Electron storage rings

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Modern synchrotron radiation theory was formulated by many physicists; in particular, its foundation was laid by **J. Schwinger**. Some of his many important results are summarized below: [J. Schwinger, Phys. Rev. **70**, 798 (1946); **75**, 1912 (1949); Proc. Nat. Acad. Sci. **40**, 132 (1954).]

[1] The angular distribution of synchrotron radiation is sharply peaked in the direction of the electron's velocity vector within an angular width of $1/\gamma$, where γ is the relativistic energy factor. The radiation is plane polarized on the plane of the electron's orbit, and elliptically polarized outside this plane.

[2] The radiation spans a continuous spectrum. The power spectrum produced by a high energy electron extends to a critical frequency $\omega_c=3\gamma^3\omega_\rho/2$, where $\omega_\rho=c/\rho$ is the cyclotron frequency for electron moving at the speed of light. [D.H. Tomboulin and P.L. Hartman experimentally verified that electrons at high energy (70 MeV then) could emit extreme ultraviolet (XUV) photons; Phys. Rev. **102**, 1423 (1956).]

[3] Quantum mechanical correction becomes important only when the critical energy of the radiated photon, $\hbar\omega=(3/2)\hbar c\gamma^3/\rho$ is comparable to the electron beam energy, $E=\gamma mc^2$. This occurs when the electron energy reaches $mc^2(mc\rho/\hbar)^{1/2} \sim 10^6$ GeV. The beamstrahlung parameter, defined as $Y=(2/3)\,\hbar\omega_c/E$ is a measure of the importance of quantum mechanical effects.

Physics of Electron Storage Rings

According to Larmor's theorem, the instantaneous radiated power from an accelerated electron is

$$P = \frac{e^2 \dot{v}^2}{6\pi\varepsilon_0 c^3} = \frac{2r_0}{3mc}\left(\frac{dp_i}{d\tau}\cdot\frac{dp_i}{d\tau}\right) = \frac{2r_0}{3mc}\left(\left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right)$$

$d\tau = dt/\gamma$ is the proper time. The radiation power from circular motion in a dipole becomes

$$P = \frac{2r_0}{3mc}\left(\frac{d\vec{p}}{d\tau}\right)^2 = \frac{2r_0}{3mc}\gamma^2\omega_\rho^2\,|\,\vec{p}\,|^2 = \frac{2r_0}{3mc}\gamma^2\,|\,F_\perp\,|^2 = \frac{\beta^4 c}{2\pi}C_\gamma\frac{E^4}{\rho^2}$$

$$C_\gamma = \frac{4\pi}{3}\frac{r_0}{(mc^2)^3} = \begin{cases} 8.846\times10^{-5}\ \mathrm{m/(GeV)^3} & \text{electron} \\ 4.840\times10^{-14}\ \mathrm{m/(GeV)^3} & \text{muon} \\ 7.783\times10^{-18}\ \mathrm{m/(GeV)^3} & \text{proton} \end{cases}$$

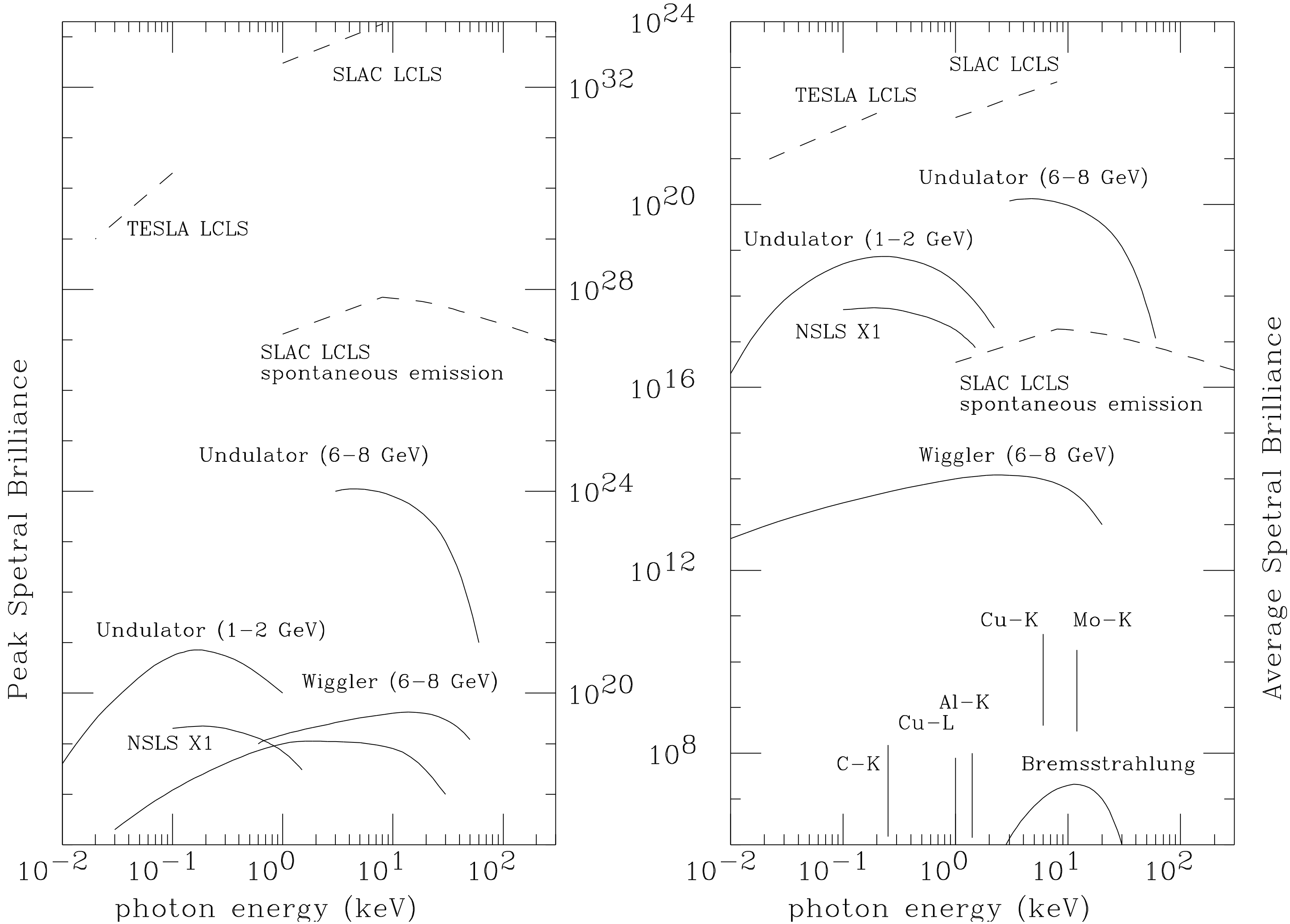
$$U_0 = \oint P\,dt = C_\gamma\beta^3E_0^4R\left\langle\frac{1}{\rho^2}\right\rangle = \frac{C_\gamma\beta^3E_0^4}{\rho}$$

$$\langle P\rangle = \frac{U_0}{T} = \frac{cC_\gamma\beta^3E_0^4}{2\pi R\rho}$$

Table 4.1: Properties of some electron storage rings.

	Colliders					Light Sources	
	BEPC	CESR	LER(e^+)	HER(e^-)	LEP	APS	ALS
E [GeV]	2.2	6	3.1	9	55	7	1.5
ν_x	5.8	9.38	32.28	25.28	76.2	35.22	14.28
ν_z	6.8	9.36	35.18	24.18	70.2	14.3	8.18
ρ [m]	10.35	60	30.6	165.0	3096.2	38.96	4.01
α [$\times 10^{-4}$]	400	152	14.9	24.4	3.866	2.374	14.3
C [m]	240.4	768.4	2199.3	2199.3	26658.9	1104	196.8
h	160	1281	3492	3492	31320	1296	328
f_{rf} [MHz]	199.5	499.8	476	476	352.2	352.96	499.65
ν_s	0.016	0.064	0.034	0.0522	0.085	0.006	0.0082
$\frac{\Delta E}{E_0}$ [$\times 10^{-4}$]	4.0	6.3	9.5	6.1	8.4	9.6	7.1
$\mathcal{A}$ [$\times 10^{-4}$eV-s]	3.5	7.2	3.1	5.7	78.	4.1	0.43
ϵ_x [nm]	450	240	64	48	51	8	4.8
ϵ_z [nm]	35	8	3.86	1.93	0.51	0.08	0.48

$$B = \frac{d^4 N}{dt\, d\Omega\, dS\, (d\lambda/\lambda)}$$ [Photons/(s-mm²-mrad²-0.1%bandwidth)]



$$\frac{dP(\omega)}{d\omega d\Omega} = \frac{\beta^2 e^2 \omega^2}{16\pi^3 \epsilon_0 c} \left| -\hat{\mathsf{e}}_{\parallel} G_{\parallel}(\omega) + \hat{\mathsf{e}}_{\perp} G_{\perp}(\omega) \right|^2 \tag{4.58}$$

where the amplitudes are

$$G_{\parallel} = \frac{1}{\omega_\rho \gamma^2}(1+X^2) \int_{-\infty}^{\infty} x e^{j\frac{3}{2}\xi(x+\frac{1}{3}x^3)} dx = \frac{2(1+X^2)}{\sqrt{3}\omega_\rho \gamma^2} K_{2/3}(\xi), \tag{4.59}$$

$$G_{\perp} = \frac{1}{\omega_\rho \gamma^2} X (1+X^2)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{j\frac{3}{2}\xi(x+\frac{1}{3}x^3)} dx = \frac{2X(1+X^2)^{1/2}}{\sqrt{3}\omega_\rho \gamma^2} K_{1/3}(\xi). \tag{4.60}$$

Thus the energy radiated per unit frequency interval per unit solid angle becomes

$$\frac{dP}{d\omega d\Omega} = \frac{3e^2}{16\pi^3 \epsilon_0 c} \gamma^2 (\frac{\omega}{\omega_c})^2 (1+X^2)^2 \left[K_{2/3}^2(\xi) + \frac{X^2}{1+X^2} K_{1/3}^2(\xi) \right], \tag{4.61}$$

where the first term in the brackets arises from the polarization vector on the plane of the orbiting electron and the second from the polarization perpendicular to the orbital plane. The angular distribution has been verified experimentally.[11] On the orbital plane, where $X = 0$, the radiation is purely plane polarized. Away from the orbital plane, the radiation is elliptically polarized.

