Homework 12

Problem 1. 10 points, 2D distribution function and RMS beam sizes

For the case of fully coupled transverse oscillations with eigen vectors

$$Y_1 = \begin{bmatrix} w_{1x} e^{i\varphi_{1x}} \\ \left(u_{1x} + i\dfrac{q}{w_{1x}} \right) e^{i\varphi_{1x}} \\ w_{1y} e^{i\varphi_{1y}} \\ \left(u_{1y} + i\dfrac{1-q}{w_{1y}} \right) e^{i\varphi_{1y}} \end{bmatrix}; \quad Y_2 = \begin{bmatrix} w_{2x} e^{i\varphi_{2x}} \\ \left(u_{2x} + i\dfrac{1-q}{w_{2x}} \right) e^{i\varphi_{2x}} \\ w_{2y} e^{i\varphi_{2y}} \\ \left(u_{2y} + i\dfrac{q}{w_{2y}} \right) e^{i\varphi_{2y}} \end{bmatrix}$$

and known values of eigen emittances $\varepsilon_{1,2} \equiv I_{1,2} = \dfrac{\langle a_{1,2}^2 \rangle}{2}$ of stationary Gaussian distribution (solution of Fokker-Plank equation)

(a) **6 points**; Write explicit expression for the distribution function in terms of x, Px, y and Py.

(b) **4 points**; Write expression of the RMS beam sizes

$$\sigma_x = \sqrt{< x^2 >}; \sigma_y = \sqrt{< y^2 >}$$

using beam emittances and necessary components of eigen vectors.

Solution: (a) The normalized distribution function in phase-action variable is

$$f(I,\varphi) = \frac{1}{(2\pi)^2 \varepsilon_1 \varepsilon_2} \exp\left(-\frac{I_1}{\varepsilon_1} - \frac{I_2}{\varepsilon_2} \right);$$

$$\int_0^\infty dI_1 \int_0^\infty dI_2 \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \cdot f(I,\varphi) = 1.$$

Now we need to find explicit expression for $I_{1,2}$ using

$$Y_1 = \begin{bmatrix} w_{1x} e^{i\varphi_{1x}} \\ (u_{1x} + iv_{1x}) e^{i\varphi_{1x}} \\ w_{1y} e^{i\varphi_{1y}} \\ (u_{1y} + iv_{1y}) e^{i\varphi_{1y}} \end{bmatrix}; \quad Y_2 = \begin{bmatrix} w_{2x} e^{i\varphi_{2x}} \\ (u_{2x} + iv_{2x}) e^{i\varphi_{2x}} \\ w_{2y} e^{i\varphi_{2y}} \\ (u_{2y} + iv_{2y}) e^{i\varphi_{2y}} \end{bmatrix};$$

$$v_{1x} = \frac{q}{w_{1x}}; v_{1y} = \frac{1-q}{w_{1y}}; v_{2x} = \frac{1-q}{w_{2x}}; v_{2y} = \frac{q}{w_{2y}}.$$

were c.c. stand for complex conjugate components:

$$X=\mathrm{Re}\left(a_1Y_1e^{i\psi_1}+a_2Y_2e^{i\psi_2}\right)=\frac{1}{2}\left(a_1Y_1e^{i\psi_1}+a_2Y_2e^{i\psi_2}+c.c.\right);Y_j^{*T}SY_k=2i\delta_{jk};Y_j^{*T}SY^*{}_k=0;$$

$$Y_k^{*T}SX=\frac{1}{2}a_kY_k^{*T}SY_k;\;Y_k^{*T}SY_k=2i\to a_i=-iY_i^{*T}SX;I_k=\frac{\left|a_k\right|^2}{2}=\frac{\left|Y_k^{*T}SX\right|^2}{2}\equiv\frac{\left|X^TSY_k\right|^2}{2};$$

$$X^TS=\left[x,P_x,y,P_y\right]\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}=\left[-P_x,x,-P_y,y\right];$$

