \begin{bmatrix} \sigma & 0 & .. & 0 \\ 0 & \sigma & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \sigma \end{bmatrix}; \sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4}$$

$$\mathbf{S} = [s_{ij}]; s_{2k-1,2k} = -s_{2k,2k-1} = 1;\ \textit{othewise}\ 0$$

or even more compact form

$$\frac{dX}{ds} = \mathbf{D}(s) \cdot X;\quad \mathbf{D}(s) = \mathbf{S} \cdot \mathbf{H}(s). \tag{5}$$

One more time, when all component of the vector bellow are small

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \equiv \begin{bmatrix} x \\ P_1 \\ y \\ P_3 \\ \tau \\ \delta \end{bmatrix} \equiv \begin{bmatrix} x \\ P_x \\ y \\ P_y \\ z \\ P_z \end{bmatrix}; \qquad (28)$$

we can expand Hamiltonian to the form in eq. (2) – all linear terms are killed by assumption that reference particles has the designed trajectory:

$$\begin{aligned} \tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L\left(xP_3 - yP_1\right) + \\ \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta + g_y y\delta + F_x x\tau + F_y y\tau \end{aligned}; \qquad (29)$$

with

$$\frac{F}{p_o} = \left[-K\cdot\frac{e}{p_oc}B_y - \frac{e}{p_oc}\frac{\partial B_y}{\partial x} + \left(\frac{eB_s}{2p_oc}\right)^2\right] - \frac{e}{p_o\mathrm{v}_o}\frac{\partial E_x}{\partial x} - 2K\frac{eE_x}{p_o\mathrm{v}_o} + \left(\frac{meE_x}{p_o^2}\right)^2;$$

$$\frac{G}{p_o} = \left[\frac{e}{p_oc}\frac{\partial B_x}{\partial y} + \left(\frac{eB_s}{2p_oc}\right)^2\right] - \frac{e}{p_o\mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{meE_z}{p_o^2}\right)^2; \qquad ;(30)$$

$$\frac{2N}{p_o} = \left[\frac{e}{p_oc}\frac{\partial B_x}{\partial x} - \frac{e}{p_oc}\frac{\partial B_y}{\partial y}\right] - K\cdot\frac{e}{p_oc}B_x - \frac{e}{p_o\mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o\mathrm{v}_o} + \left(\frac{meE_z}{p_o^2}\right)\left(\frac{meE_x}{p_o^2}\right)$$

$$L = \kappa + \frac{e}{2p_oc}B_s; \qquad \frac{U}{p_o} = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \quad g_x = \frac{(mc)^2\cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; g_y = \frac{(mc)^2\cdot eE_y}{p_o^3};$$

$$F_x = \frac{e}{c}\frac{\partial B_y}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_x}{\partial ct}; F_y = -\frac{e}{c}\frac{\partial B_x}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_y}{\partial ct}.$$

$$\mathbf{H} = \begin{bmatrix} F & 0 & N & L & F_x & g_x \\ 0 & 1/p_o & -L & 0 & 0 & 0 \\ N & -L & G & 0 & F_y & g_y \\ L & 0 & 0 & 1/p_o & 0 & 0 \\ F_x & 0 & F_y & 0 & U & 0 \\ g_x & 0 & g_y & 0 & 0 & \dfrac{m^2c^2}{p_o^3} \end{bmatrix}; \mathbf{D} = \begin{bmatrix} 0 & 1/p_o & -L & 0 & F_x & 0 \\ -F & 0 & -N & -L & 0 & -g_x \\ L & 0 & 0 & 1/p_o & F_y & 0 \\ -N & L & -G & 0 & 0 & -g_y \\ g_x & 0 & g_y & 0 & 0 & \dfrac{m^2c^2}{p_o^3} \\ -F_x & 0 & -F_y & 0 & -U & 0 \end{bmatrix} \quad (31)$$

$$\det[D-\lambda I]=\det\begin{bmatrix} -\lambda & 1 & -L & 0 \\ -f & -\lambda & -n & -L \\ L & 0 & -\lambda & 1 \\ -n & L & -g & -\lambda \end{bmatrix}$$

Let's find the solutions for 4x4 matrixes of arbitrary element. First, let solve characteristic equation for D:

$$\det[D-\lambda I]=\lambda^4+\lambda^2\left(f+g+2L^2\right)+fg+L^4-L^2(f+g)-n^2=0 \qquad (45)$$

with easy roots:

$$\lambda^2=a\pm b;\ \ a=-\frac{f+g+2L^2}{2};\ \ b^2=\frac{(f-g)^2}{4}+2L^2(f+g)+n^2 \qquad (46)$$

Before starting classification of the cases, let's note that

$$f+g=K^2+2\left(\frac{eB_s}{2p_oc}\right)^2+\left(\frac{meE_x}{p_o{}^2}\right)^2+\left(\frac{meE_z}{p_o{}^2}\right)^2\geq 0$$

i.e. $a\leq 0;\ \ b^2\geq 0;\ \ \mathrm{Im}(b)=0$.

$$\lambda^2 = a \pm b; a = -\frac{f+g+2L^2}{2}; b^2 = \frac{(f-g)^2}{4} + 2L^2(f+g) + n^2$$

Before starting classification of the cases, let's note that

$$f+g = K^2 + 2\left(\frac{eB_s}{2p_o c}\right)^2 + \left(\frac{meE_x}{p_o{}^2}\right)^2 + \left(\frac{meE_z}{p_o{}^2}\right)^2 \geq 0$$

i.e. $a \leq 0;\ b^2 \geq 0;\ \operatorname{Im}(b) = 0$. The solutions can be classified as following: remember that the full set of eigen values is $\lambda_1, -\lambda_1, \lambda_2, -\lambda_2$:

