Brookhaven National Laboratory

U.S. DEPARTMENT OF ENERGY

USPAS'23 | Hadron Beam Cooling

# Stochastic Cooling: Introduction

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# Plan for today:

| | |
|---|---|
| **9:00-10:20am:** | Lecture #1: Introduction, general idea of Stochastic Cooling (SC) |
| **10:20-10:40am:** | Break |
| **10:40am-12:00pm**: | Lecture #2: An in-depth look at SC, experimental realizations of SC |
| **12:00-2:00pm:** | Lunch |
| **2:00-3:20pm:** | Lecture #3: Optical Stochastic Cooling (OSC) and its experimental realization |
| **3:20-4:30pm:** | Break |
| **4:30-6:00pm:** | HW Q&A |
| **6:00-7:00pm:** | Dinner |
| **7:00pm-12:00am:** | HW in the study room (instructors are available for discussion **7:30-9:00 pm**) |

# Lecture#1:

- Introduction to SC: general idea & single particle picture
- From 1 particle to a bunch: sampling (Nyquist theorem), coherent & incoherent effects, mixing & noise
- Types of SC: Palmer method, betatron oscillation cooling, Hereward cooling, filter method
- Frequency Domain picture of SC: Schottky noise

## References:

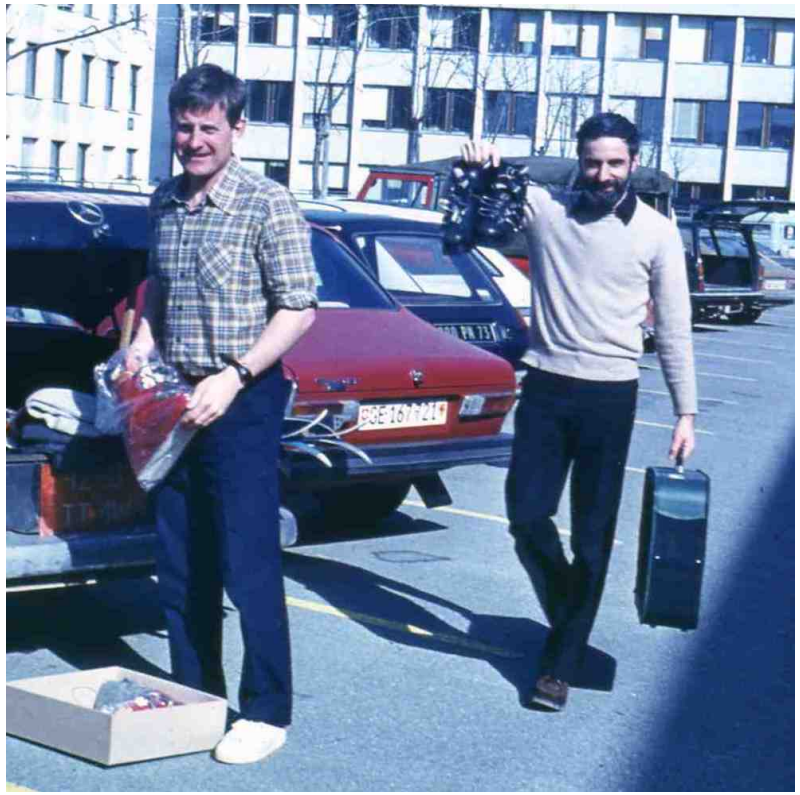
For more information, please refer to the following literature:


[1] Dieter Möhl, "*Stochastic cooling of particle beams,"* Springer, Berlin, Heidelberg, 2013.
[2] S. Nagaitsev *"Beam Cooling"* lecture, 2019.
[3] Dieter Möhl, "*Stochastic cooling for beginners"*.

# Brief History of Stochastic Cooling

- The idea was proposed by Simon van der Meer at CERN in 1969.
  *"Stochastic damping of betatron oscillations"*
  *Internal Report CERN/ISR-PO/72-31 1972.*

- The required technology was developed there, and successfully applied to the CERN proton-antiproton collider in the 70s.