$$v_{kx}=\frac{q_{kx}}{w_{kx}};v_{ky}=\frac{q_{kx}}{w_{1y}};q_{1x}=q_{2y}=q;\;q_{2x}=q_{1y}=1-q;$$

$$X^TSY_k=e^{i\varphi_{kx}}\left(x\left(u_{kx}+iv_{kx}\right)-w_{kx}P_x\right)+e^{i\varphi_{ky}}\left(y\left(u_{1y}+iv_{1y}\right)-w_{1y}P_y\right);$$

$$\left|X^TSY_k\right|^2=f_{kx}+f_{ky}+f_{kxy}$$

$$f_{kx}=\left|x\left(u_{kx}+iv_{kx}\right)-w_{kx}P_x\right|^2;f_{ky=}\left|y\left(u_{ky}+iv_{ky}\right)-w_{ky}P_y\right|^2$$

$$f_{kxy}=2\,\mathrm{Re}\,e^{i\left(\varphi_{kx}-\varphi_{ky}\right)}\left(x\left(u_{kx}+iv_{kx}\right)-w_{kx}P_x\right)\left(y\left(u_{1y}-iv_{1y}\right)-w_{1y}P_y\right)$$

with exlicit expression being (as expected) long and ugly:

$$f_{kx}=\left|x\left(u_{kx}+iv_{kx}\right)-w_{kx}P_x\right|^2=x^2v_{kx}{}^2+\left(u_{kx}x-w_{kx}P_x\right)^2=x^2\left(v_{kx}{}^2+u_{kx}{}^2\right)-2u_{kx}w_{kx}xP_x+w_{kx}{}^2P_x{}^2$$

$$f_{ky}=\left|y\left(u_{ky}+iv_{ky}\right)-w_{ky}P_y\right|^2=y^2v_{ky}{}^2+\left(u_{ky}x-w_{ky}P_y\right)^2=y^2\left(v_{ky}{}^2+u_{ky}{}^2\right)-2u_{ky}w_{ky}yP_y+w_{ky}{}^2P_y{}^2$$

$$f_{kxy}=2\,\mathrm{Re}\,e^{i\left(\varphi_{kx}-\varphi_{ky}\right)}\left(A+iB\right)$$

$$A=\left\{\left(xu_{kx}-w_{kx}P_x\right)\left(yu_{1y}-w_{1y}P_y\right)+xyv_{kx}v_{1y}\right\}$$

$$B=\left\{xv_{kx}\left(yu_{1y}-w_{1y}P_y\right)-yv_{1y}\left(xu_{kx}-w_{kx}P_x\right)\right\}$$

$$f_{kxy}=2\cos\left(\varphi_{kx}-\varphi_{ky}\right)\left\{\left(xu_{kx}-w_{kx}P_x\right)\left(yu_{1y}-w_{1y}P_y\right)+xyv_{kx}v_{1y}\right\}$$
$$-2\sin\left(\varphi_{kx}-\varphi_{ky}\right)\left\{xv_{kx}\left(yu_{1y}-w_{1y}P_y\right)-yv_{1y}\left(xu_{kx}-w_{kx}P_x\right)\right\}$$

with 4D distribution function:

$$f\left(X\right)=\frac{1}{\left(2\pi\right)^2\varepsilon_1\varepsilon_2}e^{-\frac{f_{1x}+f_{1y}+f_{1xy}}{2\varepsilon_1}}e^{-\frac{f_{2x}+f_{2y}+f_{2xy}}{2\varepsilon_2}};$$

$$\int\iiint f\left(X\right)dx\,dP_x\,dy\,dP_y=\int dX^4=1.$$

One can try to introduce α and β components similar to 1D motion

$$f_{kx} = \frac{x^2\left(q_{kx}{}^2 + \alpha_{kx}{}^2\right)}{\beta_{kx}} + 2\alpha_{kx}P_x + \beta_{kx}P_x^2;\ \beta_{kx} = w_{kx}{}^2; \alpha_{kx} = -w_{kx}u_{kx};$$

$$f_{ky} = \frac{y^2\left(q_{ky}{}^2 + \alpha_{ky}{}^2\right)}{\beta_{ky}} + 2\alpha_{ky}P_y + \beta_{ky}P_y^2;\ \beta_{ky} = w_{ky}{}^2; \alpha_{ky} = -w_{ky}u_{ky};$$

but it does not make expressions better, especially for f_{xy}.