I. $\lambda_1 = \lambda_2 = 0;\quad a = 0;\ b = 0;$

II. $\lambda_1 = \lambda_2 = i\omega;\quad a = -\omega^2;\ b = 0;$

III. $\lambda_1 = 0;\quad \lambda_2 = i\omega;\quad a + b = 0;\quad 2b = \omega^2$

IV. $\lambda_1 = i\omega_1;\quad \lambda_2 = i\omega_2;\quad \omega_1{}^2 = -a - b;\ \omega_2{}^2 = -a + b;\ |a| > b$

V. $\lambda_1 = i\omega_1;\quad \lambda_2 = \omega_2;\ \omega_1{}^2 = -a - b;\ \omega_2{}^2 = b - a;\ b > |a|$

Before going into the discussion of the parameterization of the motion, we need to finish discussion of few remaining topics for 6x6 matrix of an accelerator. First is multiplication of the 6x6 matrixes for purely magnetic elements:

$$\mathbf{M}_{k\,(6x6)} = \begin{bmatrix} \mathbf{M}_{k\,(4x4)} & 0 & R_k \\ Q_k & 1 & R_{56_k} \\ 0 & 0 & 1 \end{bmatrix};$$
$$\mathbf{M}_{2\,(6x6)}\mathbf{M}_{1\,(6x6)} = \begin{bmatrix} \mathbf{M}_{(4x4)} & 0 & R \\ Q & 1 & R_{56} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_2\mathbf{M}_1 & 0 & R_2 + \mathbf{M}_2 R_1 \\ Q_2 + Q_1\mathbf{M}_2 & 1 & R_{56_1} + R_{56_2} + Q_2 R_1 \\ 0 & 0 & 1 \end{bmatrix} \tag{51}$$

i.e. having transformation rules for mixed size objects: a 4x4 matrix M, 4-elemetn column R, 4 element line L, and a number R_{56}. As you remember, L is dependent (L4-7) and expressed as $\mathbf{Q} = \mathbf{R}^T\mathbf{S}\mathbf{M}$. Thus:

$$\mathbf{M}_{(4x4)} = \mathbf{M}_2\mathbf{M}_1;\ R = \mathbf{M}_2 R_1 + R_2;\ Q = Q_2\mathbf{M}_1 + Q_1;\ R_{56} = R_{56_1} + R_{56_2} + Q_2 R_1 \tag{52}$$

One thing is left without discussion so far – the energy change. Thus, we should look into a particle passing through an RF cavity, which has alternating longitudinal field. Again, for simplicity we will assume that equilibrium particle does not gain energy, i.e. p_o stays constant and we can continue using reduced variables. We will also assume that the is no transverse field, neither AC or DC. In this case the Hamiltonian reduces to a simple, fully decoupled:

$$\tilde{h} = \frac{\pi_1^2 + \pi_3^2}{2} + \frac{\pi_o^2}{2} \cdot \frac{m^2 c^2}{p_o^2} + u\frac{\tau^2}{2};; \tag{53}$$

$$\frac{dX}{ds} = \mathbf{D} \cdot X;\ \mathbf{D} = \begin{bmatrix} D_x & 0 & 0 \\ 0 & D_y & 0 \\ 0 & 0 & D_l \end{bmatrix};\ D_x = D_y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; D_l = \begin{bmatrix} 0 & \frac{m^2 c^2}{p_o^2} \\ -u & 0 \end{bmatrix};$$

$$\mathbf{M} = \begin{bmatrix} M_x & 0 & 0 \\ 0 & M_y & 0 \\ 0 & 0 & M_l \end{bmatrix}; M_x = M_y = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}; \qquad \omega = \sqrt{\left|\det D_l\right|} = \frac{mc}{p_o}\sqrt{|u|} \tag{53}$$

$$M_l = \begin{bmatrix} \cos\omega s & \frac{m^2 c^2}{p_o^2} \sin\omega s / \omega \\ -u \sin\omega s / \omega & \cos\omega s \end{bmatrix}; u > 0;\ M_l = \begin{bmatrix} \cosh\omega s & \frac{m^2 c^2}{p_o^2} \sinh\omega s / \omega \\ -u \sinh\omega s / \omega & \cosh\omega s \end{bmatrix}; u < 0;$$

Just for fun, let's look at 1D matrices of quadrupole:

$$\tilde{h} = \left(\frac{\pi_3{}^2}{2} + K_1 \frac{y^2}{2} \right) + \left(\frac{\pi_1{}^2}{2} - K_1 \frac{x^2}{2} \right) + \frac{\pi_\delta{}^2}{2} \cdot \frac{m^2 c^2}{p_o{}^2}; K_1 = \frac{e}{p_o c} \frac{\partial B_y}{\partial x}; \qquad (56)$$

Thus, it is non-degenerated case only when $\det[D] \neq 0$ we have a simple two-piece expression :

$$\exp[\mathbf{D}s] = e^{\lambda s} \frac{\mathbf{D} - \lambda \mathbf{I}}{2\lambda} - e^{-\lambda s} \frac{\mathbf{D} + \lambda \mathrm{I}}{2\lambda} \qquad (57)$$

while (37) bring it home right away:

$$\exp[\mathbf{D}s] = \mathbf{I} \cdot \frac{e^{\lambda s} + e^{-\lambda s}}{2} + \mathbf{D} \frac{e^{\lambda s} - e^{-\lambda s}}{2\lambda};$$

$$\exp[\mathbf{D}s] = \mathbf{I} \cdot \cosh|\lambda| s + \frac{\mathbf{D} \sinh|\lambda| s}{|\lambda|}; \quad \det[\mathbf{D}] < 0; \quad |\lambda| = \sqrt{-\det[\mathbf{D}]} \qquad (58)$$

$$\exp[\mathbf{D}s] = \mathbf{I} \cdot \cos|\lambda| s + \frac{\mathbf{D} \sin|\lambda| s}{|\lambda|}; \quad \det[\mathbf{D}] > 0; \quad |\lambda| = \sqrt{\det[\mathbf{D}]}$$