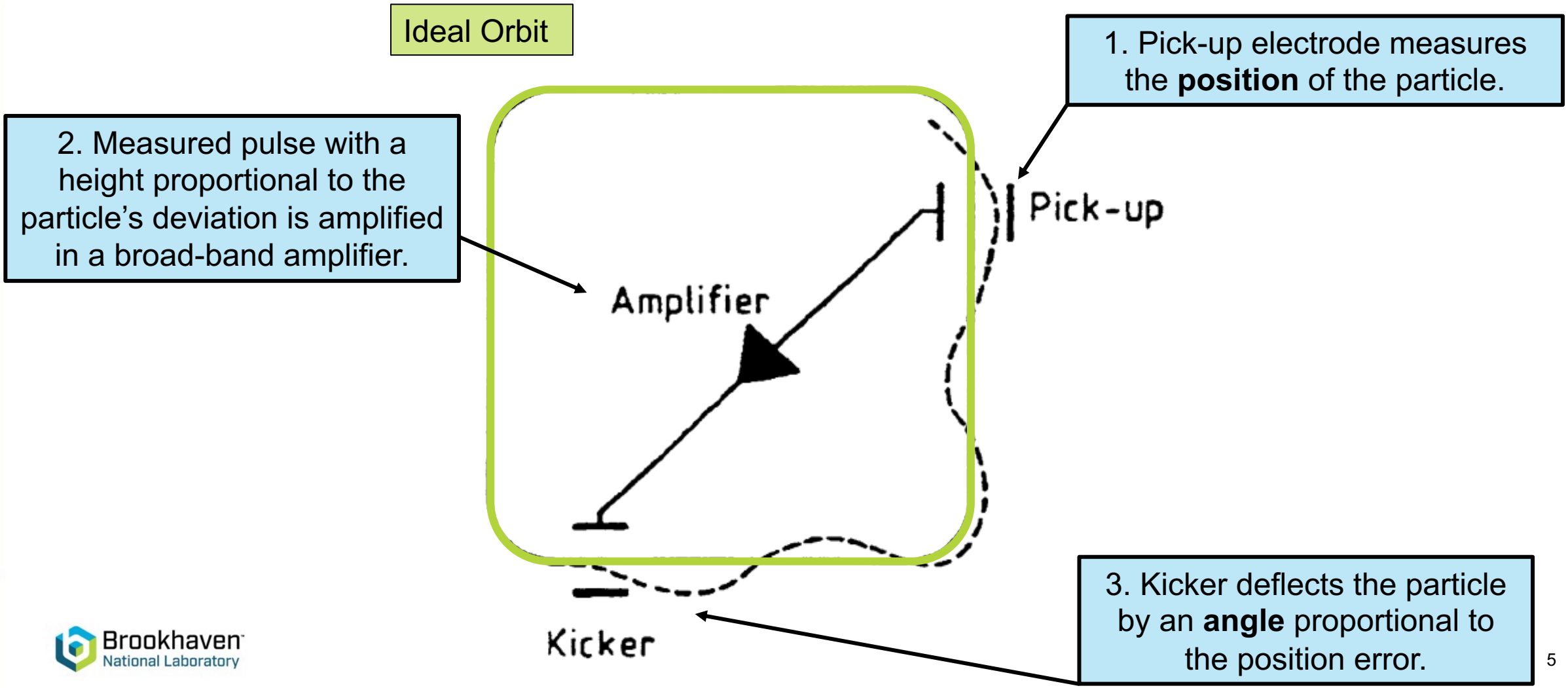


- 1975 – first experimental demonstration of emittance cooling at ISR (Intersecting Storage Rings) by Lars Thorndahl and Dieter Möhl.

- S. van der Meer shared the Nobel Prize in Physics in 1984 with Carlo Rubbia for contributions to the CERN project which led to the discovery of the W and Z particles.

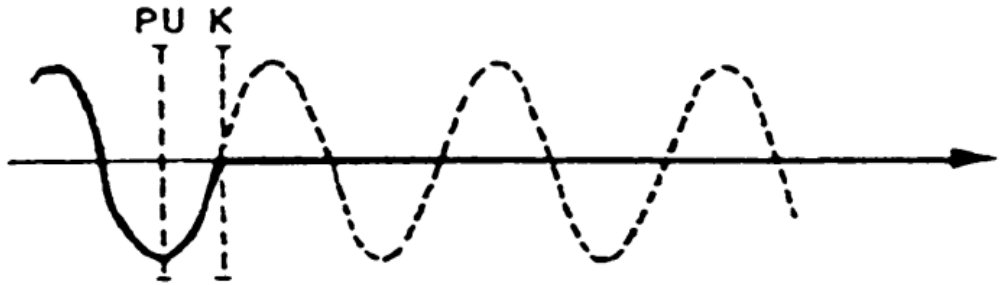
# The Basic Set-Up: Single Particle Model

- Circulating **particle has some small error in position and angle** with respect to the ideal orbit.
- We will approximate betatron oscillations by sinusoidal motion.
- **Goal**: damp this oscillation!