E. Frequency spectrum of radiated energy flux

Integrating Eq. (4.61) over the entire angular range, we obtain the energy flux[12]

$$I(\omega) = \frac{\sqrt{3}e^2}{4\pi\epsilon_0 c}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(y)dy = \frac{2e^2}{9\epsilon_0 c}\gamma S(\frac{\omega}{\omega_c}), \tag{4.67}$$

where

$$S(y) = \frac{9\sqrt{3}}{8\pi}y\int_y^{\infty} K_{5/3}(y')dy', \quad \int_0^{\infty} S(y) = 1, \tag{4.68}$$

also shown in Fig. 4.4. The total instantaneous radiation power becomes

$$P_\gamma = \frac{1}{2\pi\rho}\int_0^{\infty} I(\omega)d\omega = \frac{4e^2}{36\pi\epsilon_0\rho}\gamma\omega_c = \frac{cC_\gamma}{2\pi}\frac{E_0^4}{\rho^2}, \tag{4.69}$$

where $C_\gamma = 8.85 \times 10^{-5}$ meter/(GeV)^3. This result was obtained by Liénard in 1898. The *instantaneous* power spectrum becomes

$$\tilde{I}(\omega) = \frac{1}{2\pi\rho}I(\omega) = \frac{P_\gamma}{\omega_c}S(\frac{\omega}{\omega_c}). \tag{4.70}$$

Since the energy of the photon is $\hbar\omega$, the photon flux density is

$$\frac{d\mathcal{F}}{d\Omega} = \left[\frac{I}{e}\right]\frac{3e^2}{16\pi^3\epsilon_0\hbar c}\frac{\delta\omega}{\omega}\left(\frac{\omega}{\omega_c}\right)^2(1+X^2)^2\left[K_{\frac{2}{3}}^2(\xi) + \frac{X^2}{1+X^2}K_{\frac{1}{3}}^2(\xi)\right]. \tag{4.71}$$

I.4 Quantum Fluctuation

Electromagnetic radiation is emitted in quanta of energy $u = \hbar\omega$, where $\hbar$ is Planck's constant. Let $n(u)du$ be the number of photons per unit time emitted in the frequency interval $d\omega = du/\hbar$ at frequency ω, i.e.

$$un(u)du = I(\omega)d\omega = I(\omega)\frac{du}{\hbar} \tag{4.76}$$

or

$$n(u) = \frac{P_\gamma}{u_c^2}F(\frac{u}{u_c}) = \frac{9\sqrt{3}}{8\pi}\frac{P_\gamma}{u_c^2}\int_{u/u_c}^{\infty} K_{5/3}(y)dy, \tag{4.77}$$

where

$$F(y) = \frac{1}{y}S(y), \quad \int_0^\infty F(y) = \frac{15\sqrt{3}}{8}. \tag{4.78}$$

The total number of photons emitted per second, $\mathcal{N}$, is

$$\mathcal{N} = \int_0^\infty n(u)du = \frac{15\sqrt{3}}{8}\frac{P_\gamma}{u_c} = \frac{5\alpha c\gamma}{2\sqrt{3}\rho}, \tag{4.79}$$

where $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine structure constant. The average number of photons emitted per revolution becomes

$$N_\gamma = \mathcal{N}2\pi\frac{\rho}{c} = \frac{5\pi}{\sqrt{3}}\alpha\gamma. \tag{4.80}$$

The moments of energy distribution become

$$\langle u\rangle = \frac{1}{\mathcal{N}}\int_0^\infty u n(u)du = \frac{8}{15\sqrt{3}}u_c,$$
$$\langle u^2\rangle = \frac{1}{\mathcal{N}}\int_0^\infty u^2 n(u)du = \frac{11}{27}u_c^2,$$

$$\mathcal{N}\langle u^2\rangle = C_u u_c P_\gamma = \frac{3C_uC_\gamma}{4\pi}\frac{\hbar c^2}{(mc^2)^3}\frac{E_0^7}{\rho^3}, \quad C_u = \frac{55}{24\sqrt{3}}.$$

At a fixed bending radius, the quantum fluctuation varies as the seventh power of the energy.

Table 4.2: Properties of some high energy storage rings

	BEPC	CESR	LER	HER	APS	ALS	LEP	LHC
E [GeV]	2.2	6	3.2	9	7	1.5	55	7000
ρ [m]	10.35	60	30.6	165	38.96	4.01	3096.2	3096.2
C [m]	240.4	768.4	2199.3	2199.3	1104	196.8	26658.9	26658.9
T_0 [μs]	0.80	2.56	7.34	7.34	3.68	0.66	89.	89.
U_0 [MeV]	0.20	1.91	0.30	3.52	5.45	0.11	261.	0.00060
$\tau_\parallel$ [ms]	8.8	8.0	77.	19.	4.7	8.8	19.	1.0×10^9
$\tau_\perp$ [ms]	18.	16.	155.	38.	9.4	18.	38.	2.0×10^9
u_c [keV]	2.28	7.97	2.37	9.78	19.50	1.86	119.00	0.040
N_γ	285	777	415	1166	907	194	7125	494

II Radiation Damping and Excitation

The instantaneous power radiated by a relativistic electron at energy E is

$$P_\gamma = \frac{\beta^4 c C_\gamma}{2\pi}\frac{E^4}{\rho^2} = \frac{\beta^2 e^2 c^3}{2\pi} C_\gamma E^2 B^2,$$

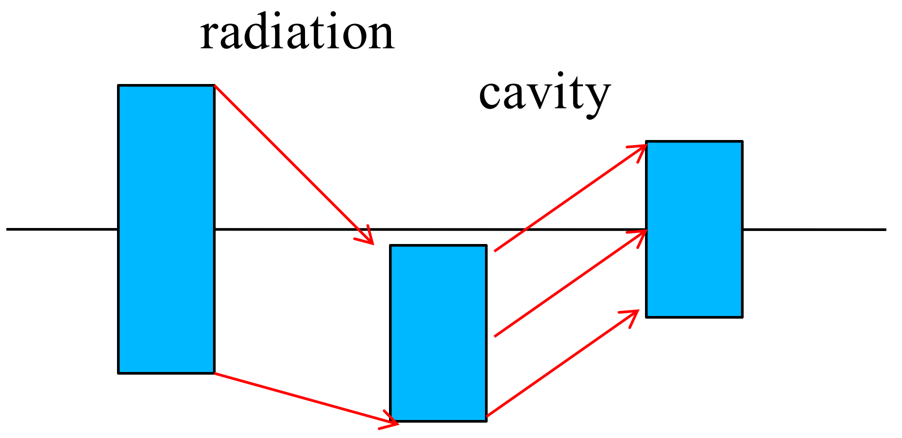
where B is the magnetic field strength, ρ is the local radius of curvature, and $C_\gamma = 8.85 \times 10^{-5}$ m/(GeV)^3 is given by Eq. (4.5). The total energy radiated in one revolution becomes

$$U_0 = \frac{C_\gamma \beta^2 E^4}{2\pi}\oint \frac{ds}{\rho^2} \xrightarrow{\text{isomagnetic}} \frac{C_\gamma E^4}{\rho} = 26.5\ (E[\mathrm{GeV}])^3\ B[\mathrm{T}]\ [\mathrm{keV}].$$

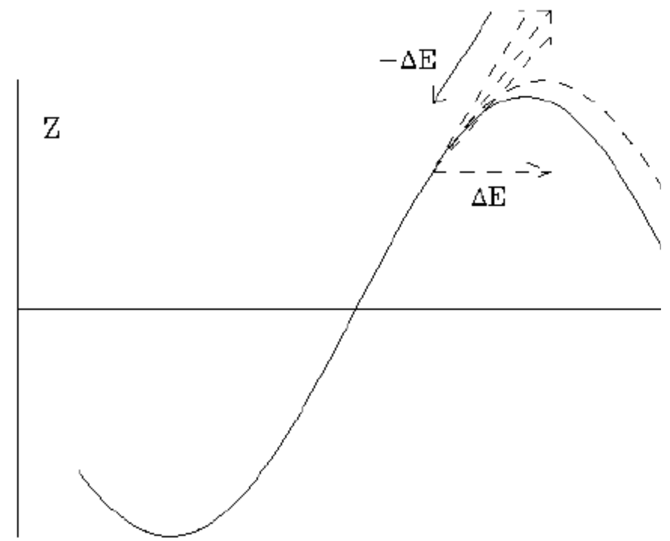
The average radiation power for an isomagnetic ring is $\langle P_\gamma \rangle = \dfrac{U_0}{T_0} = \dfrac{c C_\gamma E^4}{2\pi R \rho},$

where $T_0 = \beta c/2\pi R$ is the revolution period, and R is the average radius of a storage ring. An electron at 50 GeV in the LEP at CERN (ρ = 3.096 km) will lose 0.18 GeV per turn. The energy loss per revolution at 100 GeV is 2.9 GeV, i.e. 3% of its total energy. The energy of circulating electrons is compensated by rf cavities with longitudinal electric field.

Since higher energy electrons lose more energy than lower energy electrons and the average beam energy is compensated by longitudinal electric field, there is radiation damping (cooling) in the longitudinal phase space.



Electrons lose energy in a cone with an angle about $1/\gamma$ of their instantaneous velocity vector, and gain energy through rf cavities in the longitudinal direction. This mechanism provides transverse phase-space damping.



The damping (e-folding) time is generally equal to the time it takes for the beam to lose all of its energy.

II.1 Damping of Synchrotron Motion

$$U(E) = U_0 + W\Delta E, \quad W = \left.\frac{dU}{dE}\right|_{E=E_0},$$

($c\tau_s$, 0): the longitudinal phase-space coordinates of a synchronous particle.

($c(\tau+\tau_s)$,ΔE): of a particle with energy deviation ΔE from the synchronous energy.

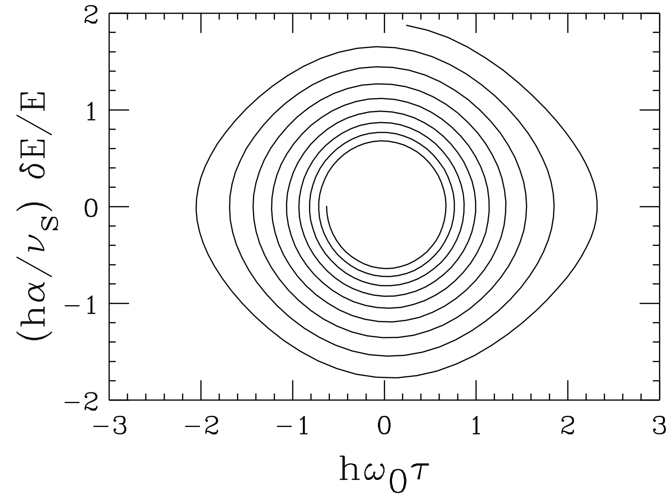
The path length difference between these two particles is $\Delta C = \alpha_c C\, \Delta E/E$, where α_c is the momentum compaction factor, C is the accelerator circumference, and the difference in arrival time is

During one revolution, the electron loses energy U(E) by radiation, and gains energy eV (τ) from the rf system. Thus the net energy change is

$$V(\tau) = V_0 \sin\phi = V_0 \sin\omega_{\rm rf}(\tau + \tau_s),$$

$$U_{\rm rf} = eV(\tau) = U_0 + e\dot{V}\tau$$

$$U_0 = eV_0\sin(\omega_{\rm rf}\tau_s), \quad \dot{V} = \omega_{\rm rf}V_0\cos(\omega_{\rm rf}\tau_s).$$



$$\tau(t) = Ae^{-\alpha_E t}\cos(\omega_s t - \theta_0).$$

$$\Delta\tau = \alpha_c\frac{C}{c}\frac{\Delta E}{E} = \alpha_c T_0\frac{\Delta E}{E}.$$

$$\frac{d\tau}{dt} = \alpha_c\frac{\Delta E}{E}$$

$$\frac{d(\Delta E)}{dt} = \frac{eV(\tau) - U(E)}{T_0},$$

$$\frac{d(\Delta E)}{dt} = \frac{1}{T_0}(e\dot{V}\tau - W\Delta E).$$

$$\boxed{\frac{d^2\tau}{dt^2} + 2\alpha_E\frac{d\tau}{dt} + \omega_s^2\tau = 0,}$$

$$\alpha_E = \frac{W}{2T_0}, \quad \omega_s^2 = -\frac{\alpha_c e\dot{V}}{T_0 E}.$$

$$\frac{d^2\tau}{dt^2} + 2\alpha_E \frac{d\tau}{dt} + \omega_s^2 \tau = 0, \qquad \alpha_E = \frac{W}{2T_0}, \quad \omega_s^2 = -\frac{\alpha_c e\dot{V}}{T_0 E}.$$

$$\tau(t) = Ae^{-\alpha_E t}\cos(\omega_s t - \theta_0). \qquad U(E) = U_0 + W\Delta E, \quad W = \left.\frac{dU}{dE}\right|_{E=E_0},$$

The damping re-partition

To evaluate the damping rate, we need to evaluate W. Since the radiation energy loss per revolution is

$$U_{\rm rad} = \oint P_\gamma dt = \oint P_\gamma \frac{dt}{ds} ds = \frac{1}{c}\oint P_\gamma (1 + \frac{x}{\rho}) ds = \frac{1}{c}\oint P_\gamma (1 + \frac{D}{\rho}\frac{\Delta E}{E_0}) ds,$$