The case $\det[\mathbf{D}]=0$ means in this case that D is nilpotent: eqs (37) look like follows

$$\det\mathbf{D}=0 \Rightarrow \lambda_1=-\lambda_2=0;\ d(\lambda)=\det[\mathbf{D}-\lambda I]=(\lambda_1-\lambda)(-\lambda_1-\lambda)=\lambda^2 \quad \Rightarrow \mathbf{D}^2=0$$

hence

$$\exp[\mathbf{D}s]=\mathbf{I}+\mathbf{D}s;\quad \det[\mathbf{D}]=0; \tag{59}$$

For non-scaled case is just a change of variables:

$$\tilde{h}=\left(\frac{P_3^{\ 2}}{2p_o}+p_oK_1\frac{y^2}{2}\right)+\left(\frac{P_1^{\ 2}}{2p_o}-p_oK_1\frac{x^2}{2}\right)+\frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^{\ 2}};K_1=\frac{e}{p_oc}\frac{\partial B_y}{\partial x}; \tag{60}$$

$$D_x=\begin{bmatrix}0 & 1/p_o\\ p_oK_1 & 0\end{bmatrix};\ D_y=\begin{bmatrix}0 & 1/p_o\\ -p_oK_1 & 0\end{bmatrix};\ \phi=s\sqrt{K_1}$$

$$M_F=\begin{bmatrix}\cos\phi & \sin\phi/p_o\sqrt{K_1}\\ -p_o\sqrt{K_1}\sin\phi & \cos\phi\end{bmatrix};\ M_D=\begin{bmatrix}\cosh\phi & \sinh\phi/p_o\sqrt{K_1}\\ p_o\sqrt{K_1}\sinh\phi & \cosh\phi\end{bmatrix}$$

In a case when length of the quadrupole is very short, but the strength is finite. It is called thin-lens approximations:

$$\varphi = s\sqrt{K_1} \to 0; K_1 s = const = \frac{1}{F}$$

$$M_F \to \begin{bmatrix} 1 & 0 \\ -\frac{p_o}{F} & 1 \end{bmatrix}; M_D \to \begin{bmatrix} 1 & 0 \\ \frac{p_o}{F} & 1 \end{bmatrix}$$

$$\{x, x'\}, \{y, y'\}$$

$$M_F \to \begin{bmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{bmatrix}; M_D \to \begin{bmatrix} 1 & 0 \\ \frac{1}{F} & 1 \end{bmatrix} \qquad (61)$$

In majority of the cases ωs<<1 (mc/p_o ~ 1/γ) and RF cavity can be represented as a thin lens located in its center:

$$\mathbf{M} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M_l \end{bmatrix}; \quad M_l = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix}; \quad q = u \cdot l_{RF} = -\frac{e}{p_o c}\frac{\partial V_{rf}}{\partial t} \qquad (55)$$

Lecture 14. Periodic systems and parameterization of linearized particle's motion.

Periodic linear Hamiltonian systems are of special interest for accelerators. First, one of most popular accelerator designs is a circular accelerator (called synchrotrons and storage rings) where particles going around bot millions and billions of turns. At each path they go through the same sequence of the elements, e.g. they see periodic structure with period equal to accelerator circumference. Stability of the particle's motion is of a paramount importance for their proper operation.

Furthermore, for an accelerator beam-lines (e.g. part of an accelerator) comprising hundreds (or even thousands) of magnets, physicist and engineers like using a relatively simple cell and repeat it multiple times. This allows one to study this cell in detail and then "match" the beam into the entire beamline. A FODO cell comprised of two quadrupoles F and D, separated by drift spaces O. It is customary to call F quadrupole focusing in horizontal (x, radial) direction and, naturally, defocusing in vertical (y) direction. Vice versa, D is a defocusing quadrupole focuses in y direction and defocuses in x. FODO is a simple and still very popular cell. For example, eRHIC energy recovery linac (ERL) arcs will be comprised of many hundreds of FODO cells.

Stability and Parameterization of motion in periodic systems

A Hamiltonian periodic system with period *C*, is described by periodic Hamiltonian: $H(X,s+C)=H(X,s)$. For linear Hamiltonian system is means that (elements of) matrix of the Hamiltonian is (are) a periodic function of s.:

$$H=\frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n}h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T\cdot\mathbf{H}(s)\cdot X,\ \mathbf{H}(s+C)=\mathbf{H}(s); \qquad (1)$$

In this case, a one-turn (or one period) transport matrix

$$\mathbf{T}(s)=\mathbf{M}\left(s\middle|s+C\right) \qquad (2)$$

plays a very important role. Its eigen values, λ_i,

$$\det\left[\mathbf{T}-\lambda_i\cdot\mathbf{I}\right]=0 \qquad (3)$$

determine is the motion is stable, e.g. that all $|\lambda_i|\le 1$ or is unstable, e.g. some $|\lambda_i|>1$. Before making specific statements about the stability, we look at the properties of the eigen vectors.