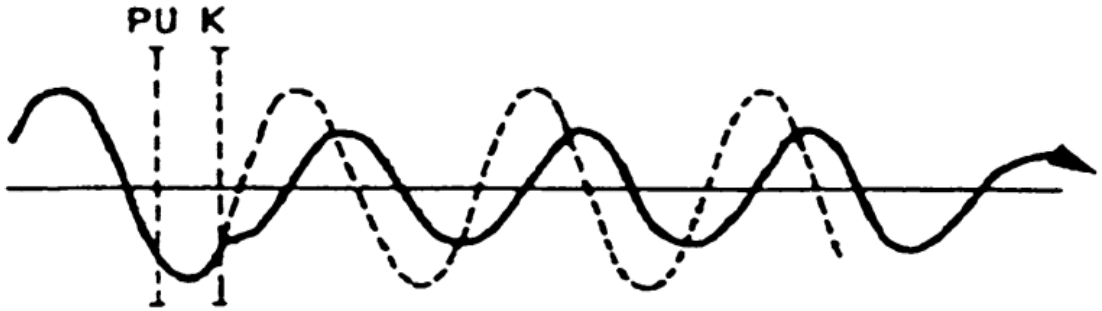


# The Basic Set-Up: Single Particle Model

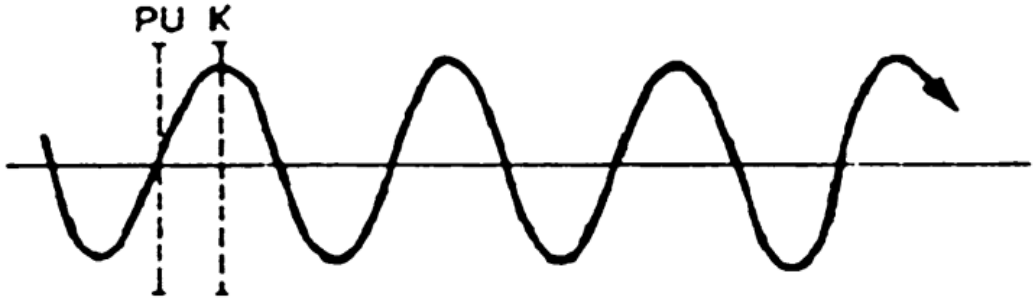
The **pick-up detects position**, and the **kicker corrects the angle** → **separate pick-up and kicker by a quarter of the betatron oscillation** (plus an integer number of half wavelengths, if necessary).



Particle crosses the pick-up with max displacement. Its oscillation is completely cancelled in the kicker.



Particle crosses the pick-up at an intermediate phase. Its oscillation is partially eliminated – will take several passages to completely eliminate the oscillations.



Particle crosses the pick-up in the most unfavorable phase. Its oscillation is not affected by the system.

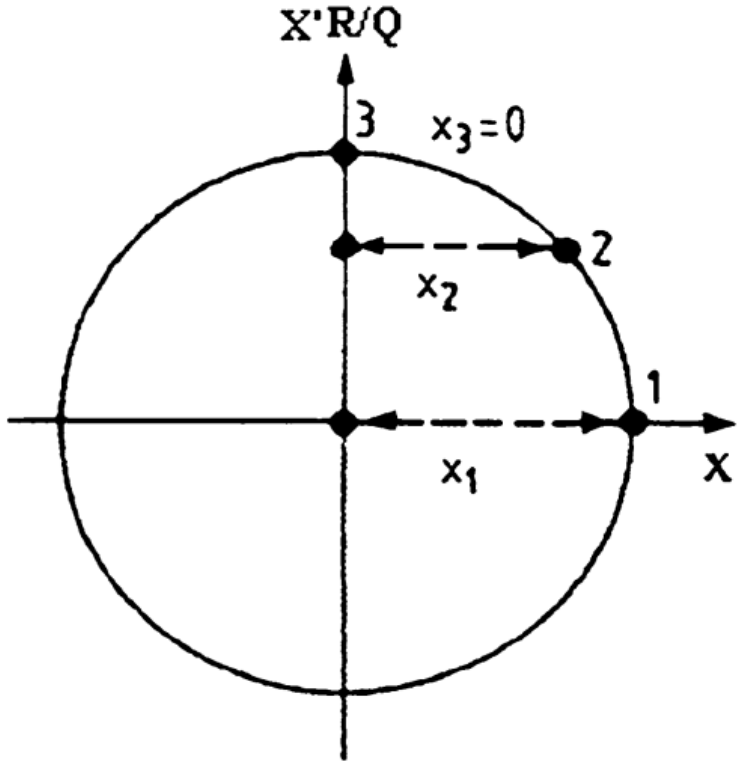
# The Basic Set-Up: Single Particle Model

**Phase space representation of betatron cooling:**

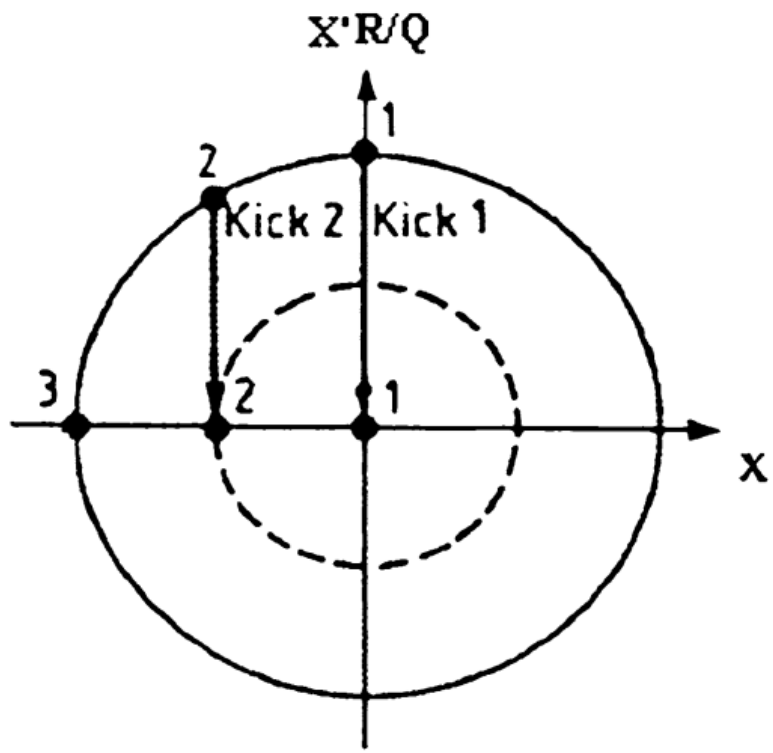
- The undisturbed particle motion is given by a circle with the radius equal to the betatron amplitude $\tilde{x}$.