$cdt/ds = (1+x/\rho)$,

$$W = \frac{dU_{\rm rad}}{dE} = \frac{1}{c}\oint \left\{ \frac{dP_\gamma}{dE} + \frac{D}{\rho}\frac{P_\gamma}{E} \right\}_{E_0} ds.$$

$$\left.\frac{dP_\gamma}{dE}\right|_{E_0} = 2\frac{P_\gamma}{E_0} + 2\frac{P_\gamma}{B_0}\frac{dB}{dE} = 2\frac{P_\gamma}{E_0} + 2\frac{P_\gamma}{B_0}\frac{dx}{dE}\frac{dB}{dx} = 2\frac{P_\gamma}{E_0} + 2\frac{P_\gamma}{B_0}\frac{D}{E_0}\frac{dB}{dx},$$

$$\frac{dU_{\rm rad}}{dE} = \frac{1}{c}\oint \left\{ 2\frac{P_\gamma}{E_0} + 2\frac{P_\gamma}{B_0}\frac{D}{E_0}\frac{dB}{dx} + \frac{P_\gamma}{E_0}\frac{D}{\rho} \right\}_{E_0} ds = \frac{U_0}{E_0}\left[2 + \frac{1}{cU_0}\oint \left\{ DP_\gamma \left(\frac{1}{\rho} + \frac{2}{B}\frac{dB}{dx} \right) \right\}_{E_0} ds \right].$$

$$\alpha_E = \frac{1}{2T_0}\frac{dU_{\rm rad}}{dE} = \frac{U_0}{2T_0 E}(2 + \mathcal{D}).$$

$$\mathcal{D} = \frac{1}{cU_0}\oint \left\{ DP_\gamma \left(\frac{1}{\rho} + \frac{2}{B}\frac{dB}{dx} \right) \right\}_{E_0} ds = \left[\oint \frac{D}{\rho}\left(\frac{1}{\rho^2} + 2K(s) \right) ds \right] \left[\oint \frac{ds}{\rho^2} \right]^{-1}$$

$$\alpha_E = \frac{1}{2T_0}\frac{dU_{\rm rad}}{dE} = \frac{U_0}{2T_0 E}(2+\mathcal{D}).$$

$$\begin{aligned}\mathcal{D} &= \frac{1}{cU_0}\oint\left\{DP_\gamma\left(\frac{1}{\rho}+\frac{2}{B}\frac{dB}{dx}\right)\right\}_{E_0} ds \\ &= \left[\oint\frac{D}{\rho}\left(\frac{1}{\rho^2}+2K(s)\right)ds\right]\left[\oint\frac{ds}{\rho^2}\right]^{-1},\end{aligned}$$

where K(s) = B1/Bρ is the quadrupole gradient function with B1 = ∂B/∂x. The damping partition number D is a property of lattice configuration. For an isomagnetic ring,

$$\mathcal{D} = \frac{1}{2\pi}\oint D(s)\left(\frac{1}{\rho^2}+2K(s)\right)_{\rm dipole} ds.$$

1. For an isomagnetic ring with **separate function magnets**, where K(s) = 0 in dipoles

$$\mathcal{D} = \frac{1}{2\pi\rho}\oint\frac{D(s)}{\rho}ds = \frac{\alpha_c R}{\rho} \ll 1$$

$$\alpha_E = \frac{\langle P_\gamma\rangle}{2E}\left(2+\frac{\alpha_c R}{\rho}\right) \approx \frac{U_0}{ET_0} = \frac{\langle P_\gamma\rangle}{E}.$$

The damping time constant, which is the inverse of α_E, is nearly equal to the time it takes for the electron to radiate away its total energy.

2. For an isomagnetic combined function accelerator, we find $\mathcal{D} = 2 - \frac{\alpha_c R}{\rho}$

$$\alpha_E \approx 2\langle P_\gamma\rangle/E.$$

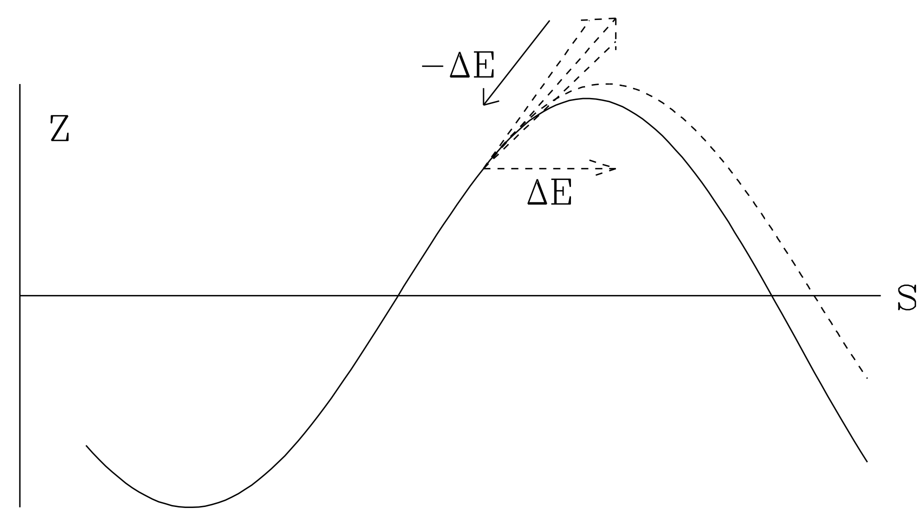
II.2 Damping of Betatron Motion

A relativistic electron emits synchrotron radiation primarily along its direction of motion within an angle $1/\gamma$. The momentum change resulting from recoil of synchrotron radiation is exactly opposite to the direction of particle motion. Figure 4.6 illustrates betatron motion with synchrotron radiation, where vertical betatron coordinate z is plotted as a function of longitudinal coordinate s. The betatron phase-space coordinates are

$$z = A\cos\phi, \quad z' = -\frac{A}{\beta}\sin\phi, \quad A^2 = z^2 + (\beta z')^2,$$

where A is betatron amplitude, ϕ is betatron phase, and β is betatron function.

Schematic drawing of the damping of vertical betatron motion due to synchrotron radiation. The energy loss through synchrotron radiation along the particle trajectory with an opening angle of $1/\gamma$. Energy is replenished in the rf cavity along the longitudinal direction. This process damps the vertical betatron oscillation to a very small value.



When an electron loses an amount of energy u by radiation, the momentum vector P changes by δP , such that δP is parallel and opposite to P with |cδP | = u. Since the radiation loss changes neither slope nor position of the trajectory, the betatron amplitude is unchanged except for a small increment in effective focusing force.

$$z'|_{\text{before}} = \frac{p_z}{p} = z'|_{\text{after}} = \frac{p_z}{p-\delta p}\left(1-\frac{\delta p}{p}\right) \xrightarrow{\text{after cavity}} \frac{p_z}{p}\left(1-\frac{\delta p}{p}\right)$$

Now the energy gain from rf accelerating force is on the average parallel to the designed orbit, i.e.

$$z' \to z'(1-\frac{\delta p}{p}), \quad \text{or} \quad \Delta z' = -z'\frac{\delta p}{p} = -z'\frac{u}{E}.$$

The corresponding change of amplitude A in one revolution becomes

$$A\delta A = \langle \beta^2 z' \Delta z' \rangle = -\langle (\beta z')^2 \rangle \frac{U_0}{E},$$

where <..> averages over betatron oscillations in one revolution, and U_0 is synchrotron radiation energy per revolution. Since the betatron motion is sinusoidal, we obtain $<(\beta z')^2> = A^2/2$,

$$\frac{\delta A}{A} = -\frac{U_0}{2E}, \qquad \frac{1}{A}\frac{dA}{dt} = \frac{1}{T_0}\frac{\delta A}{A} = -\frac{U_0}{2ET_0}, \qquad \boxed{\alpha_z = \frac{U_0}{2ET_0} = \frac{\langle P_\gamma \rangle}{2E}}.$$

The damping rate applies also to the horizontal betatron motion.

Horizontal betatron motion

The horizontal motion of an electron is complicated by the off-momentum closed orbit. The horizontal displacement from the reference orbit is

$$x = x_\beta + x_e, \quad x_e = D(s)\frac{\Delta E}{E}, \quad x' = x'_\beta + x'_e, \quad x'_e = D'(s)\frac{\Delta E}{E},$$

$$\delta x_\beta = -\delta x_e = -D(s)\frac{u}{E}, \qquad \delta x'_\beta = -\delta x'_e = -D'(s)\frac{u}{E}.$$

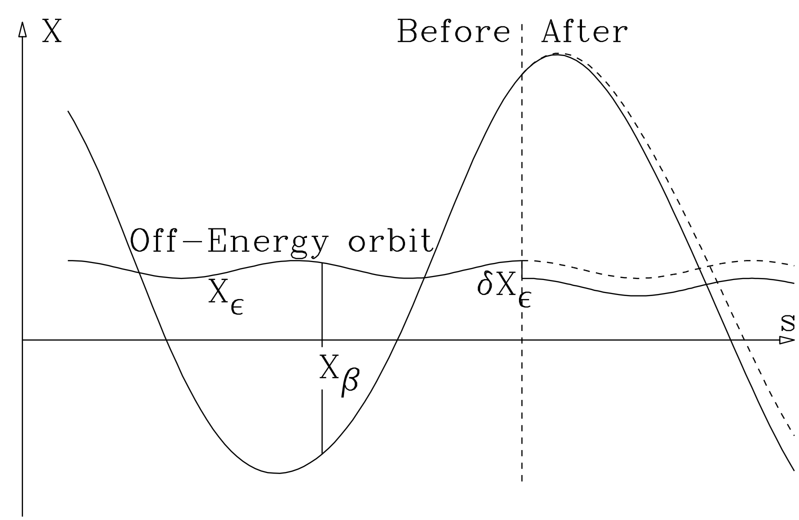
$$x_\beta = A\cos\phi, \quad x'_\beta = -\frac{A}{\beta}\sin\phi, \quad A^2 = x_\beta^2 + (\beta x'_\beta)^2;$$

$$A\delta A = x_\beta \delta x_\beta + \beta^2 x'_\beta \delta x'_\beta = -(Dx_\beta + \beta^2 D' x'_\beta)\frac{u}{E}.$$

$$u = -\frac{P_\gamma(x_\beta)}{c}\delta\ell = -\frac{1}{c}\left(P_\gamma + 2\frac{P_\gamma}{B}\frac{dB}{dx}x_\beta\right)\left(1 + \frac{x_\beta}{\rho}\right)ds$$

$$A\delta A = x_\beta D\left(1 + \frac{2}{B}\frac{dB}{dx}x_\beta + \frac{x_\beta}{\rho}\right)\frac{P_\gamma}{cE}ds.$$

We neglected all terms linear in x_β', because their average over the betatron phase is zero. The time average over the betatron phase gives $\langle x_\beta \rangle = 0$ and $\langle x_\beta^2 \rangle = A^2/2$.