Another way of writing it is to use real and imaginary parts of the eigen vectors

$$Y_k = R_k + iQ_k;$$

$$\left|a_k\right|^2 = \left|X^T S R_k + i X^T S Q_k\right| = \left(X^T S R_k\right)^2 + \left(X^T S Q_k\right)^2;$$

$$\left|a_k\right|^2 = \left(x r_{k2} - P_x r_{k1} + y r_{k4} - P_y r_{k3}\right)^2 + \left(x q_{k2} - P_x q_{k1} + y q_{k4} - P_y q_{k3}\right)^2;$$

$$f\left(X\right) = f_1\left(X\right) f_2\left(X\right);$$

$$f_k\left(X\right) = \frac{1}{2\pi\varepsilon_k}\exp\left(-\frac{\left(x r_{k2} - P_x r_{k1} + y r_{k4} - P_y r_{k3}\right)^2 + \left(x q_{k2} - P_x q_{k1} + y q_{k4} - P_y q_{k3}\right)^2}{2\varepsilon_x}\right).$$

(b) This is much easier task:

$$x = a_1 w_{1x}\cos\varphi_{1x} + a_2 w_{2x}\cos\varphi_{2x};$$

$$y = a_1 w_{1y}\cos\varphi_{1y} + a_2 w_{2y}\cos\varphi_{2y};$$

$$\left\langle x^2\right\rangle = a_1{}^2 w_{1x}{}^2\left\langle\cos^2\varphi_{1x}\right\rangle + 2a_1 w_{1x} a_2 w_{2x}\left\langle\cos\varphi_{1x}\cos\varphi_{2x}\right\rangle + a_2{}^2 w_{2x}{}^2\left\langle\cos^2\varphi_{2x}\right\rangle;$$

$$\left\langle y^2\right\rangle = a_1{}^2 w_{1y}{}^2\left\langle\cos^2\varphi_{1y}\right\rangle + 2a_1 w_{1y} a_2 w_{2y}\left\langle\cos\varphi_{1y}\cos\varphi_{2y}\right\rangle + a_2{}^2 w_{2y}{}^2\left\langle\cos^2\varphi_{2y}\right\rangle;$$

Since phases of oscillation are random are not correlated between to modes, we have

$$\left\langle\cos^2\varphi_{kx}\right\rangle = \left\langle\cos^2\varphi_{ky}\right\rangle = \frac{1}{2}; \left\langle\cos\varphi_{1x}\cos\varphi_{2x}\right\rangle = 0;$$

$$\left\langle x^2\right\rangle = \frac{a_1{}^2 w_{1x}{}^2 + a_2{}^2 w_{2x}{}^2}{2} = \varepsilon_1\beta_{1x} + \varepsilon_2\beta_{2x}; \sigma_x = \sqrt{\varepsilon_1\beta_{1x} + \varepsilon_2\beta_{2x}};$$

$$\left\langle y^2\right\rangle = \frac{a_1{}^2 w_{1y}{}^2 + a_2{}^2 w_{2y}{}^2}{2} = \varepsilon_1\beta_{1y} + \varepsilon_2\beta_{2y}; \sigma_y = \sqrt{\varepsilon_1\beta_{1y} + \varepsilon_2\beta_{2y}}.$$