$$\begin{cases} x = \tilde{x} \cdot \sin(Q(s/R) + \mu_0), \\ x' = \tilde{x} \cdot (Q/R) \cdot \cos(Q(s/R) + \mu_0). \end{cases}$$

- Kicks correspond to jumps in $x'$ .
- The cooling system tries to put particles onto smaller circles.




At pick-up




At kicker

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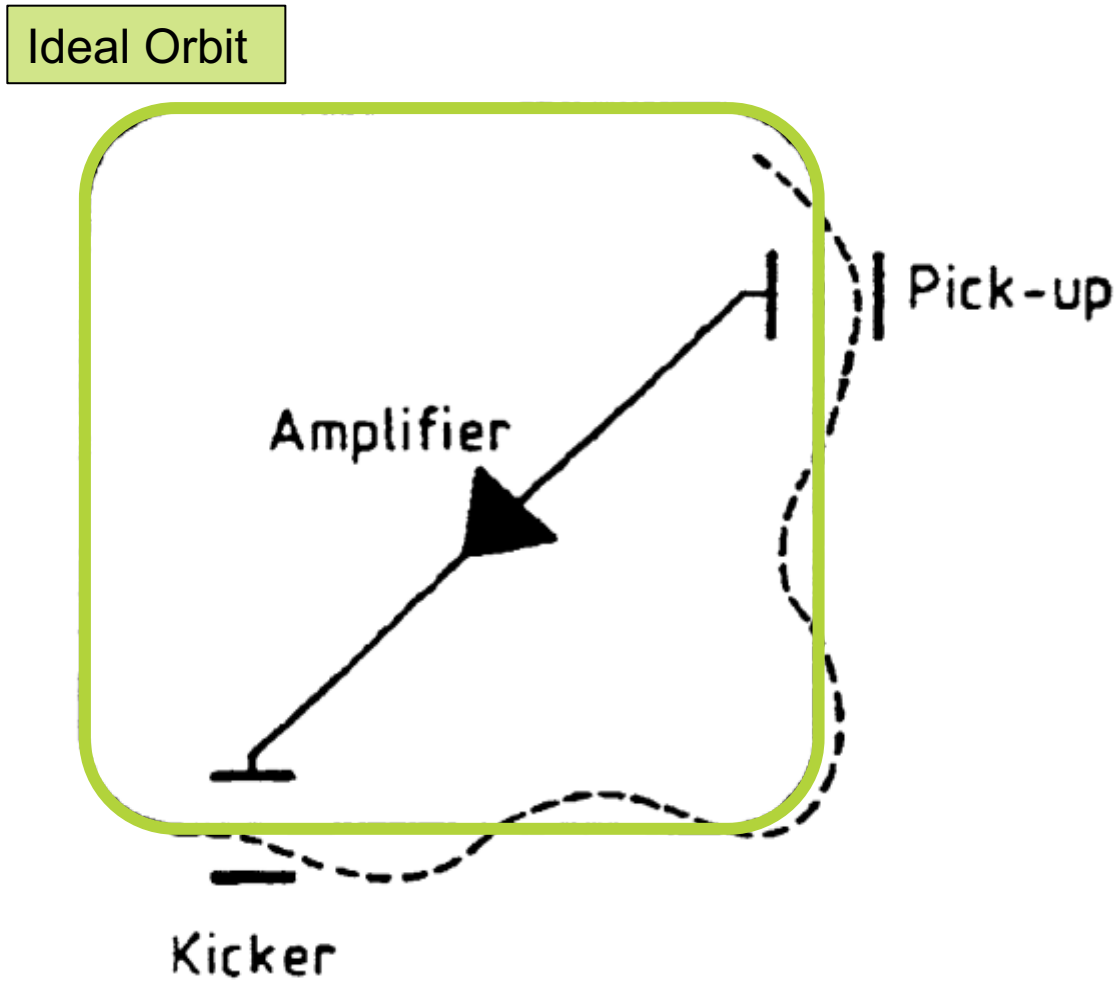
# The Basic Set-Up: Single Particle Model

**The correction signal has to arrive at the kicker at the same time as the test particle.**




- Signal is delayed in the cables and the amplifier.
- The cooling path must take a short cut across the ring.
- Only at low and medium energy $v/c < 0.5$ is a parallel path feasible.

