Homework for Lecture 14, due Apr.7 2014

1. Calculate the relative relation between $\Delta V/V$, $\Delta P/P$ and $\Delta E/E$. Note V is the velocity of the particle, P is the amplitude of the momentum and E is the energy of the particle.

   Answer: From the relation $E = \sqrt{m^2c^4 + P^2c^2}$, we have $dE = c^2PdP/E$. Divide E^2 on both side we have $dE/E = (cP/E)^2 dP/P$. Note that $\beta = cP/E$, we get $\frac{dE}{E} = \frac{\beta^2 dP}{P}$. Then we start from the relation $\beta = cP/E$, the relative change is $d\beta/\beta = dP/P - dE/E = (1-\beta^2)dP/P = (1/\gamma^2)(dP/P)$

2. In class, we transform the longitudinal map

$$\begin{aligned}\delta_{n+1} - \delta_n &= \frac{eV}{\beta^2 E_0}(\sin\phi_n - \sin\phi_s) \\ \phi_{n+1} - \phi_n &= 2\pi h\eta\delta_{n+1}\end{aligned}$$

   to longitudinal effective Hamiltonian. Actually we can also establish the one turn matrix for longitudinal motion if assume $\phi_n = \phi_s + \Delta_n$, where $|\Delta_n| \ll 1$. Find this matrix for $(\delta_{n+1}, \Delta_{n+1})$ from (δ_n, Δ_n) . Find the tune for this map, by assuming the tune is very close to zero, which is true in ring accelerator.

   Answer: The matrix can be built as:

$$\begin{aligned}\begin{pmatrix}\delta_{n+1} \\ \phi_{n+1}\end{pmatrix} &= \begin{pmatrix}1 & 0 \\ 2\pi h\eta & 1\end{pmatrix}\begin{pmatrix}\delta_{n+1} \\ \phi_n\end{pmatrix} \\ &= \begin{pmatrix}1 & 0 \\ 2\pi h\eta & 1\end{pmatrix}\begin{pmatrix}1 & \frac{eV\cos(\phi_s)}{\beta^2E_0} \\ 0 & 1\end{pmatrix}\begin{pmatrix}\delta_n \\ \phi_n\end{pmatrix} \\ &= \begin{pmatrix}1 & \frac{eV\cos(\phi_s)}{\beta^2E_0} \\ 2\pi h\eta & 1+\frac{2\pi eVh\eta\cos(\phi_s)}{\beta^2E_0}\end{pmatrix}\begin{pmatrix}\delta_n \\ \phi_n\end{pmatrix}\end{aligned}$$

   The trace of the one-turn longitudinal matrix is $Tr = 2+2\pi eV\eta\cos(\phi_s)/\beta^2E_0$. Compare with the transverse case the one turn map is

$$\begin{pmatrix}\cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\gamma\sin\psi & \cos\psi - \alpha\sin\psi\end{pmatrix}$$

   where $\psi = 2\pi\nu_s$. The longitudinal phase advance can be found from the relation $2\cos\psi = Tr = 2 + 2\pi eV\eta\cos(\phi_s)/\beta^2E_0$. Again we know that only if $\eta\cos\phi_s < 0$, we can find tune, i.e. the motion is stable. Taylor expend $\cos\psi \sim 1 - \psi^2/2$, we can easily find

$$\nu_s = \sqrt{\frac{-eVh\eta\cos\phi_s}{2\pi\beta^2E_0}}$$

also we can find 'beta function' for longitudinal motion.

$$\begin{aligned}\beta \sin\psi &= \frac{eV\cos\phi_s}{\beta^2 E_0} \\ \beta &= \sqrt{\frac{-eV\cos\phi_s}{2\pi h\eta\beta^2 E_0}}\end{aligned}$$

The ellipse of phase space is near-up-right one ($\alpha \sim \nu_s$, small).

3. For the example in class, find the synchrotron tune for both 100GeV case and 15GeV proton ring. The relative parameter is the cavity has 5MV voltage, 360 harmonic. Compaction factor $\alpha_c = 0.002$. The RF phase is zero or π. How does the number change if the same ring is for 3GeV electron beam.

   Answer: Find the gamma then beta of the the particle with the mass, proton mass is 938 MeV and electron mass is 0.511MeV, then plug in the number to find ν_s.