Problem 2. 10 points, 3D distribution function and RMS beam sizes

(a) 5 points: For the case of fully coupled transverse oscillations with eigen vectors

$$Y_k(s) = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + i\dfrac{q_{kx}}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + i\dfrac{q_{ky}}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{w}_{k\tau} e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau} + i\dfrac{q_{k\tau}}{\mathrm{w}_{k\tau}} \right) e^{i\chi_{k\tau}} \end{bmatrix}; k = 1,2,3$$

and known values of eigen emittances $\varepsilon_k \equiv I_k = \dfrac{\left\langle a_k^2 \right\rangle}{2}; k = 1,2,3$ of stationary Gaussian distribution, write expression of the RMS beam sizes

$$\sigma_x = \sqrt{< x^2 >}; \sigma_y = \sqrt{< y^2 >}; \sigma_\tau = \sqrt{< \tau^2 >}$$

using the beam emittances and necessary components of eigen vectors.

(b) 5 points: For the case of slow synchrotron oscillations and approximate expressions for the eigen vectors:

$$Y_k = \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + \dfrac{iq_k}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + \dfrac{i(1-q_k)}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{y}_{k\tau} = \eta^T S Y_{k\beta} \\ 0 \end{bmatrix}; k = 1,2;\ Y_\delta = \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix} = \begin{bmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \\ \chi_\tau \\ 1 \end{bmatrix};$$

and known values of eigen emittances $\varepsilon_k \equiv I_k = \frac{\langle a_k^2 \rangle}{2}; k = 1,2$ and RMS values of the relative energy spread $\sigma_\delta = \sqrt{\langle \delta^2 \rangle}$ write expressions for transverse beam sizes:

$$\sigma_x = \sqrt{< x^2 >}; \sigma_y = \sqrt{< y^2 >}$$

Solution:

(a) Treatment is identical to the RMS beam sizes in the previous problem:

$$x = a_1 w_{1x} \cos\varphi_{1x} + a_2 w_{2x} \cos\varphi_{2x} + a_3 w_{3x} \cos\varphi_{3x};$$
$$y = a_1 w_{1y} \cos\varphi_{1y} + a_2 w_{2y} \cos\varphi_{2y} + a_3 w_{3y} \cos\varphi_{3y};$$
$$\tau = a_1 w_{1\tau} \cos\varphi_{1\tau} + a_2 w_{2\tau} \cos\varphi_{2\tau} + a_3 w_{3y} \cos\varphi_{3\tau};$$
$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\frac{a_1^2 w_{1x}^2 + a_2^2 w_{2x}^2 + a_3^2 w_{3x}^2}{2}} = \sqrt{\varepsilon_1 w_{1x}^2 + \varepsilon_2 w_{2x}^2 + \varepsilon_3 w_{3x}^2};$$
$$\sigma_y = \sqrt{\langle y^2 \rangle} = \sqrt{\frac{a_1^2 w_{1y}^2 + a_2^2 w_{2y}^2 + a_3^2 w_{3y}^2}{2}} = \sqrt{\varepsilon_1 w_{1y}^2 + \varepsilon_2 w_{2y}^2 + \varepsilon_3 w_{3y}^2};$$
$$\sigma_\tau = \sqrt{\langle \tau^2 \rangle} = \sqrt{\frac{a_1^2 w_{1\tau}^2 + a_2^2 w_{2\tau}^2 + a_3^2 w_{3\tau}^2}{2}} = \sqrt{\varepsilon_1 w_{1\tau}^2 + \varepsilon_2 w_{2\tau} + \varepsilon_3 w_{3\tau}};$$

(b) In this specific case we have

$$Y_k = \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + \frac{iq_k}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + \frac{i(1-q_k)}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{y}_{k\tau} = \eta^T S Y_{k\beta} \\ 0 \end{bmatrix}; k = 1,2;\ Y_\delta = \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix} = \begin{bmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \\ \chi_\tau \\ 1 \end{bmatrix}$$

$$\sigma_x^2 = \varepsilon_1 w_{1x}^2 + \varepsilon_2 w_{2x} + \eta_x^2 \sigma_\delta^2$$
$$\sigma_y^2 = \varepsilon_1 w_{1y}^2 + \varepsilon_2 w_{2y} + \eta_y^2 \sigma_\delta^2$$
$$\sigma_\tau^2 = \varepsilon_1 \left(\eta^T S Y_{1\beta} \right)^2 + \varepsilon_2 \left(\eta^T S Y_{2\beta} \right)^2 + + \chi_\tau^2 \sigma_\delta^2;$$