# The Basic Set-Up: Single Particle Model

**Two constraints on the distance from the pickup to the kicker:**

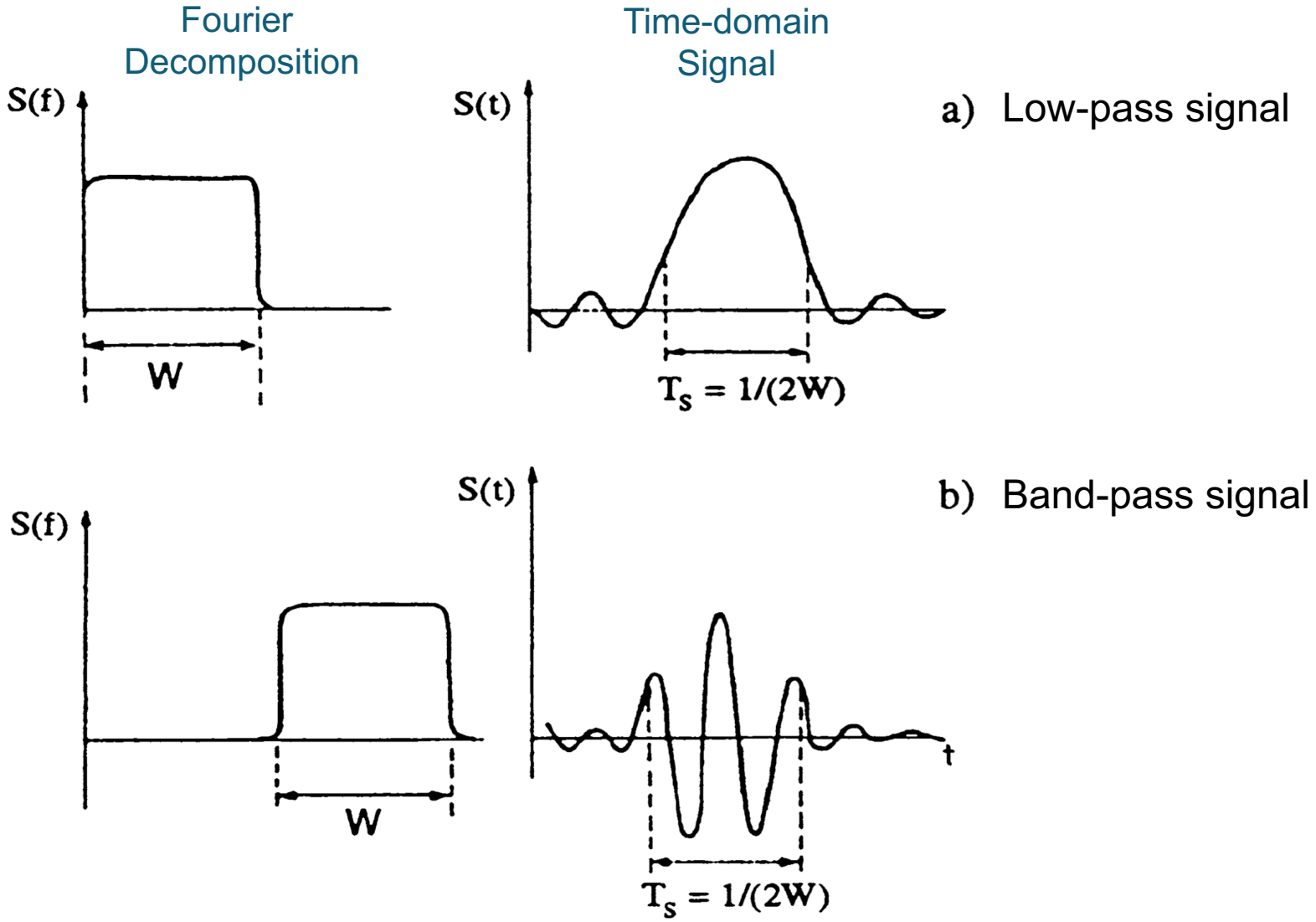
1. **Along the beam:** the distance quantized due to the required phase relationship of the betatron oscillation.
2. **Along the cooling path:** the distance is fixed due to the synchronism between the particle and the signal.

## What happens when we move from the single particle model to a beam of particles?

- Particles in the beam will oscillate incoherently—they will have different phases and amplitudes.
- Let us group particles in the beam into samples that will receive the same correcting kick during a passage through the system.