First, eigen values are a function of periodic system and do not depend on the azimuth, *s*. It is easy to show that a one-turn matrix is transformed by the transport matrix as

$$\mathbf{T}(s_1) = \mathbf{M}(s|s_1)\mathbf{T}(s)\mathbf{M}^{-1}(s|s_1) \tag{4}$$

$$\mathbf{T}(s_1) = \mathbf{M}(s_1|s_1+C) = \mathbf{M}(s+C|s_1+C)\mathbf{M}(s_1|s+C) = \mathbf{M}(s+C|s_1+C)\mathbf{M}(s|s+C)\mathbf{M}(s_1|s)$$

$$\mathbf{M}(s+C|s_1+C) \equiv \mathbf{M}(s|s_1);\ \mathbf{M}(s|s+C),\ \mathbf{M}(s_1|s) \equiv \mathbf{M}^{-1}(s|s_1) \Rightarrow \mathbf{T}(s_1) = \mathbf{M}(s|s_1)\mathbf{T}(s)\mathbf{M}^{-1}(s|s_1)\#$$

It means that $\mathbf{T}(s_1)$ has the same eigen values (3); thus, the eigen values of $\mathbf{T}(s)$ do not depend upon *s* because

$$\det\left[\mathbf{MTM}^{-1} - \lambda_i \cdot \mathbf{I}\right] = \det\left[\mathbf{M}(\mathbf{T} - \lambda_i \cdot \mathbf{I})\mathbf{M}^{-1}\right] = \det[\mathbf{T} - \lambda_i \cdot \mathbf{I}]$$
$$\Rightarrow \left[\mathbf{T}(s_1) - \lambda_i \cdot \mathbf{I}\right] = \left[\mathbf{T}(s) - \lambda_i \cdot \mathbf{I}\right] = 0 \tag{5}$$

The matrix $\mathbf{T}$ is a real, complex conjugate of eigen value λ_i^* which is also eigen value of $\mathbf{T}$

$$\left[\mathbf{T} - \lambda_i \cdot \mathbf{I}\right]^* = \left[\mathbf{T} - \lambda^*{}_i \cdot \mathbf{I}\right] = 0$$

Furthermore, the symplecticity of $\mathbf{T}$ requires that λ_i^{-1} also is eigen value of $\mathbf{T}$. Proving that the inverse matrix $\mathbf{T}^{-1}$ has λ_i^{-1} as a eigen value is easy.

$$\mathbf{T}Y_i = \lambda_i Y_i; \;\; \mathbf{T}^{-1}\mathbf{T} = \mathbf{I} \to \left(\mathbf{T}^{-1}\mathbf{T}\right)Y_i = \mathbf{I}Y_i = Y_i$$

$$\mathbf{T}^{-1}\mathbf{T}Y_i = \lambda_i \mathbf{T}^{-1}Y_i = Y_i \to \mathbf{T}^{-1}Y_i = \lambda_i^{-1}Y_i$$

At the same time

$$0 = \det\left[\mathbf{T}^{-1} - \lambda_i^{-1}\mathbf{I}\right] = \det\left(\mathbf{S}\left[\mathbf{T}^T - \lambda_i^{-1}\mathbf{I}\right]\mathbf{S}^{-1}\right) = \det\left[\mathbf{T}^T - \lambda_i^{-1}\mathbf{I}\right] = \det\left[\mathbf{T} - \lambda_i^{-1}\mathbf{I}\right] \qquad (5')$$

and here, symplectic conditions help us again. Thus, the real symplectic matrix has n pairs of eigen values as follows: a) inverse $\{\lambda_i, \lambda_i^{-1}\}$, and b) complex conjugate $\{\lambda_i, \lambda_i^*\}$. We assume that matrix T can be diagonalized (see note on the following page for the general case of Jordan normal form).

In general case of multiplicity of eigen vectors, the matrix cannot be diagonalized but can be brought to Jordan normal form http://en.wikipedia.org/wiki/Jordan_normal_form#Generalized_eigenvectors , Glenn James and Robert C. James, Mathematics. In this case, there is a subset of generalized eigen vectors $\left\{Y_{k,1},...,Y_{k,h}\right\}$ that belong to a eigen value λ_k with multiplicity h:

$$\mathbf{T}\cdot Y_{k,h} = \lambda_k Y_{k,h}; \; \mathbf{T}\cdot Y_{k,m} = \lambda_k Y_{k,m} + Y_{k,m+1}; \; m = 1...h-1.$$

The result is even stronger than in the diagonal case; motion is unstable even when $\lambda_k = 1$:

$$\mathbf{T}\cdot Y_{k,h-1} = \lambda_k Y_{k,h-1} + Y_{k,h} \Rightarrow \mathbf{T}^n \cdot Y_{k,h-1} = Y_{k,h-1} + n\cdot Y_{k,h}$$

Therefore, repeating the matrix T again and again undoubtedly will cause an exponentially growing solution if $|\lambda_i| > 1$. This statement is readily verified, but in so, we introduce some useful term and matrices. The set of eigen vectors Y_i of matrix **T**

$$\mathbf{T} \cdot Y_i = \lambda_i \cdot Y_i; \qquad i = 1,2....2n \tag{6}$$

is complete and an arbitrary vector X can be expanded about this basis:

$$X = \sum_{i=1}^{2n} a_i Y_i \equiv \mathbf{U} \cdot A, \quad \mathbf{U} = [Y_1 ......Y_{2n}], \qquad A^T = [a_1 ......a_{2n}]. \tag{7}$$

where we introduces matrix **U** built from eigen vector of the matrix **T**:

$$\mathbf{T} \cdot \mathbf{U} = \mathbf{U} \cdot \Lambda, \ \ \Lambda = \begin{bmatrix} \lambda_1 & ... & 0 \\ ... & ... & .... \\ 0 & .... & \lambda_{2n} \end{bmatrix} \tag{8}$$