The fractional betatron amplitude increment in one turn becomes

$$\frac{\langle\delta A\rangle}{A} = \frac{U_0}{2E}\boxed{\left(\oint \frac{D}{\rho}\left[2K(s)+\frac{1}{\rho^2}\right] ds\right)\left(\oint \frac{1}{\rho^2}ds\right)^{-1} = \mathcal{D}}\frac{U_0}{2E},$$

Including the phase space damping, the net horizontal amplitude change:

$$\frac{\Delta A}{A} = -(1-\mathcal{D})\frac{U_0}{2E}. \qquad \alpha_x = (1-\mathcal{D})\frac{U_0}{2T_0E}$$

In summary, radiation damping coefficients for the three degrees of freedom in a bunch are

$$\alpha_x = \mathcal{J}_x\alpha_0, \quad \alpha_z = \mathcal{J}_z\alpha_0, \quad \alpha_E = \mathcal{J}_E\alpha_0,$$

where $\alpha_0 = \langle P_\gamma\rangle/2E$, and the damping partition numbers

$$\mathcal{J}_x = 1-\mathcal{D}, \quad \mathcal{J}_z = 1, \quad \mathcal{J}_E = 2+\mathcal{D}$$

Robinson theorem $\sum \mathcal{J}_i = \mathcal{J}_x + \mathcal{J}_z + \mathcal{J}_E = 4 \quad \text{or} \quad \mathcal{J}_x + \mathcal{J}_E = 3$

$$\tau_x = \frac{2E}{\mathcal{J}_x\langle P_\gamma\rangle} = \frac{4\pi R\rho}{cC_\gamma\mathcal{J}_xE^3} = \frac{2E}{\mathcal{J}_xU_0}T_0,$$
$$\tau_z = \frac{2E}{\mathcal{J}_z\langle P_\gamma\rangle} = \frac{4\pi R\rho}{cC_\gamma\mathcal{J}_zE^3} = \frac{2E}{\mathcal{J}_zU_0}T_0,$$
$$\tau_E = \frac{2E}{\mathcal{J}_E\langle P_\gamma\rangle} = \frac{4\pi R\rho}{cC_\gamma\mathcal{J}_EE^3} = \frac{2E}{\mathcal{J}_EU_0}T_0,$$

II.3 Damping Rate Adjustment

A. Increase U to increase damping rate (damping wiggler)

$$U_{\rm w} = U_0 + U_{\rm wiggler}, \qquad \alpha_{\rm w} = \frac{U_{\rm w}}{2ET_0} = \alpha_0 + \alpha_{\rm wiggler}.$$

The damping time is shortened by a factor of $(1+U_{\rm wiggler}/U_0)^{-1}$.

B. Change D to repartition the partition number

Many early synchrotrons, such as 8 GeV synchrotron (DESY) in Hamburg, 28 GeV CERN-PS, 33 GeV AGS, etc., used combined function isomagnetic magnets, where $D \approx 2$. Thus the energy oscillations are strongly damped ($J_E \approx 4$) and the horizontal oscillations become anti-damped ($J_x \approx -1$).

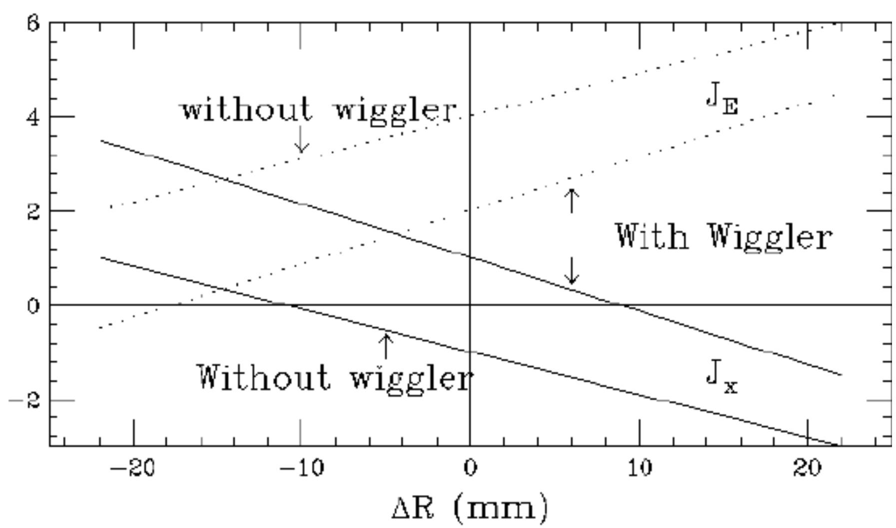


Figure 4.8: The variation of the damping partition number of the CERN PS with the strength of the Robinson wiggler. Without the Robinson wiggler, a fairly large change in ΔR is needed to attain $\mathcal{J}_x = 1$, with loss of useful aperture. From K. Hubner, CERN 85-19, p. 226 (1985).

Robinson wiggler

If the gradient and dipole field of each magnet satisfy $K\rho < 0$, as shown in Fig. 4.9, the damping partition of Eq. (4.100) can be made negative.

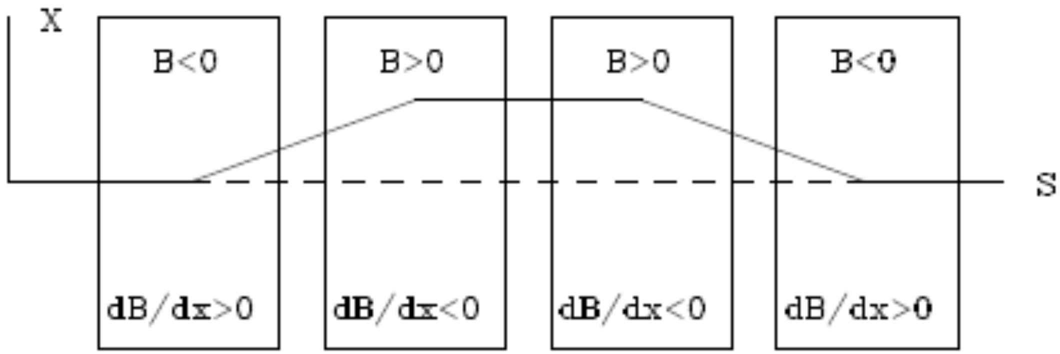


Figure 4.9: Schematic drawing of a Robinson wiggler, where gradient dipoles with $B\frac{dB}{dx} < 0$ are used to change the damping partition number.

$$\Delta\mathcal{D} = 2\langle D\rangle \frac{1}{B_{\rm w}}\frac{dB_{\rm w}}{dx}\frac{4L_{\rm w}\rho}{2\pi\rho_{\rm w}^2}\left(1+\frac{4L_{\rm w}\rho}{2\pi\rho_{\rm w}^2}\right)^{-1},$$

The Robinson wiggler has been successfully employed in the CERN PS to obtain $J_x \approx 2$, which enhances damping of horizontal emittance and reduces damping in energy oscillation. The resulting line density of beam bunches is likewise reduced to prevent collective instabilities.

II.4 Radiation Excitation and Equilibrium Energy Spread

The time during which a quantum is emitted is about $\frac{\rho\Theta}{c} \approx \frac{\rho}{c\gamma} \approx \frac{6}{B[\text{Tesla}]} \times 10^{-12}$ s

Emission of individual quanta are statistically independent because the energy of each photon [keV] is a very small fraction of electron energy.

Discontinuous quantized photon emission disturbs electron orbits. The cumulative effect of many such small disturbances introduces diffusion similar to random noise. The amplitude of oscillation will grow until the rates of quantum excitation and radiation damping are on the average balanced. The damping process depends only on the average rate of energy loss, whereas the quantum excitation fluctuates about its average rate.

A. Effects of quantum excitation

$$\Delta E = A_0 e^{j\omega_s(t-t_0)}$$

$$\Delta E = A_0 e^{j\omega_s(t-t_0)} - u e^{j\omega_s(t-t_1)} = A_1 e^{j\omega_s(t-t_1)}, \quad (t > t_1),$$

$$A_1^2 = A_0^2 + u^2 - 2A_0 u \cos\omega_s(t_1 - t_0).$$

$$\delta A^2 = \langle A^2 - A_0^2 \rangle_t = u^2,$$

$$\boxed{\langle \frac{dA^2}{dt} \rangle = \frac{d\langle A^2 \rangle}{dt} = \mathcal{N} u^2,}$$

where $\mathcal{N}$ is the rate of photon emission.

B. Equilibrium rms energy spread

$$\frac{d\langle A^2\rangle}{dt} = -2\frac{\langle A^2\rangle}{\tau_E} + \mathcal{N}u^2, \qquad {\sigma_E}^2 = \frac{\langle A^2\rangle}{2} = \frac{1}{4}\mathcal{N}u^2\tau_E.$$

$$\frac{d\langle A^2\rangle}{dt} = \int_0^\infty u^2 n(u)du = \mathcal{N}\langle u^2\rangle, \quad \mathcal{N} = \int_0^\infty n(u)du.$$

$$G_E = \langle\mathcal{N}\langle u^2\rangle\rangle_s = \frac{1}{2\pi R}\oint \mathcal{N}\langle u^2\rangle ds, \qquad \sigma_E^2 = \frac{1}{4}G_E\tau_E.$$

$$P_\gamma\Big|_{\text{designed orbit}} = \frac{cC_\gamma}{2\pi}\frac{E^4}{\rho^2} = \frac{\langle P_\gamma\rangle}{\langle 1/\rho^2\rangle\rho^2}. \qquad \mathcal{N}\langle u^2\rangle\Big|_{\text{designed orbit}} = \frac{3}{2}C_u\hbar c\frac{\gamma^3}{\rho^3}\frac{\langle P_\gamma\rangle}{\langle 1/\rho^2\rangle},$$

$$G_E = \frac{3}{2}C_u\hbar c\gamma^3\frac{\langle P_\gamma\rangle}{\langle 1/\rho^2\rangle}\langle 1/\rho^3\rangle \qquad \sigma_E^2 = \frac{3C_u\hbar mc^3\gamma^4}{4\mathcal{J}_E\langle 1/\rho^2\rangle}\langle 1/\rho^3\rangle.$$

$$C_q = \frac{3C_u\hbar}{4mc} = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} = 3.83\times 10^{-13}\ \text{m}. \qquad (\frac{\sigma_E}{E})^2 = \frac{C_q\gamma^2}{\mathcal{J}_E\langle 1/\rho^2\rangle}\langle 1/\rho^3\rangle,$$

$$(\frac{\sigma_E}{E})^2 = \frac{55}{48\sqrt{3}}\frac{\hbar\omega_c}{\mathcal{J}_E E} = C_q\frac{\gamma^2}{\mathcal{J}_E\rho} \quad \text{or} \quad \frac{\sigma_E}{E} \sim (0.62\times 10^{-6})\frac{\gamma}{\sqrt{\mathcal{J}_E\rho[\text{m}]}}.$$

C. Adjustment of rms momentum spread

$$\sigma_E^2 = \sigma_{E0}^2 \left(1 + \frac{I_{3w}}{I_3}\right)\left(1 + \frac{I_{2w}}{I_2}\right)^{-1}$$

$$I_3 = \int \frac{1}{|\rho|^3} ds, \quad I_2 = \int \frac{1}{|\rho|^2} ds, \quad I_{3w} = \int \frac{1}{|\rho_{\rm w}|^3} ds, \quad I_{2w} = \int \frac{1}{|\rho_{\rm w}|^2} ds$$

D. Beam distribution function in momentum

$$\Psi(\theta) = \frac{1}{\sqrt{2\pi}\sigma_E} e^{-\theta^2/2\sigma_{\rm E}^2}, \quad \Psi(\Delta E) = \frac{1}{\sqrt{2\pi}\sigma_E} e^{-(\Delta E)^2/2\sigma_{\rm E}^2}.$$

The normalized phase-space coordinates are (ΔE, $\theta = E\omega_s\tau/\alpha_c$).

$$\sigma_\tau = \frac{\alpha_c}{E\omega_s}\sigma_E$$

Central Limit Theorem: If the probability P(u) of each quantum emission is statistically independent, and the probability function falls off rapidly as $|u| \to \infty$, then the probability distribution function for the emission of n photons is a Gaussian,

$$\mathcal{P}_n(w) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-(w-w_n)^2/2\sigma_n^2}$$

$$w_n = n\langle u\rangle, \quad \langle u\rangle = \int uP(u)du, \quad \sigma_n^2 = n\sigma_u^2, \quad \sigma_u^2 = \int (u - \langle u\rangle)^2 P(u)du.$$

II.5 Radial Bunch Width and Distribution Function

Emission of discrete quanta in synchrotron radiation also excites random betatron motion. The emission of a quantum of energy u results in a change of betatron coordinates, i.e. $\delta x_\beta = -D\ (u/E_0), \qquad \delta x'_\beta = -D'\ (u/E_0).$