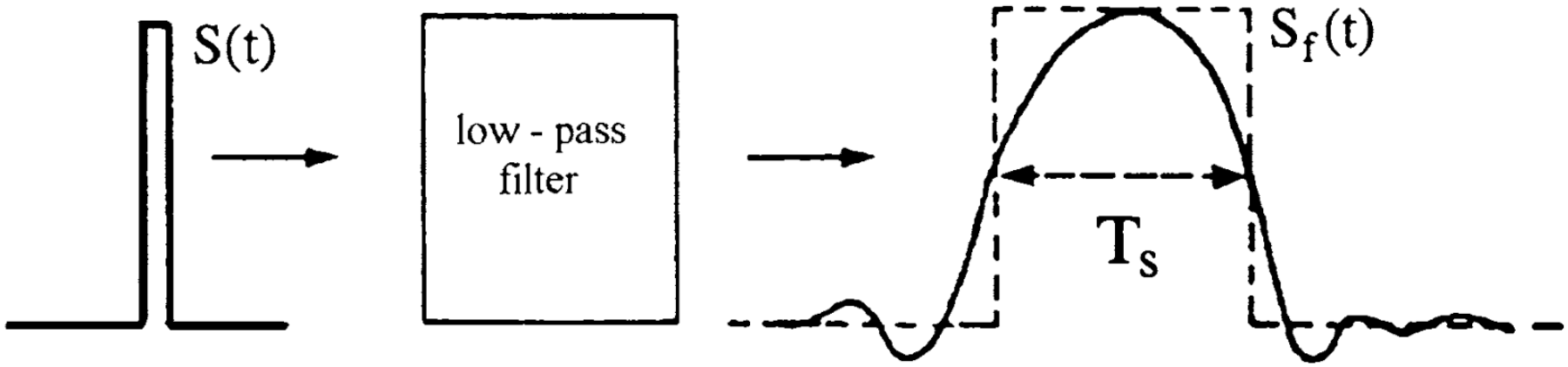
# Nyquist Theorem: relationship between bandwidth and pulse length

*If a signal has a Fourier decomposition of bandwidth $\Delta f = W$, then it's typical time duration will be $T_s = 1/(2W)$*



# Nyquist Theorem: relationship between bandwidth and pulse length

*When a short pulse is filtered by a low-pass or band-pass filter of bandwidth $W$, the resulting pulse has a typical time width $T_s = 1/(2W)$*

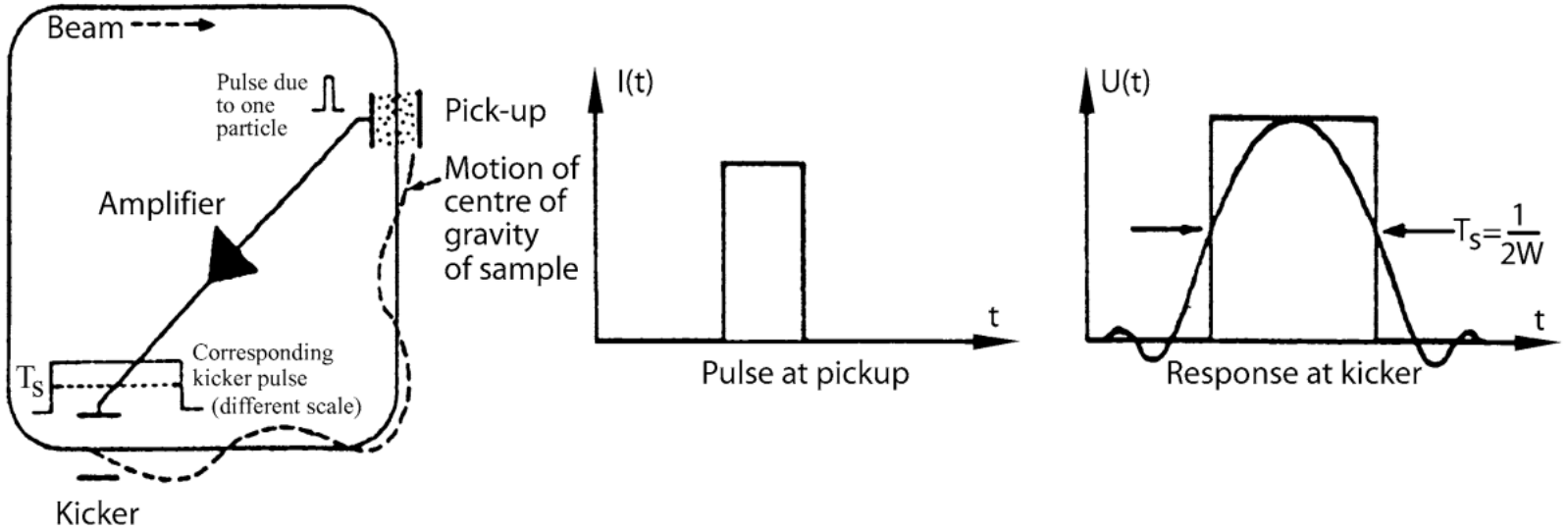



Input and Output signals of a low-pass system

**How is it applicable in the case of a cooling system?**

# Beam Samples

- Passing through the pick-up, particle induces a short pulse with a length given by the transit time.
- Due to the finite bandwidth of the cooling system, the kicker signal will broaden into a pulse of length $T_s$ .