The later equation is equivalent to diagonalization of the matrix **T**:

$$\mathbf{U}^{-1} \cdot \mathbf{T} \cdot \mathbf{U} = \Lambda, \ or \ \ \mathbf{T} = \mathbf{U} \cdot \Lambda \cdot \mathbf{U}^{-1} \tag{9}$$

Multiple application of matrix **T** (i.e., passes around the ring)

$$\mathbf{T}^n \cdot X = \sum_{i=1}^{2n} \lambda_i^{\,n} a_i Y_i \tag{10}$$

exhibit exponentially growing terms if the module of even one eigen value is larger than 1, $\lambda_k = |\lambda| e^{i\mu}$, $|\lambda| > 1$; we easily observe that a solution with the initial condition $X_o = \operatorname{Re} a_k Y_k$ grows exponentially:

$$\mathbf{T}^n X_o = |\lambda|^n \operatorname{Re} a_k Y_k e^{in\mu}.$$

Immediately this suggests that the only possible stable system is when all eigen values are uni-modular

$$|\lambda_i| = 1. \tag{11}$$

otherwise assuming $|\lambda_i| < 1$ means that there is eigen value $\lambda_k = \lambda_i^{-1}$; $|\lambda_k| = 1/|\lambda_i| > 1$. We also consider only cases when all eigen vectors differ. Then, there are *n* pairs of eigen vectors, which we term modes of oscillations:

$$\lambda_k \equiv 1/\lambda_{k+n} \equiv \lambda^*{}_{k+n} \equiv e^{i\mu_k}; \;\; \mu_k \equiv 2\pi\nu_k, \;\; \{k = 1,..n\}. \tag{12}$$

where the complex conjugate pairs are identical to the inverse pairs.

Eq. (9) can be rewritten as

$$\mathbf{T}(s) = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_1^* & & 0 \\ & & \ldots & 0 \\ 0 & 0 & 0 & \lambda_n^* \end{pmatrix};\ \mathbf{T}(s)\cdot\mathbf{U}(s) = \mathbf{U}(s)\cdot\Lambda \tag{13}$$

and matrix **U** built from complex conjugate eigen vectors of **T**:

$$\mathbf{U}(s) = \left[Y_1, Y_1^* \ldots Y_n, Y_n^*\right]; \quad \mathbf{T}(s)Y_k(s) = \lambda_k Y_k(s) \quad \Leftrightarrow \quad \mathbf{T}(s)Y_k^*(s) = \lambda_k^* Y_k^*(s) \tag{14}$$

Thus, eigen vectors can be transported from one azimuth to another by the transport matrix:

$$\tilde{Y}_k(s_1) = \mathbf{M}(s|s_1)\tilde{Y}_k(s) \ \Leftrightarrow \ \frac{d}{ds}\tilde{Y}_k = \mathbf{D}(s)\cdot\tilde{Y}_k \tag{15}$$

It is eigen vector of $\mathbf{T}(s_1)$. - just add (4) to (14):

$$\mathbf{T}(s_1)\tilde{Y}_k(s_1) = \mathbf{M}(s|s_1)\mathbf{T}(s)\mathbf{M}^{-1}(s|s_1)\mathbf{M}(s|s_1)\tilde{Y}_k(s) = \mathbf{M}(s|s_1)\mathbf{T}(s)\tilde{Y}(s) = \lambda_k\mathbf{M}(s|s_1)\tilde{Y}(s) = \lambda_k\tilde{Y}_k(s_1)\#$$

Similarly,

$$\tilde{\mathbf{U}}(s_1) = \mathbf{M}(s|s_1)\tilde{\mathbf{U}}(s) \ \Leftrightarrow \ \frac{d}{ds}\tilde{\mathbf{U}} = \mathbf{D}(s)\cdot\tilde{\mathbf{U}} \tag{16}$$

with the obvious follow-up by

$$\tilde{\mathbf{U}}(s+C) = \tilde{\mathbf{U}}(s)\cdot\Lambda,\ \tilde{Y}_k(s+C) = \lambda_k\tilde{Y}_k(s) = e^{i\mu_k}\tilde{Y}_k(s) \tag{17}$$

The k^{th} eigen vectors are multiplied by $e^{i\mu_k}$ after each pass through the period. Hence, we can write

$$\tilde{Y}_k(s) = Y_k(s)e^{\psi_k(s)}; \quad Y_k(s+C) = Y_k(s); \qquad \psi_k(s+C) = \psi_k(s) + \mu_k \tag{18}$$

$$\tilde{\mathbf{U}}(s) = \mathbf{U}(s)\cdot\Psi(s),\ \Psi(s) = \begin{pmatrix} e^{i\psi_1(s)} & 0 & & 0 \\ 0 & e^{-i\psi_1(s)} & & 0 \\ & & \dots & 0 \\ 0 & 0 & 0 & e^{-i\psi_n(s)} \end{pmatrix} \tag{19}$$

It is remarkable that the symplectic products (12) of the eigen vectors are non-zero-only complex conjugate pairs: in other words, the structure of the Hamiltonian metrics is preserved here. $Y_k^T \cdot \mathbf{S} \cdot Y_k^T \equiv 0$ is obvious. Using only the symplecticity of $\mathbf{T}$ gives us desirable yields

$$Y_k^T \cdot \mathbf{S} \cdot Y_j^T = Y_k^T \cdot \mathbf{T}^T\mathbf{S}\mathbf{T} \cdot Y_j^T = \lambda_k\lambda_j\left(Y_k^T \cdot \mathbf{S} \cdot Y_j^T\right) \Rightarrow (1-\lambda_k\lambda_j)\left(Y_k^T \cdot \mathbf{S} \cdot Y_j^T\right) = 0$$

for $\lambda_k\lambda_j \neq 1$

$$Y_k^{T*} \cdot \mathbf{S} \cdot Y_{j\neq k} = 0; \quad Y_k^T \cdot \mathbf{S} \cdot Y_j = 0; \ . \tag{20}$$

and only the nonzero products for $\lambda_k = 1/\lambda_j = \lambda^*{}_j$ are clearly pure imaginary:

$$Y_k^{T*} \cdot \mathbf{S} \cdot Y_k = 2i, \tag{21}$$

where we chose the calibration of purely imaginary values as *2i* for the following expansion to be symplectic.