The resulting change in the Courant-Snyder invariant is

$$\delta a^2 = \frac{2}{\beta_x}\left[D x_\beta + (\beta_x D' - \frac{\beta'_x}{2}D)(\beta_x x' - \frac{\beta'_x}{2}x)\right]\frac{u}{E_0} + \frac{1}{\beta_x}\left[D^2 + (\beta_x D' - \frac{\beta'_x}{2}D)^2\right](\frac{u}{E_0})^2$$

$$\delta\langle a^2\rangle = \mathcal{H}(\frac{u}{E_0})^2, \qquad \mathcal{H} = \frac{1}{\beta_x}\left[D^2 + (\beta_x D' - \frac{1}{2}\beta'_x D)^2\right]$$

$$\frac{d\langle a^2\rangle}{dt} \equiv G_x = \frac{1}{2\pi R E^2}\oint \mathcal{N}\langle u^2\rangle \mathcal{H} ds = \frac{\langle \mathcal{N}\langle u^2\rangle \mathcal{H}\rangle_s}{E^2}$$

$$\frac{d\langle a^2\rangle}{dt} = -2\frac{\langle a^2\rangle}{\tau_x} + G_x$$

$$\langle a^2\rangle = \frac{1}{2}\tau_x G_x \quad \text{and} \quad \sigma^2_{x\beta_x} = \frac{1}{2}\beta_x\langle a^2\rangle$$

Using Eq. (4.71) for N<u^2>, we obtain

$$G_x = \frac{3}{2}C_u\hbar c\gamma^3\frac{\langle P_\gamma\rangle\langle\mathcal{H}/|\rho|^3\rangle}{E^2\langle 1/\rho^2\rangle} = \frac{3C_q c r_0\gamma^5\langle\mathcal{H}/|\rho^3|\rangle}{3\langle\rho^2\rangle\langle 1/\rho^2\rangle},$$

$$\epsilon_x = \frac{\sigma_{x\beta_x}^2}{\beta_x} = \frac{1}{4}\tau_x G_x = C_q\frac{\gamma^2\langle\mathcal{H}/|\rho|^3\rangle}{\mathcal{J}_x\langle 1/\rho^2\rangle} \xrightarrow{\text{isomagnetic}} C_q\frac{\gamma^2\langle\mathcal{H}\rangle_{\text{mag}}}{\mathcal{J}_x\rho},$$

$$C_q = \frac{3C_u\hbar}{4mc} = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} = 3.83\times10^{-13}\ \text{m}.$$

This is called the **natural emittance**. Since the H-function is proportional to Lθ^2 ~ $\rho\theta^3$, where θ is the dipole angle of a half cell, the natural emittance of an electron storage ring is proportional to $\gamma^2\theta^3$. The normalized emittance is proportional to $\gamma^3\theta^3$. Unless the orbital angle of each dipole is inversely proportional to γ, the normalized natural emittance of an electron storage ring increases with energy.

The horizontal distribution function

$$\Psi(x_\beta, x'_\beta) = \frac{1}{\sqrt{2\pi}\sigma_{x\beta_x}}\exp\left\{-\frac{x_\beta^2 + (\beta_x x'_\beta - (\beta'_x/2)x_\beta)^2}{2\sigma_{x\beta_x}^2}\right\}.$$

The total radial beam width has contributions from both betatron and energy oscillations. The rms beam width is Gaussian quadrature

$$\sigma_x^2 = \sigma_{x\beta_x}^2 + \sigma_{x\epsilon}^2 \xrightarrow{\text{isomagnetic}} C_q\frac{\gamma^2}{\rho}\left[\frac{\beta_x(s)\langle\mathcal{H}\rangle_{\text{mag}}}{\mathcal{J}_x} + \frac{D^2(s)}{\mathcal{J}_E}\right]$$

II.6 Vertical Beam Width

The transverse kick is then equal to $\theta_\gamma u/c$. The transverse angular kicks on phase-space coordinates become

$$\delta x = 0,\ \delta x' = \frac{u}{E_0}\theta_x, \quad \delta z = 0,\ \delta z' = \frac{u}{E_0}\theta_z$$

where θ_x, θ_z are projections of θ_γ onto x, z axes respectively. Since $\delta x'$ is small compared with that of Eq. (4.159), we neglect it.

$$\delta\langle a_z^2\rangle = (u/E_0)^2\theta_z^2\beta_z$$
$$\sigma_{z\beta}^2 = \frac{1}{4}\tau_z G_z \beta_z,$$
$$G_z = \frac{\langle \mathcal{N}\langle u^2\theta_z^2\rangle\beta_z\rangle_s}{E^2} \approx \frac{\langle \mathcal{N}\langle u^2\rangle\langle\theta_z^2\rangle\beta_z\rangle_s}{E^2} \approx \frac{\langle \mathcal{N}\langle u^2\rangle\rangle\langle\beta_z\rangle}{\gamma^2 E^2},$$

$$\sigma_z^2 \approx C_q\langle\beta_z^2\rangle/\rho \quad \text{or} \quad \epsilon_z \approx C_q\langle\beta_z^2\rangle^{1/2}/\rho,$$

Emittances in the presence of linear coupling

$$\epsilon_x = \frac{1}{1+\kappa}\epsilon_{\text{nat}}, \quad \epsilon_z = \frac{\kappa}{1+\kappa}\epsilon_{\text{nat}},$$

II.7 Radiation Integrals

$$B = \frac{d^4 N_{\mathrm{ph}}}{dt\,d\Omega\,dS(d\lambda/\lambda)} \qquad \epsilon_x = C_q \gamma^2 \frac{\langle \mathcal{H}/|\rho|^3 \rangle}{\mathcal{J}_x \langle 1/\rho^2 \rangle},$$

$I_1 = \int \frac{D}{\rho}\, ds$	$\alpha_c = \frac{I_1}{2\pi R}$
$I_2 = \int \frac{1}{\rho^2}\, ds$	$U_0 = \frac{C_\gamma}{2\pi} E^4 I_2$
$I_3 = \int \frac{1}{\|\rho\|^3}\, ds$	$\left(\frac{\sigma_E}{E}\right)^2 = \frac{C_q \gamma^2 I_3}{2 I_2 + I_4} = \frac{C_q \gamma^2 I_3}{2 I_2 \mathcal{J}_E}$
$I_{3a} = \int \frac{1}{\rho^3}\, ds$	$\langle S \rangle = P_{\mathrm{ST}} \frac{I_{3a}}{I_3}$ $\qquad P_{ST} = -\frac{8}{5\sqrt{3}}$
$I_4 = \int \frac{D}{\rho} \left(\frac{1}{\rho^2} + 2K \right) ds$	$\mathcal{D} = \frac{I_4}{I_2}; \qquad \mathcal{J}_x = 1 - \mathcal{D}, \quad \mathcal{J}_E = 2 + \mathcal{D}, \quad \mathcal{J}_z = 1$
$K = \frac{1}{B\rho} \frac{\partial B_z}{\partial x}$	$\alpha_x = \frac{U_0}{2 T_0 E} \mathcal{J}_x, \qquad \tau_x = \frac{2E}{\mathcal{J}_x U_0} T_0,$
	$\alpha_z = \frac{U_0}{2 T_0 E} \mathcal{J}_z, \qquad \tau_z = \frac{2E}{\mathcal{J}_z U_0} T_0,$
	$\alpha_E = \frac{U_0}{2 T_0 E} \mathcal{J}_E, \qquad \tau_E = \frac{2E}{\mathcal{J}_E U_0} T_0,$
$I_5 = \int \frac{\mathcal{H}}{\|\rho\|^3}\, ds$	$\epsilon_x = \frac{C_q \gamma^2 I_5}{I_2 - I_4} = \frac{C_q \gamma^2 I_5}{I_2 \mathcal{J}_x}$

III Emittance in Electron Storage Rings

$$\epsilon_x = C_q \gamma^2 \frac{\langle \mathcal{H}/|\rho|^3 \rangle}{\mathcal{J}_x \langle 1/\rho^2 \rangle} \xrightarrow{\text{isomagnetic}} C_q \gamma^2 \frac{\langle \mathcal{H} \rangle_{\text{dipole}}}{\mathcal{J}_x \rho},$$

Since H ~ $L\theta^2 = \rho\theta^3$, the <H> and the resulting natural emittance obey the scaling laws:

$$\langle \mathcal{H} \rangle / \mathcal{J}_x = \mathcal{F}_{\text{lattice}} \rho \theta^3$$

$$\epsilon_x = \mathcal{F}_{\text{lattice}} C_q \gamma^2 \theta^3$$

where the scaling factor F_{lattice} depends on the design of the storage ring lattices, and θ is the total dipole bending angle in a bend-section. The resulting normalized emittance is

$$\epsilon_{\text{n}} = \gamma \epsilon_x = \mathcal{F}_{\text{lattice}} C_q (\gamma\theta)^3.$$

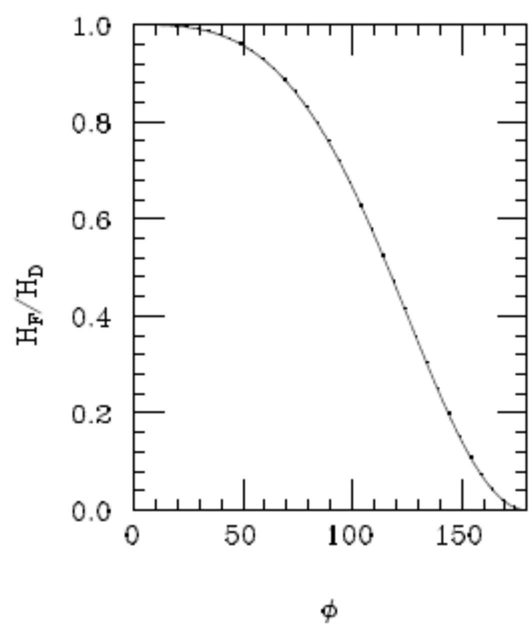
A. FODO cell lattice

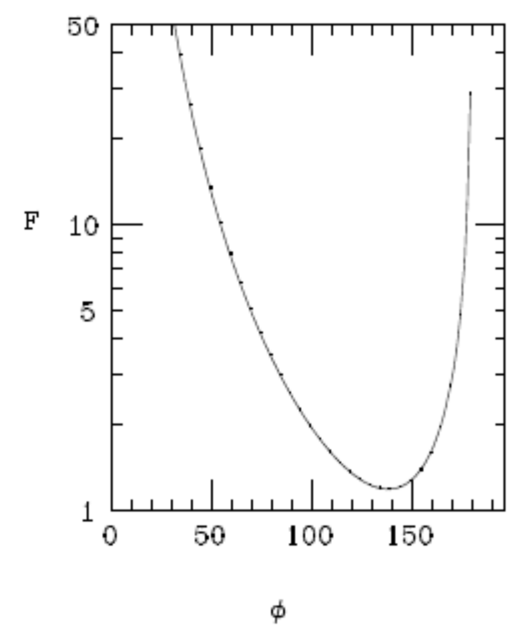
$$\mathcal{H}_{\rm F} = L\theta^2 \frac{\cos(\Phi/2)}{\sin^3(\Phi/2)(1+\sin(\Phi/2))}\left(1+\frac{1}{2}\sin\frac{\Phi}{2}\right)^2$$

$$\mathcal{H}_{\rm D} = L\theta^2 \frac{\cos(\Phi/2)}{\sin^3(\Phi/2)(1-\sin(\Phi/2))}\left(1-\frac{1}{2}\sin\frac{\Phi}{2}\right)^2$$

$$\langle\mathcal{H}\rangle \approx \frac{1}{2}\rho\theta^3\frac{\cos(\Phi/2)}{\sin^3(\Phi/2)}\left[\frac{(1+\frac{1}{2}\sin(\Phi/2))^2}{(1+\sin(\Phi/2))}+\frac{(1-\frac{1}{2}\sin(\Phi/2))^2}{(1-\sin(\Phi/2))}\right]$$

$$\mathcal{F}_{\rm FODO} = \frac{1-\frac{3}{4}\sin^2(\Phi/2)}{\sin^3(\Phi/2)\cos(\Phi/2)}\ \mathcal{J}_x^{-1}$$