Problem 3. (1) Let's prove that

$$\begin{bmatrix} \tilde{y} \\ \tilde{y}' \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} = M \begin{bmatrix} y \\ y' \end{bmatrix} = M \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \tilde{\Sigma} = M \Sigma M^T$$

by observing that

$$\Sigma_{ij} = \langle X_i X_j \rangle;$$

$$\tilde{\Sigma}_{ij} = \tilde{\Sigma}_{ji} = \langle \tilde{X}_i \tilde{X}_j \rangle = \langle M_{ik} X_k X_n M_{nj} \rangle =$$

$$= M_{ik} \langle X_k X_n \rangle M_{nj} = M_{ik} \Sigma_{ij} M_{nj}$$

(where we use the fact that one can extract constants from the averaging brackets) which in matrix form is equivalent to

$$\tilde{\Sigma} = M \Sigma M^T$$

The rest is easy since $\det M = 1$:

$$\det \tilde{\Sigma} = \det M \det \Sigma \det M^T = \det \Sigma$$

Shorter proof: $\Sigma = X \otimes X^T \to \tilde{\Sigma} = \tilde{X} \otimes \tilde{X}^T = M \cdot X \otimes X^T \cdot M^T = M \cdot \Sigma \cdot M^T$ #

(2) Let's remember that

$$y = a \mathrm{w}_y \cos\psi_y; \; y' = a \left(\mathrm{w}'_y \cos\psi_y - \frac{1}{\mathrm{w}_y} \sin\psi_y \right)$$

and calculate averages using randomness of particles' phases

$$\langle \cos^2\psi_y \rangle = \frac{1}{2}; \langle \cos\psi_y \sin\psi_y \rangle = 0; \langle \sin^2\psi_y \rangle = \frac{1}{2}; \frac{\langle a^2 \rangle}{2} = \varepsilon_y$$

$$\langle y^2 \rangle = \langle a^2 \mathrm{w}_y{}^2 \cos^2\psi_y \rangle = \mathrm{w}_y{}^2 \frac{\langle a^2 \rangle}{2} = \beta_y \frac{\langle a^2 \rangle}{2};$$

$$\langle y y' \rangle = \left\langle a^2 \mathrm{w}_y \cos\psi_y \left(\mathrm{w}'_y \cos\psi_y - \frac{1}{\mathrm{w}_y} \sin\psi_y \right) \right\rangle = \mathrm{w}_y \mathrm{w}'_y \frac{\langle a^2 \rangle}{2} = -\alpha_y \frac{\langle a^2 \rangle}{2};$$

$$\langle y'^2 \rangle = \left\langle a^2 \left(\mathrm{w}'_y \cos\psi_y - \frac{1}{\mathrm{w}_y} \sin\psi_y \right)^2 \right\rangle = \frac{1 + (\mathrm{w}'_y \mathrm{w}_y)^2}{\mathrm{w}_y{}^2} \frac{\langle a^2 \rangle}{2} = \frac{1 + \alpha_y{}^2}{\beta_y}.$$

$$\Sigma = \begin{bmatrix} \langle y^2 \rangle & \langle y y' \rangle \\ \langle y y' \rangle & \langle y'^2 \rangle \end{bmatrix} = \varepsilon_y \begin{bmatrix} \beta_y & -\alpha_y \\ -\alpha_y & \dfrac{1 + \alpha_y{}^2}{\beta_y} \end{bmatrix}.$$

Thus, for 1D case it one can use this relation to design matched lattice for a given Σ matrix of the beam – for example at injection point into a storage ring. This matching minimizes RMS amplitudes of particles oscillation in the storage ring.