$$\left(A^{\bar{T}} \cdot \mathbf{S} \cdot A\right)^* = \left(A^T \cdot \mathbf{S} \cdot A^*\right) = -\left(A^{*T} \cdot \mathbf{S} \cdot A\right)^T = -\left(A^{\bar{T}} \cdot \mathbf{S} \cdot A\right)$$

Eqs. (20-21) in compact matrix form is

$$\mathbf{U}^T \cdot \mathbf{S} \cdot \mathbf{U} \equiv \tilde{\mathbf{U}}^T \cdot \mathbf{S} \cdot \tilde{\mathbf{U}} = -2i\mathbf{S},\ \ \mathbf{U}^{-1} = \frac{1}{2i}\mathbf{S} \cdot \mathbf{U}^T \cdot \mathbf{S}. \tag{22}$$

The expressions for the transport matrices through β, α-functions, and phase advances often derived as a "miraculous" result, and hence called matrix gymnastics, is just a trivial consequence of equations (16), (19), and (22):

$$\mathbf{M}(s|s_1) = \tilde{\mathbf{U}}(s_1)\tilde{\mathbf{U}}^{-1}(s) = \frac{1}{2i}\tilde{\mathbf{U}}(s_1) \cdot \mathbf{S} \cdot \tilde{\mathbf{U}}^T(s) \cdot \mathbf{S} = \frac{1}{2i}\mathbf{U}(s_1) \cdot \Psi(s_1) \cdot \mathbf{S} \cdot \Psi^{-1}(s) \cdot \mathbf{U}^T(s_1) \tag{16'}$$

with a specific case of a one-turn matrix:

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \frac{1}{2i}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^T\mathbf{S} \tag{13'}$$

S-orthogonality (20) provides an excellent tool of finding complex coefficients in the expansion eq. (7) of an arbitrary solution X(s)

$$X_o = \sum_{i=1}^{2n} a_i Y_i \Rightarrow X(s) = \frac{1}{2}\sum_{k=1}^{n}\left(a_k\tilde{Y}_k + a_k^*\tilde{Y}_k^*\right) \equiv \mathrm{Re}\sum_{k=1}^{n} a_k Y_k e^{i\psi_k} \equiv \frac{1}{2}\tilde{\mathbf{U}} \cdot A = \frac{1}{2}\mathbf{U} \cdot \Psi \cdot A = \frac{1}{2}\mathbf{U} \cdot \tilde{A} \tag{23}$$

where *2n* complex coefficients, which are constants of motion for linear Hamiltonian system, can be found by a simple multiplications (instead of solving a system of 2n linear equations (7))

$$\begin{gathered} a_i = \frac{1}{2i}Y_i^{*T}SX;\ \tilde{a}_i \equiv a_i e^{i\psi_i} = \frac{1}{2i}Y_i^{*T}SX; \\ A = 2\tilde{\mathbf{U}}^{-1} \cdot X = -i\Psi^{-1} \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X;\ \tilde{A} = \Psi A = -i \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X. \end{gathered} \tag{24}$$

in matrix form using (16) we have $X = \frac{1}{2}\tilde{\mathbf{U}}A,\ X' = \frac{1}{2}\left(\tilde{\mathbf{U}}'A + \tilde{\mathbf{U}}A'\right), = \mathbf{D}X = \frac{1}{2}\mathbf{D}\tilde{\mathbf{U}} \cdot A = \frac{1}{2}\tilde{\mathbf{U}}' \cdot A \Rightarrow A' = 0$

Equation (23) is nothing else but a general parameterization of motion in the linear Hamiltonian system. It is very powerful tool and we will use this many times in this course.

We consider next a specific case of a 1D system with a linear periodical Hamiltonian:

$$\tilde{h} = \frac{p^2}{2} + K_1(s)\frac{y^2}{2};\ \mathbf{H} = \begin{bmatrix} K_1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{D} = \mathbf{SH} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix}. \tag{25}$$

The equations of motion are simple

$$\frac{d}{ds}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -K_1 x \end{bmatrix}\ (i.e.\ x' \equiv p). \tag{26}$$

A one-turn matrix within its determinant (*ad-bc=1*)

$$\mathbf{T}(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix}; \tag{27}$$

$$Y = \begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix}; \tilde{Y} = \begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix} e^{i\psi}; \mathbf{U} = \begin{bmatrix} \mathrm{w} & \mathrm{w} \\ u + i/\mathrm{w} & u - i/\mathrm{w} \end{bmatrix}; \tilde{\mathbf{U}} = \mathbf{U} \cdot \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \tag{28}$$

where w(s) and u(s) are real functions and calibration was used for (21).

[1] We are free to multiply the eigen vector **Y** by $e^{i\phi}$ to make **a** real number. In other words we define the choice of our phase as $\tilde{Y}(s) = \begin{pmatrix} \tilde{y}_1(s) \\ \tilde{y}_2(s) \end{pmatrix}$; $\mathrm{w}(s) = |\tilde{y}_1(s)|;\ \psi(s) = \arg(\tilde{y}(s))$.