B. Double-bend achromat (Chasman-Green lattice)

$$D = \rho(1-\cos\phi) + D_0\cos\phi + \rho D_0'\sin\phi,$$

$$D' = \left(1 - \frac{D_0}{\rho}\right)\sin\phi + D_0'\cos\phi\ , \qquad \mathcal{H}_0 = \gamma_0 D_0^2 + 2\alpha_0 D_0 D_0' + \beta_0 D_0'^2$$

$$\begin{aligned}\mathcal{H}(\phi) &= \mathcal{H}_0 + 2(\alpha_0 D_0 + \beta_0 D_0')\sin\phi - 2(\gamma_0 D_0 + \alpha_0 D_0')\rho(1-\cos\phi) \\ &\quad +\beta_0\sin^2\phi + \gamma_0\rho^2(1-\cos\phi)^2 - 2\alpha_0\rho\sin\phi(1-\cos\phi),\end{aligned}$$

$$\begin{aligned}\langle\mathcal{H}\rangle &= \mathcal{H}_0 + (\alpha_0 D_0 + \beta_0 D_0')\theta E(\theta) - \frac{1}{3}(\gamma_0 D_0 + \alpha_0 D_0')\rho\theta^2 F(\theta) \\ &\quad +\frac{\beta_0}{3}\theta^2 A(\theta) - \frac{\alpha_0}{4}\rho\theta^3 B(\theta) + \frac{\gamma_0}{20}\rho^2\theta^4 C(\theta),\end{aligned}$$

$$E(\theta) = 2(1-\cos\theta)/\theta^2, \quad F(\theta) = 6(\theta - \sin\theta)/\theta^3, \quad A(\theta) = (6\theta - 3\sin 2\theta)/(4\theta^3),$$

$$B(\theta) = (6 - 8\cos\theta + 2\cos 2\theta)/\theta^4, \quad C(\theta) = (30\theta - 40\sin\theta + 5\sin 2\theta)/\theta^5.$$

$$d_0 = \tfrac{D_0}{L\theta},\ d_0' = \tfrac{D_0'}{\theta},\ \tilde\beta_0 = \tfrac{\beta_0}{L},\ \tilde\gamma_0 = \gamma_0 L,\ \tilde\alpha_0 = \alpha_0, \qquad L = \rho\theta$$

$$\begin{aligned}\langle\mathcal{H}\rangle &= \rho\theta^3\left\{\tilde\gamma_0 d_0^2 + 2\tilde\alpha_0 d_0 d_0' + \tilde\beta_0 d_0'^2 + (\tilde\alpha_0 E - \frac{\tilde\gamma_0}{3}F)d_0\right. \\ &\quad \left. +(\tilde\beta_0 E - \frac{\tilde\alpha_0}{3}F)d_0' + \frac{\tilde\beta_0}{3}A - \frac{\tilde\alpha_0}{4}B + \frac{\tilde\gamma_0}{20}C\right\}.\end{aligned}$$

Achromatic: d_0=0, d_0'=0,

$$
\begin{aligned}
\langle \mathcal{H} \rangle \;=\; \rho\theta^3 \Big\{ & \cancel{\left[\tilde{\gamma}_0 d_0^2 + 2\tilde{\alpha}_0 d_0 d_0' + \tilde{\beta}_0 d_0'^2\right] + \left[\tilde{\alpha}_0 E - \frac{\tilde{\gamma}_0}{3} F\right] d_0} \\
& + \cancel{\left[\tilde{\beta}_0 E - \frac{\tilde{\alpha}_0}{3} F\right] d_0'} + \frac{\tilde{\beta}_0}{3} A - \frac{\tilde{\alpha}_0}{4} B + \frac{\tilde{\gamma}_0}{20} C \Big\}.
\end{aligned}
$$

$$\tilde{\beta}_0 \tilde{\gamma}_0 = 1 + \tilde{\alpha}_0^2$$

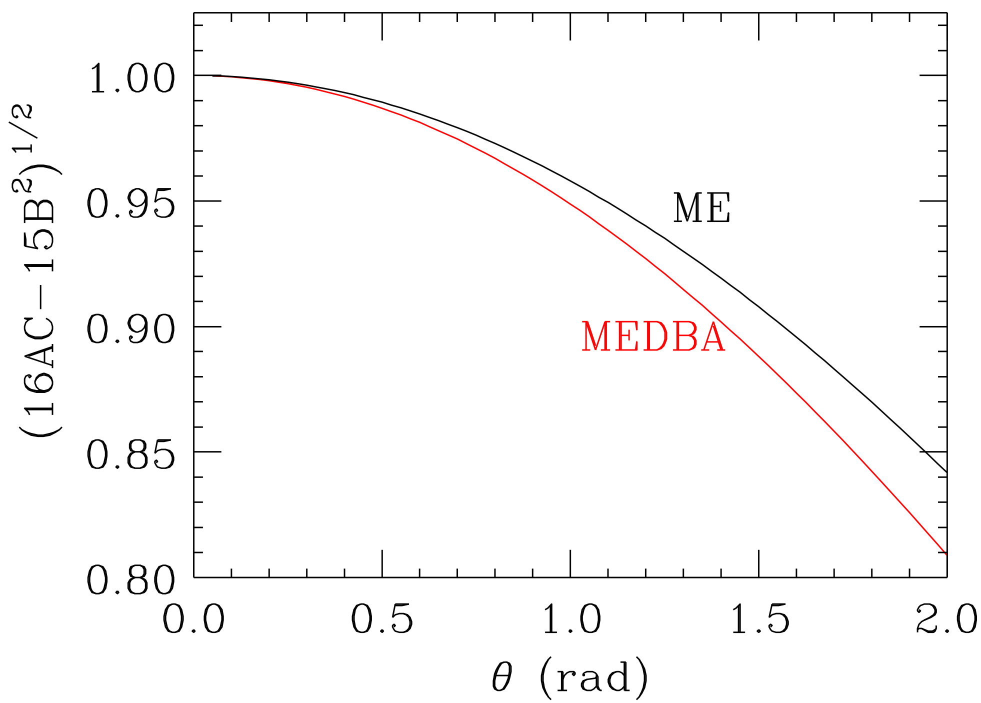
$$\langle \mathcal{H} \rangle_{\text{MEDBA}} = \frac{G}{4\sqrt{15}} \rho\theta^3$$

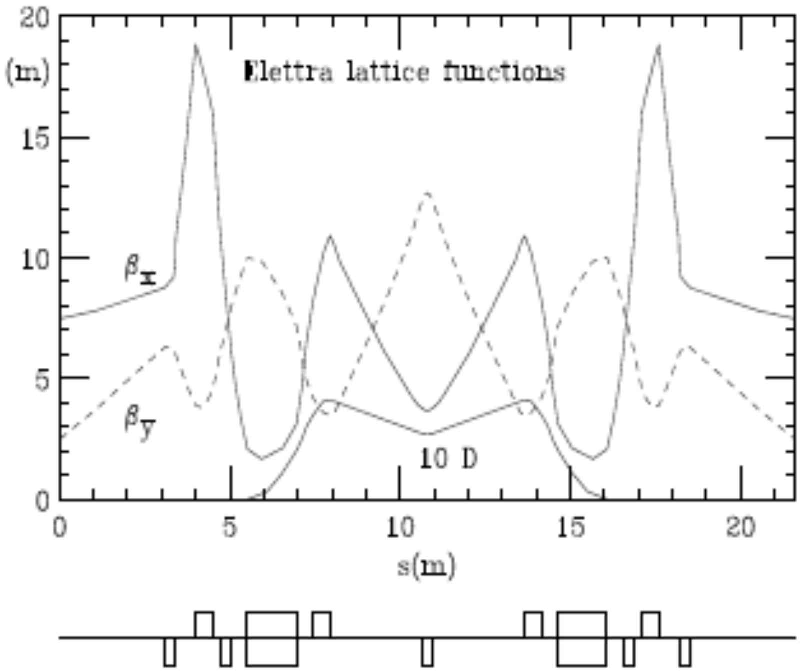
$$G = \sqrt{16AC - 15B^2}$$

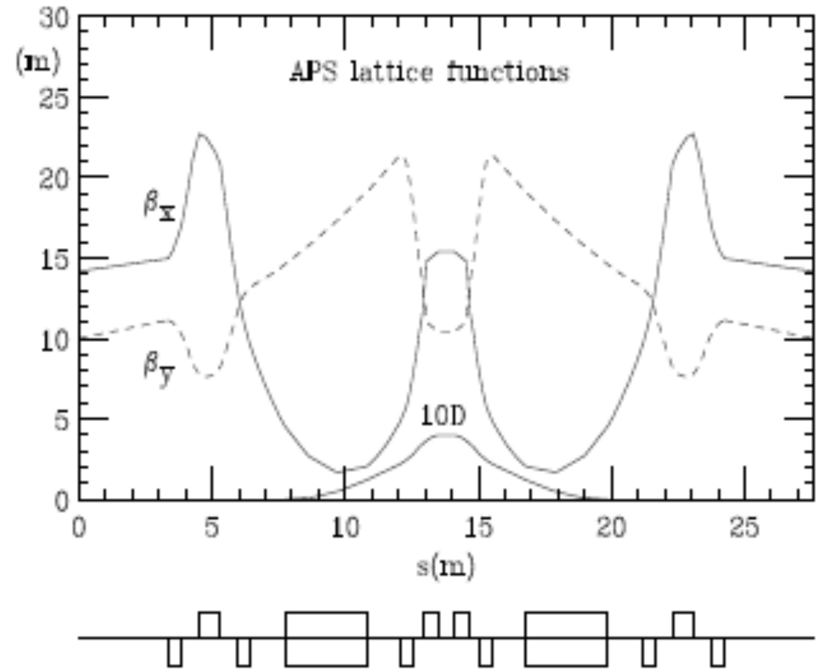
$$\tilde{\beta}_0 = \frac{\sqrt{12}C}{\sqrt{5}G}, \quad \tilde{\alpha}_0 = \frac{\sqrt{15}B}{G},$$

$$\beta^*_{\text{MEDBA}} = \frac{3}{4\sqrt{60}} L$$

$$s^*_{\text{MEDBA}} = \frac{3}{8} L$$







$$\frac{\epsilon_{\text{elettra}}}{\epsilon_{\text{MEDBA}}} \approx 1.38 \qquad \text{and} \qquad \frac{\epsilon_{\text{aps}}}{\epsilon_{\text{MEDBA}}} \approx 3.64.$$

The H-function at the end of the dipole is

$$\mathcal{H}(\theta) = \frac{\rho\theta^3}{\sqrt{15}G}\left\{6C[\frac{\sin^2\theta}{\theta^2}] - 15B[\frac{2\sin\theta(1-\cos\theta)}{\theta^3}] + 10A[\frac{4(1-\cos\theta)^2}{\theta^4}]\right\}.$$

$$\rightarrow \quad \frac{1}{\sqrt{15}}\rho\theta^3 \quad \text{(thin lens approximation).}$$

This criterion can be used to evaluate the goodness of DBA lattice match.