T has a trace

$$Trace(\mathbf{T}) = Trace(\Lambda) = 2\cos\mu \tag{29}$$

(because $Trace(ABA^{-1}) = Trace(B)$). Thus, the stability of motion (when μ is real!) is easy to check:

$$-2 < Trace(\mathbf{T}) < 2 \tag{30}$$

where some well-know resonances are excluded: The integer $\mu = 2\pi m$, and the half-integer $\mu = 2(m+1)\pi$ as being unstable (troublesome!).

Combining (28) into the equations of motion (25)

$$\frac{d}{ds}\begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix} e^{i\psi} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix} \cdot \tilde{Y} = \begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix} e^{i\psi} \Rightarrow \begin{matrix} \mathrm{w}' + \mathrm{i}\mathrm{w}\psi' = u + i/\mathrm{w} \\ u' - i\mathrm{w}'/\mathrm{w}^2 + \mathrm{i}\psi'\left(u + i/\mathrm{w}\right) = -K_1\mathrm{w} \end{matrix} . \tag{31}$$

Then, separating the real and imaginary parts, we have from the first equation:

$$u = \mathrm{w}'; \quad \psi' = 1/\mathrm{w}^2 . \tag{32}$$

Plugging these into the second equation yields one nontrivial equation on the envelope function, w(s):

$$\mathrm{w}'' + K_1(\mathrm{s})\mathrm{w} = \frac{1}{\mathrm{w}^3} . \tag{33}$$

Thus, the final form of the eigen vector can be rewritten as

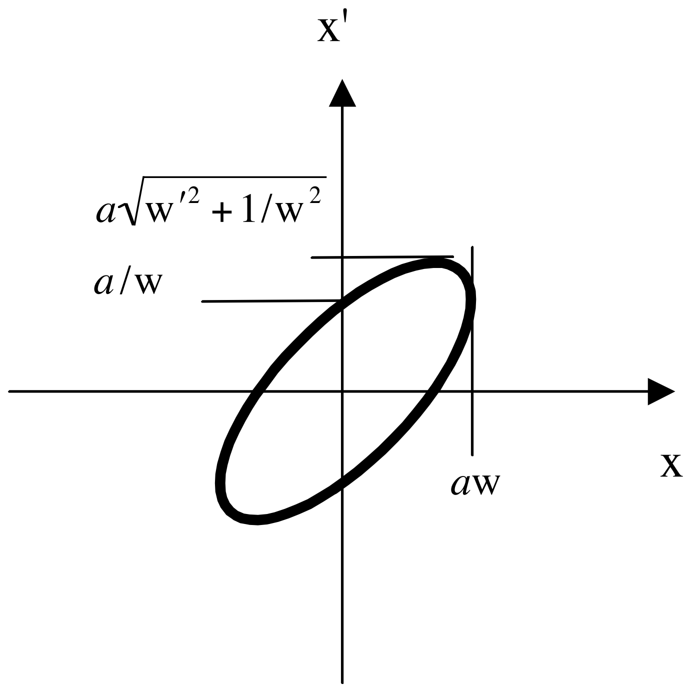
$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix}; \psi' = \frac{1}{\mathrm{w}^2}; \tilde{Y} = Y e^{i\psi} \tag{34}$$

The parameterization of the linear 1D motion is

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathrm{Re}\left(a e^{i\varphi} \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix} e^{i\psi} \right);$$

$$\begin{aligned} x &= a \cdot \mathrm{w(s)} \cdot \cos(\psi(s) + \varphi) \\ x' &= a \cdot \left(\mathrm{w}'(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) - sin(\psi(s) + \varphi)/\mathrm{w(s)} \right) \end{aligned} \tag{35}$$

where ***a*** and ***φ*** are the constants of motion.



Tradition in accelerator physics calls for using the so-called β-function, which simply a square of the envelope function:

$$\beta \equiv \mathrm{w}^2 \;\Rightarrow\; \psi' = 1/\beta. \tag{36}$$

and a wavelength of oscillations divided by 2π. Subservient functions are defined as

$$\alpha \equiv -\beta' \equiv -\mathrm{w}\,\mathrm{w}',\; \gamma \equiv \frac{1+\alpha^2}{\beta}. \tag{37}$$

While α,β,γ are frequently used in accelerator physics, unless they are equiped with indicies $\alpha_{x,y}$,$\beta_{x,y}$,$\gamma_{x,y}$, they can be easily mistaken with relativistic factors β and γ. Beware of this possibility and see in what contest α,β,γ are used.

Manipulations with them is much less transparent, and oscillation (35) looks like

$$\begin{aligned} x &= a \cdot \sqrt{\beta(\mathrm{s})} \cdot \cos(\psi(s)+\varphi) \\ x' &= -\frac{a}{\sqrt{\beta(\mathrm{s})}} \cdot \big(\alpha(\mathrm{s}) \cdot \cos(\psi(s)+\varphi) + \sin(\psi(s)+\varphi)\big) \end{aligned} \tag{38}$$

Finally, (13') gives us a well-known feature in AP parameterization of a one-turn matrix:

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu; \quad \mathbf{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix} \tag{39}$$

We will discuss the features of **J**=ln(**T**) *for a general case in next class*

Action-angle variables

Another important transformation used (not-only!) in accelerator physics is the transformation to the action-angle variables $\left\{ \varphi_k, I_k = \frac{a_k^{\ 2}}{2} \right\}$. Usually this requires two steps: The first is

$$\left\{ \tilde{q}_k = \frac{a_k e^{i\varphi_k}}{\sqrt{2}}, \tilde{p}_k = i \frac{a_k e^{-i\varphi_k}}{\sqrt{2}} \right\}. \tag{40}$$

The second step is very simple, It is known from classical theory of harmonic oscillators. A Canonical transformation

$$\left\{ q_k = \varphi_k; p_k \equiv I_k = \frac{a^2}{2} \right\} \Leftrightarrow \left\{ \tilde{q}_k = \frac{a_k e^{i\varphi_k}}{\sqrt{2}}, \tilde{p}_k = i \frac{a_k e^{-i\varphi_k}}{\sqrt{2}} \right\}$$

$$F(q,\tilde{q}) = -\sum_{k=1}^{n} i \frac{\tilde{q}_k^2}{2} e^{-2i\varphi_k}; \frac{\partial F}{\partial s} = 0 \to \tilde{H} = H \qquad . \tag{41}$$

$$I_k = \frac{\partial F}{\partial q_k} \equiv \frac{\partial F}{\partial \varphi_k} = \tilde{q}_k^2 e^{-2i\varphi_k} = \frac{a_k^2}{2};\ \tilde{p}_k = -\frac{\partial F}{\partial \tilde{q}_k} = i\tilde{q}_k e^{-2i\varphi_k} i \frac{a_k e^{-i\varphi_k}}{\sqrt{2}}.$$

First step we already went when we discussed HW9. Let us now, again, demonstrate that symplectic transformation $X(s) \Rightarrow \tilde{X}(s)$

$$X(s) = \mathbf{V}(s)\tilde{X}, \quad \mathbf{V}'(s) = \mathbf{SH}(s)\mathbf{V}(s). \tag{42}$$

is canonical. Beginning from a Hamiltonian composed of two parts, a linear part and an arbitrary one

$$\mathcal{H} = \frac{1}{2} X^T \mathbf{H}(s) X + \mathcal{H}_1(X,s). \tag{43}$$

The equation of motion

$$\frac{dX}{ds} = \mathbf{S} \cdot \frac{\partial \mathcal{H}}{\partial X} = \mathbf{SH}(s) \cdot X + \mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial X}. \tag{44}$$

becomes with substitution (41)

$$\left(\mathbf{V}\tilde{X}\right)' = \mathbf{SHV} \cdot \tilde{X} + \mathbf{V}\tilde{X}' = \mathbf{SH}(s) \cdot \mathbf{V}\tilde{X} + \mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial X} \Rightarrow \mathbf{V}\tilde{X}' = \mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial X}. \tag{45}$$

equivalent to the equations of motion with the new Hamiltonian: $\mathcal{H}_1(\mathbf{V}\tilde{X},s)$

$$\tilde{X}' = \mathbf{V}^{-1}\mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial X}; \frac{\partial}{\partial X} = \mathbf{V}^{-1T} \frac{\partial}{\partial \tilde{X}} \Rightarrow \tilde{X}' = \left(\mathbf{V}^{-1}\mathbf{S}\mathbf{V}^{-1T}\right) \cdot \frac{\partial \mathcal{H}_1}{\partial \tilde{X}} \Rightarrow \tilde{X}' = \mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial \tilde{X}}. \tag{46}$$

This result (even though expected) has long-lasting consequences – the trivial (linear) part in the Hamiltonian can be removed from equations of motion, so allowing one to use this in perturbation theory or at least to focus only on non-trivial part of the motion.

$$\mathbf{V} = \frac{1}{\sqrt{2}}[Y_1, iY_1......] \Rightarrow \mathbf{V}^T\mathbf{S}\mathbf{V} = \mathbf{S} \ \# \tag{47}$$

Finally, we know that for any canonical transformation:

$$X^T \equiv \{q_k, p_k\} \Leftrightarrow A^T \equiv \left\{\tilde{q}_k = \frac{a_k e^{i\varphi_k}}{\sqrt{2}}, \tilde{p}_k = i\frac{a_k e^{-i\varphi_k}}{\sqrt{2}}\right\}. \tag{48}$$
$$\tilde{H} = H + \frac{\partial F(q,\tilde{q},s)}{\partial s}$$

But by design for a linear Hamiltonian system,

$$H_L = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T \cdot \mathbf{H}(s) \cdot X \tag{49}$$

A^T = consts. It means that

$$\frac{\partial F(q,\tilde{q},s)}{\partial s} = -H_L \tag{50}$$

Thus, if we are applying transformation of the action-angle Canonical variables of an arbitrary (in general case, nonlinear) Hamiltonian system

$$H(X,s) = H_L(X,s) + H_1(X,s) \tag{51}$$

we will come to the reduced equations of motion with the Hamiltonian:

$$\begin{gathered}\tilde{H} = H + \frac{\partial F}{\partial s} = H - H_L = H_1(X,s);\\ \tilde{H}(A,s) = H_1(X(A,s),s).\end{gathered} \tag{51}$$

where we eliminated "boring" oscillating part of the motion. Since next step of transformation to the action-angle variables (41) does not change the Hamiltonian, we finally get:

$$\begin{gathered}\tilde{H}(\varphi_k, I_k, s) = H_1(X(\varphi_k, I_k, s), s);\\ \frac{d\varphi_k}{ds} = \frac{\partial \tilde{H}}{\partial I_k}; \frac{dI_k}{ds} = -\frac{\partial \tilde{H}}{\partial \varphi_k}.\end{gathered} \tag{52}$$

These "reduced" equations of motion can be very useful when H_1 can be treated as perturbation or in studies of non-linear map. We will return to them again and again through the course.

If you fill uncomfortable using complex Canonical variables (48), you can perform identical (but a bit lengthier) exercise with $\left\{\tilde{q}_k = a_k \cos\varphi_k, \tilde{p}_k = -a_k \sin\varphi_k\right\}$ and rewriting (24) using real and imaginary part of the eigen vectors. In any case , you will reach the same conclusions.