C. Minimum emittance (ME): Minimum <H>-function lattice

$$\langle\mathcal{H}\rangle = \rho\theta^3\left\{\left[\tilde{\gamma}_0 d_0^2 + 2\tilde{\alpha}_0 d_0 d_0' + \tilde{\beta}_0 d_0'^2\right] + \left[\tilde{\alpha}_0 E - \frac{\tilde{\gamma}_0}{3}F\right]d_0 + \left[\tilde{\beta}_0 E - \frac{\tilde{\alpha}_0}{3}F\right]d_0' + \frac{\tilde{\beta}_0}{3}A - \frac{\tilde{\alpha}_0}{4}B + \frac{\tilde{\gamma}_0}{20}C\right\}.$$

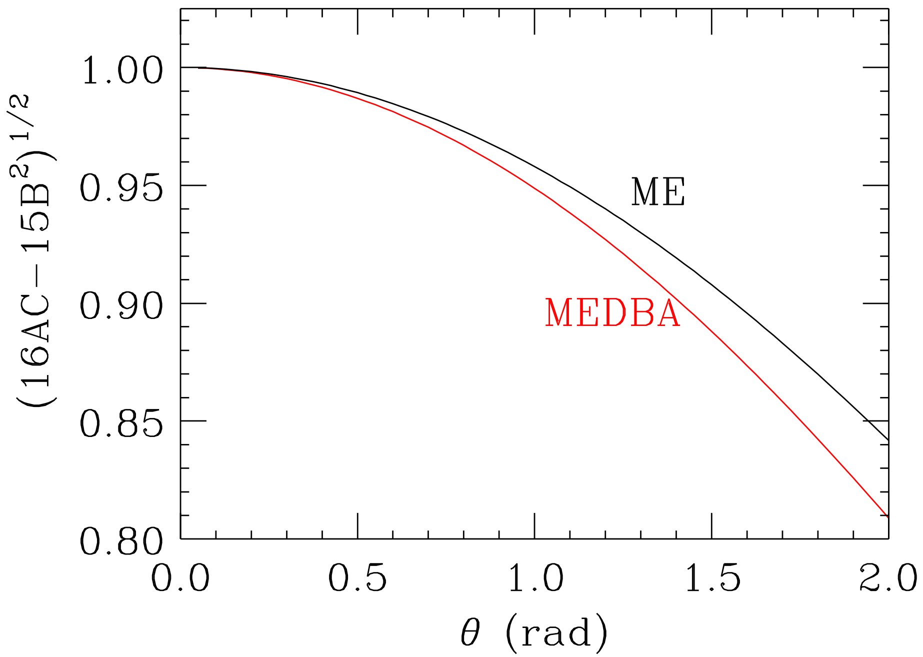
$$\frac{\partial\langle\mathcal{H}\rangle}{\partial d_0} = 0, \quad \frac{\partial\langle\mathcal{H}\rangle}{\partial d_0'} = 0, \qquad d_{0,\min} = \frac{1}{6}F, \quad d'_{0,\min} = -\frac{1}{2}E,$$

$$\langle\mathcal{H}\rangle = \frac{1}{12}\rho\theta^3\left(\tilde{\beta}_0\tilde{A} - \tilde{\alpha}_0\tilde{B} + \frac{4\tilde{\gamma}_0}{15}\tilde{C}\right) \qquad \tilde{A} = 4A - 3E^2, \; \tilde{B} = 3B - 2EF, \; \tilde{C} = \tfrac{9}{4}C - \tfrac{5}{4}F^2$$

$$\tilde{\beta}_0\tilde{\gamma}_0 = 1 + \tilde{\alpha}_0^2$$

$$\langle\mathcal{H}\rangle_{\rm ME} = \frac{\tilde{G}}{12\sqrt{15}}\rho\theta^3$$

$$\tilde{G} = \sqrt{16\tilde{A}\tilde{C} - 15\tilde{B}^2}$$



$$\tilde{\beta}_0 = \frac{8\tilde{C}}{\sqrt{15}\tilde{G}}, \quad \tilde{\alpha}_0 = \frac{\sqrt{15}\tilde{B}}{\tilde{G}}, \quad \tilde{\gamma}_0 = \frac{2\sqrt{15}\tilde{A}}{\tilde{G}}.$$

$$s^*_{\mathrm{ME}} = L/2 \qquad \beta^*_{\mathrm{ME}} = L/\sqrt{60} \qquad D^*_{\mathrm{ME}} = L\theta/24$$

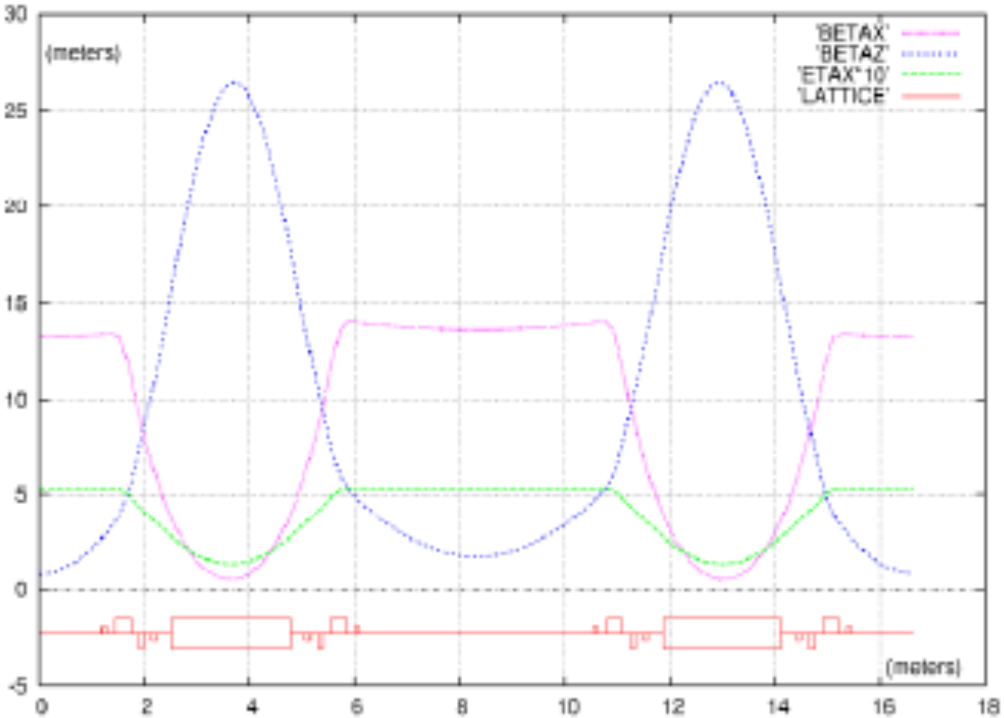
$$\mathcal{H}(0) = \mathcal{H}(\theta) = \frac{1}{3\sqrt{15}}\rho\theta^3 \left\{6\tilde{C}E^2 - \frac{15}{2}\tilde{B}EF + \frac{5}{2}\tilde{A}F^2\right\}\tilde{G}^{-1}.$$

Define the H-function at the end of TME lattice: $H_{TME} = \frac{1}{3\sqrt{15}}\rho\theta^3$

We find

$$\frac{< H >_{MEDBA}}{H_{TME}} = \frac{3}{4}$$

$$\frac{< H >_{ME}}{H_{TME}} = \frac{1}{4}$$



SESAME full period optical functions for (Qx=7.23 - Qz=6.19), **ε=26 nm**.

$$\varepsilon_{\text{nat}} = C_q \frac{\gamma^2\theta^3}{12\sqrt{15}J_x} = 12 \text{ nm}$$

SESAME design parameters.

Energy (GeV)	2.5
Circumference (m)	133.12
N. of Periods	8
Dipole field (T)	1.455
Dipole field index	11
Q_x - Q_z	7.23 – 6.19
Mom. Compaction	0.00829
N. Emitt.(nm.rad)	26.0
U_0 (keV/turn)	589.7
τ_ε, τ_x, τ_z (ms)	2.80, 2.28, 3.77
RF freq. (MHz)	499.564
Harmonic Number	222
Peak Voltage(MV)	2.4
Synch. Freq. (kHz)	37.18
σ_L (cm)	1.15
Current (mA)	400
N. of bunches	200
1/e Lifetime(hrs)	16.9

Effective emittance

$$x = x_\beta(s) + D(s)\delta, \quad x' = x'_\beta(s) + D'(s)\delta$$

$$\langle y\rangle = \int y\rho(y,y')dydy', \quad \langle y'\rangle = \int y'\rho(y,y')dydy',$$

$$\sigma_y^2 = \int (y-\langle y\rangle)^2\rho(y,y')dydy', \quad \sigma_{y'}^2 = \int (y'-\langle y'\rangle)^2\rho(y,y')dydy',$$

$$\sigma_{yy'} = \int (y-\langle y\rangle)(y'-\langle y'\rangle)\rho(y,y')dydy' = r\sigma_y\sigma_{y'},$$

$$\sigma_x^2 = \beta_x\epsilon_x + D^2\sigma_\delta^2, \quad \sigma_{x'}^2 = \gamma_x\epsilon_x + D'^2\sigma_\delta^2, \quad \sigma_{x,x'} = -\alpha_x\epsilon_x + DD'\sigma_\delta,$$

$$\epsilon_{x,\text{eff}} \equiv \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\epsilon_x\left[\epsilon_x + \mathcal{H}(s)\sigma_\delta^2\right]},$$

$$\mathcal{H}(s) = \gamma_x D^2 + 2\alpha_x DD' + \beta_x D'^2.$$

Define the dispersive emittance as

$$\epsilon_d \equiv \gamma_x (D\sigma_\delta)^2 - \beta_x'(D\sigma_\delta)(D'\sigma_\delta) + \beta_x(D'\sigma_\delta)^2 = \mathcal{H}(0)\sigma_\delta^2,$$

$$\sigma_\delta^2 = (\sigma_E/E)^2 = C_q\gamma^2/\rho\mathcal{J}_E$$

$$\epsilon_d = \frac{1}{3\sqrt{15}}\frac{C_q\gamma^2\theta^3}{\mathcal{J}_E} \qquad \mathcal{J}_E \approx 2,\ \mathcal{J}_x \approx 1 \text{ or } \mathcal{J}_E \approx 2\mathcal{J}_x$$

$$\epsilon_{x,1\mathrm{D}} = \epsilon_x + \mathcal{H}_{\mathrm{ID}}\sigma_\delta^2 = \epsilon_{\mathrm{ME}} + \epsilon_d = \frac{1}{4\sqrt{15}}\frac{C_q\gamma^2\theta^3}{\mathcal{J}_x} = \epsilon_{\mathrm{MEDBA}}$$

$$\epsilon_{x,\mathrm{eff}} = \sqrt{\epsilon_x\epsilon_{x,1\mathrm{D}}}\ , \qquad \epsilon_{z,\mathrm{eff}} = \epsilon_z\,,$$

D. Three-bend achromat

Table I BESSY II Basic Parameters

Energy [GeV]	1.7 (0.9-1.9)
Electron Current [mA]	> 100
Lattice Type	TBA (10)
Circumference [m]	194.39
Harmonic Number	324
Bunch Length [ps]	20
Tune Q_x/Q_z	14.16/6.18
Chromaticity ξ_x/ξ_z	-39/-13
Momentum Compaction α	$1.6 \cdot 10^{-2}$
Natural Emittance [nmrad]	6.2
β_x/straight/max/min [m]	8.0/17.2/0.27
β_z/straight/max/min [m]	3.0/10.8/2.9
Energy Spread $\Delta E/E$	$7 \cdot 10^{-4}$
Beam Lifetime [h]	> 6
Defl. Radius ρ [m]	4.2
Dipole Field [T]	1.5
Length Straight [m]	5.8
Insertion Devices N	8 - 10
Pre-Accelerator:	50 MeV-Microtron
Injector:	Full Energy Synchrotron

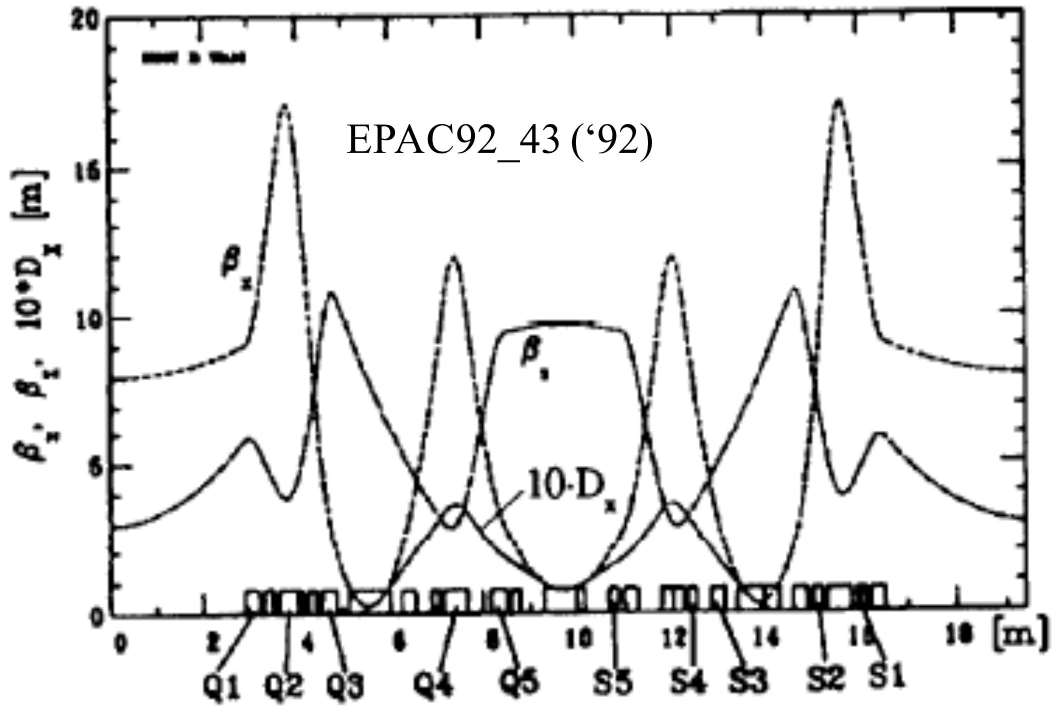


Fig. 2: Lattice functions β_x, β_z and $10 \cdot D_x$ for one superperiode of BESSY II.

Theoretical ε_{MEDBA}=1.85 nm compared to 6.2 nm.

Problem: Dispersion function mis-match

$$X_{\mathrm{MEDBA}} = \frac{D}{\sqrt{\beta}} = \frac{(15)^{1/4}}{8}\frac{L_1^{3/2}}{\rho_1}, \qquad X_{\mathrm{ME}} = \frac{D}{\sqrt{\beta}} = \frac{\sqrt{2}(15)^{1/4}}{24}\frac{L_2^{3/2}}{\rho_2},$$

$$P_{\mathrm{MEDBA}} = \frac{\alpha D + \beta D'}{\sqrt{\beta}} = \pm\frac{7}{8(15)^{1/4}}\frac{L_1^{3/2}}{\rho_1}, \qquad P_{\mathrm{ME}} = \frac{\alpha D + \beta D'}{\sqrt{\beta}} = \mp\frac{3}{4\sqrt{2}(15)^{1/4}}\frac{L_2^{3/2}}{\rho_2},$$

$$\begin{pmatrix} X_{\mathrm{ME}} \\ P_{\mathrm{ME}} \end{pmatrix} = \begin{pmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{pmatrix} \begin{pmatrix} X_{\mathrm{MEDBA}} \\ P_{\mathrm{MEDBA}} \end{pmatrix} \qquad \frac{L_2^3}{\rho_2^2} = 3\frac{L_1^3}{\rho_1^2}$$

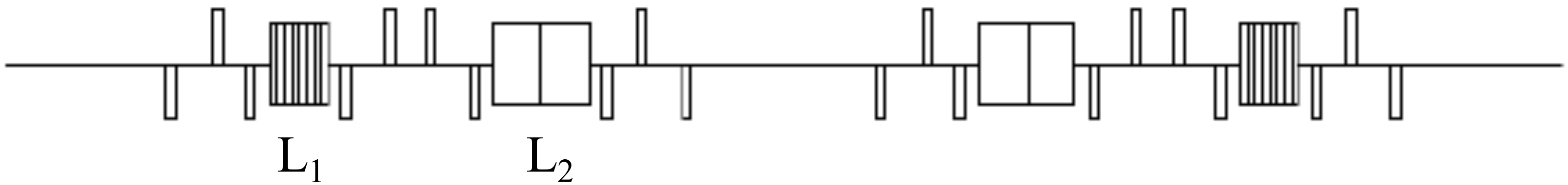
The matching condition requires $L_2 = 3^{1/3}L_1$ for isomagnetic storage rings, or $\rho_1 = \sqrt{3}\rho_2$ for storage rings with equal length dipoles.

The emittance of the matched minimum TBA (QBA, or nBA) lattice is

$$\epsilon_{\mathrm{METBA}} = \frac{C_q\gamma^2\theta_1^3}{4\sqrt{15}\mathcal{J}_x}$$

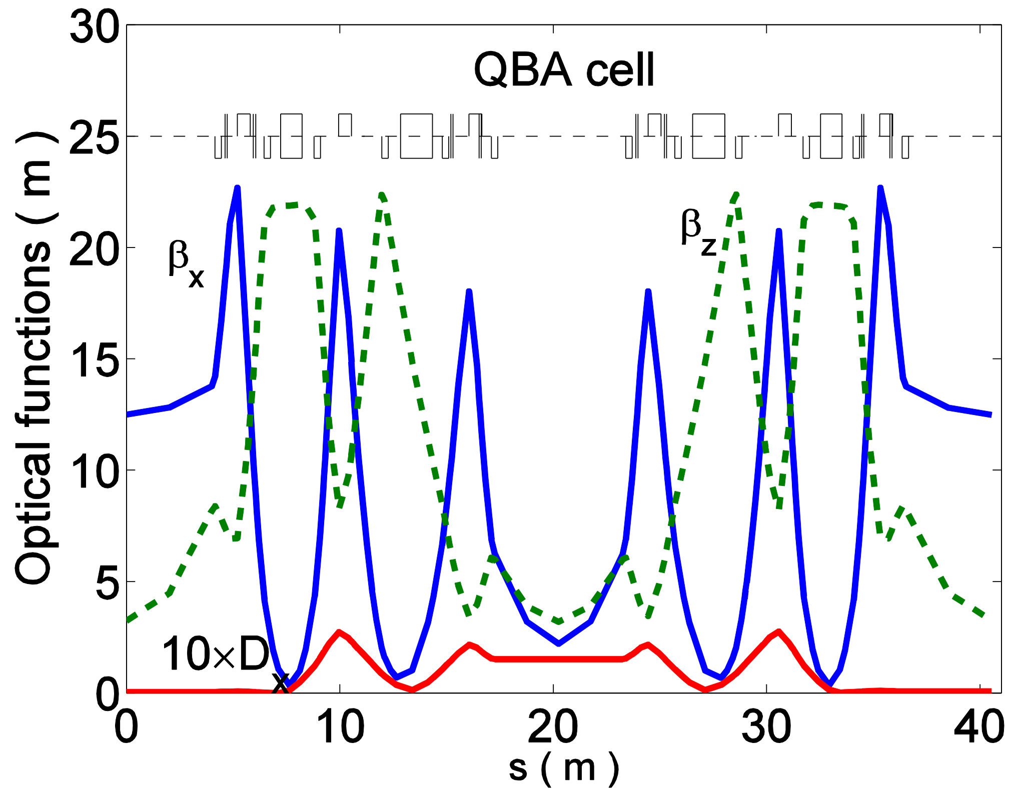
where θ_1 is the bending angle of the outer dipoles, provided the middle dipole is longer by a factor of $3^{1/3}$ than the outer dipoles. The formula for the attainable minimum emittance is identical to that for the MEDBA.

The Quadruple-bend achromat



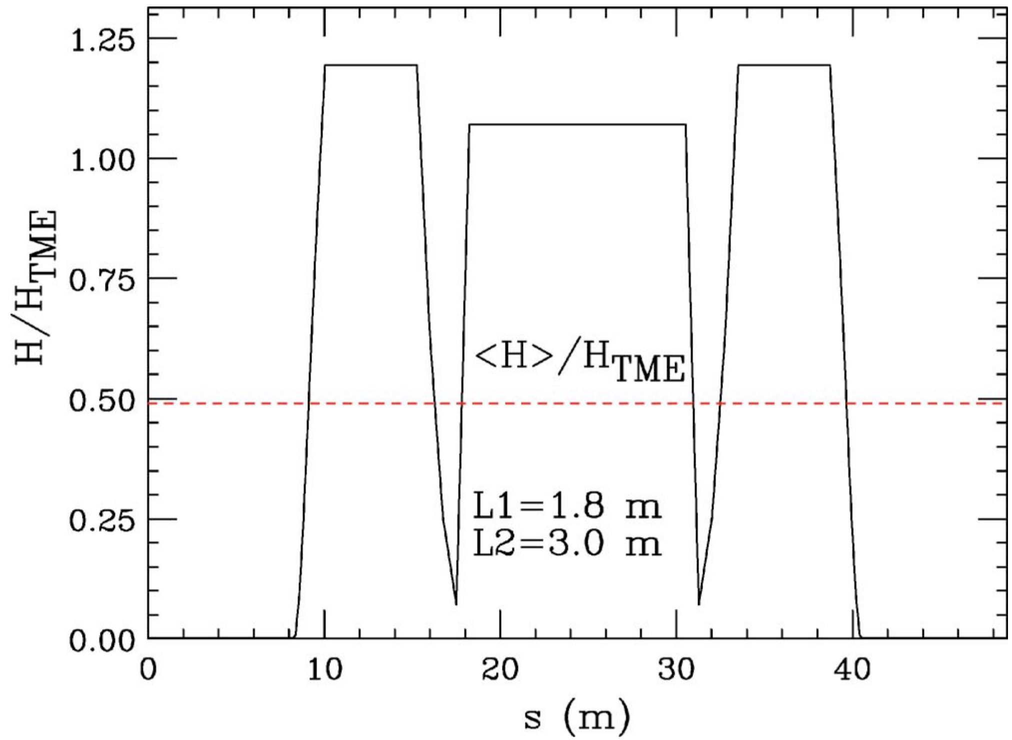
With L_2=1.45L_1, we find that the emittance should scale like $[2/(1+3^{1/3})]^3$=**0.54**.

1. Matching is relatively easy (as compared to TBA lattices)
2. The chromatic properties is as good as the DB lattice!
3. The emittance obeys the scaling law: $\gamma^2\theta^3$, i.e. 30 cells (15 QBA cells) will give about 1.24 nm emittance without any damping wiggler!



Consider a lattice with 16 QBA cells, a circumference of 780.3 m. The parameters used in this lattice are L_1=1.8 m and L_2=3.0 m.The lengths of the straight sections are 10.0 and 5.77 m. The emittance of 1.02 nm at the 3 GeV beam energy.

H/H_{TME} of one super QBA cell with The $H_{\mathrm{TME}}=\rho\theta_2^3/3\sqrt{15}$ is the maximum H-function for the theoretical minimum emittance TME lattice based on the middle dipole. The dotted line is the average H-function in dipoles, derived from the emittance of 1.02 nm at 3 GeV. The ratio is 0.25 for a theoretical MEQBA!



III.4 Beam Physics of High Brightness Storage Rings

A. Low emittance lattices and the dynamical aperture

B. Diffraction limit

$$\Delta x_{\mathrm{r}} \Delta x'_{\mathrm{r}} = \Delta z_{\mathrm{r}} \Delta z'_{\mathrm{r}} = \sigma_{\mathrm{r}} \sigma'_{\mathrm{r}} \geq \lambda/4\pi = \epsilon_{\mathrm{photon}},$$

C. Beam lifetime

D. Collective beam instabilities