- A test particle passing system at $t_0$ will be affected by the kicks due to all particles passing during the time interval $t_0 \pm T_s/2$ .
- These particles belong to the same sample of the test particle.
- In a uniform beam of length $T_{rev}$ (revolution time), there are $n_s = T_{rev}/T_s$ equally spaced samples of length with

$$N_s = N/(2WT_{rev})$$ particles per sample.

# Coherent and Incoherent Effects

In the “test particle picture”, **cooling is a competition** between:

1. The ***coherent effect*** of the test particle upon itself via the cooling loop.
2. The ***incoherent effect*** – the disturbance of the test particle by the kicks due to the sample members.

Consider a test particle with displacement $x$ and assume the corresponding correction at the kicker $\lambda x$ .
If **no other particles present** in the sample, the corrected position $x_{\mathrm{c}} = x - \lambda x$

Including other sample members, the corrected position of the test particle after one turn:

$$x_{\mathrm{c}} = x - \lambda x - \sum_{s'} \lambda_i x_i \longrightarrow \Delta x = \underbrace{-\lambda x}_{\text{coherent}} \underbrace{- \sum_{s'} \lambda_i x_i}_{\text{incoherent}} \qquad (1)$$

If $\lambda_i = \lambda$ , one can rewrite $\Delta x = \lambda x - \lambda \sum_{s'} x_i$ . Let us introduce the average sample error: $\langle x \rangle_{\mathrm{s}} = \frac{1}{N_{\mathrm{s}}} \sum_{s} x_i$

Then (1) becomes: $x_{\mathrm{c}} = x - (\lambda N_{\mathrm{s}}) \langle x \rangle_{\mathrm{s}} \longrightarrow \Delta x = -(\lambda N_{\mathrm{s}}) \langle x \rangle_{\mathrm{s}} \equiv -g \langle x \rangle_{\mathrm{s}}$

gain

$$\Delta x = \underbrace{-\frac{g}{N_{\mathrm{s}}} x}_{\text{coherent}} \underbrace{- g \langle x \rangle_{\mathrm{s}'}}_{\text{incoherent}}$$

# How Should We Treat Incoherent Effects? –Neglect It!

$$\Delta x = \underbrace{-\frac{g}{N_{\mathrm{s}}}x}_{\text{coherent}} \underbrace{- g\langle x\rangle_{\mathrm{s}'}}_{\text{incoherent}}$$

One expects an exponential form for the error of the test particle $x = x_0 e^{-t/\tau}$.

That brings us to the expression for the damping rate:

$$\frac{1}{\tau} = \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} \approx \frac{1}{x}\frac{\Delta x_{\text{per turn}}}{T_{\mathrm{rev}}}$$

Substituting this into the boxed equation without incoherent term gives:

$$\frac{1}{\tau} = \frac{g}{T_{\mathrm{rev}} N_{\mathrm{s}}}$$

Let us put $g = 1$ to make an estimate of the upper limit: Then we obtain an approximation for the cooling rate:

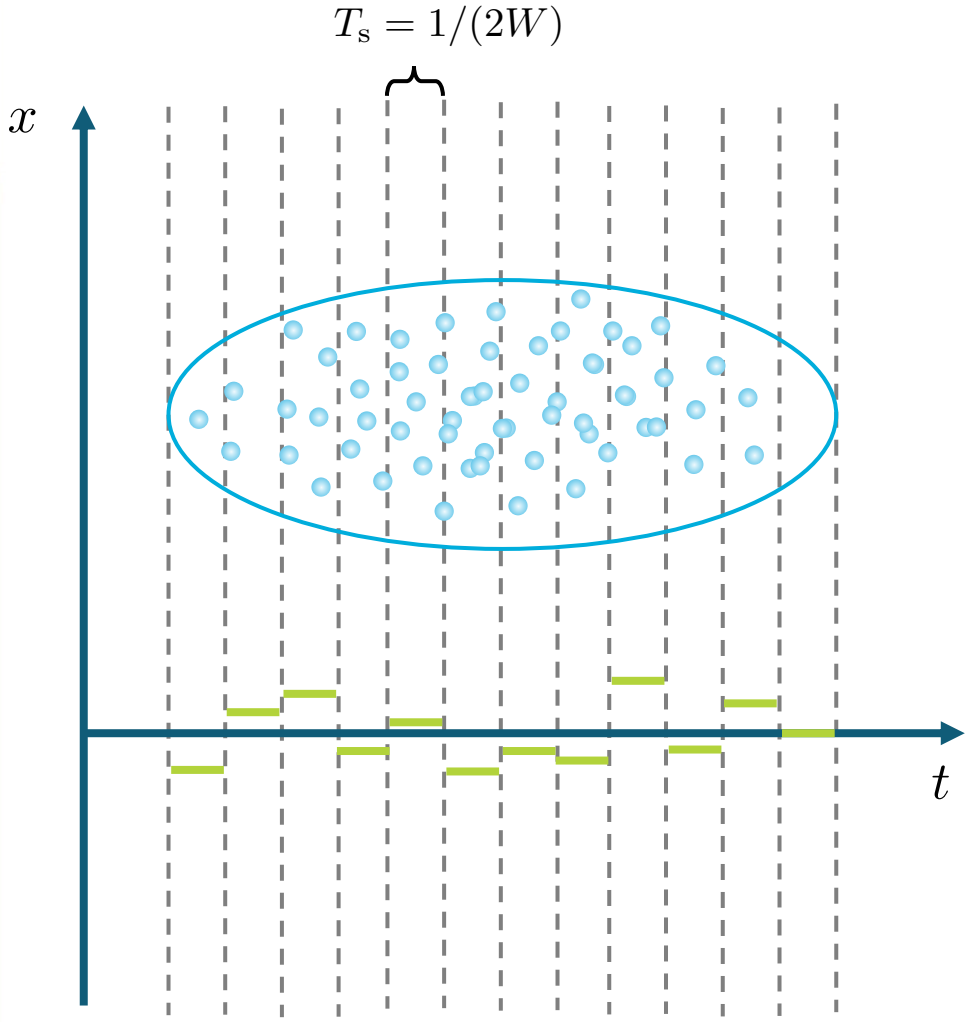
$$\frac{1}{\tau} = \frac{2W}{N}$$

Will suffer from bandwidth limitations and large number of particles!

*This simple expression overestimates the optimal cooling rate by only a factor of 2

# How can we evaluate an incoherent term?

Beam of $N$ particles characterized by an average $\langle x \rangle = 0$ and a variance $\langle x^2 \rangle = x_{\mathrm{rms}}^2$ .




If we consider $N_s$ samples, we can characterize:

- the sample average $\langle x \rangle_s$
- the sample variance $\langle x^2 \rangle_s$
- the square of the sample average $(\langle x \rangle_s)^2$

What are the most probable values of these sample characteristics?

- $E\left(\langle x \rangle_s\right) = \langle x \rangle = 0$
- $E\left(\langle x^2 \rangle_s\right) = \langle x^2 \rangle = x_{\mathrm{rms}}^2$
- $E\left(\left(\langle x \rangle_s\right)^2\right) = x_{\mathrm{rms}}^2 / N_s$

If we replace the population average by the sample average, one can expect an error:

$$\langle x \rangle_s = \langle x \rangle \pm x_{\mathrm{rms}} / \sqrt{N_s}$$

The larger the beam variance and the smaller the sample size, the more imprecise is the sampling.

*When a large number of random samples of size $N_s$ are taken from a population with statistics $\langle x \rangle = 0$ and $\langle x^2 \rangle = x_{\mathrm{rms}}^2$ then the distribution of the sample averages is approximately Gaussian with a mean equal to the population mean and a standard deviation $\sigma = x_{\mathrm{rms}} / \sqrt{N_s}$*

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# How can we evaluate an incoherent term?

Earlier we obtained: $\Delta x = -(\lambda N_\mathrm{s})\langle x\rangle_\mathrm{s} \equiv -g\langle x\rangle_\mathrm{s}$

Let us instead evaluate the change per turn of the squares: $\Delta(x^2) = x_\mathrm{c}^2 - x^2$

$$\Delta(x^2) = -2x\frac{g}{N_\mathrm{s}}\sum_\mathrm{s} x_i + \left(\frac{g}{N_\mathrm{s}}\sum_\mathrm{s} x_i\right)^2$$

$$-2x\frac{g}{N_\mathrm{s}}\sum_\mathrm{s} x_i = -2g\left(\frac{x^2}{N_\mathrm{s}} + \frac{x}{N_\mathrm{s}}\sum_{\mathrm{s}'} x_i\right)$$

$$E\left(\frac{1}{N_\mathrm{s}}\sum_{\mathrm{s}'} x_i\right) \approx E\left(\frac{1}{N_\mathrm{s}-1}\sum_{\mathrm{s}'} x_i\right) = E\left(\langle x_i\rangle_{\mathrm{s}'}\right) = 0$$

obtained using sampling relation from the previous slide

$$\left(\frac{g}{N_\mathrm{s}}\sum_\mathrm{s} x_i\right)^2 = g^2(\langle x\rangle_\mathrm{s})^2 = \frac{g^2}{N_\mathrm{s}}x_\mathrm{rms}^2$$

$$\boxed{\Delta(x^2) \to -\frac{2g}{N_\mathrm{s}}x^2 + \frac{g^2}{N_\mathrm{s}}x_\mathrm{rms}^2}$$

If a typical error equals to the beam r.m.s. $\frac{1}{x_\mathrm{rms}^2}\Delta(x_\mathrm{rms}^2) \to -\frac{1}{N_\mathrm{s}}(2g - g^2)$

The cooling rate for the beam variance: $\frac{1}{\tau_{x^2}} = \frac{1}{T}\frac{\Delta(x_\mathrm{rms}^2)}{x_\mathrm{rms}^2} = \frac{2W}{N}(2g - g^2)$

coherent incoherent

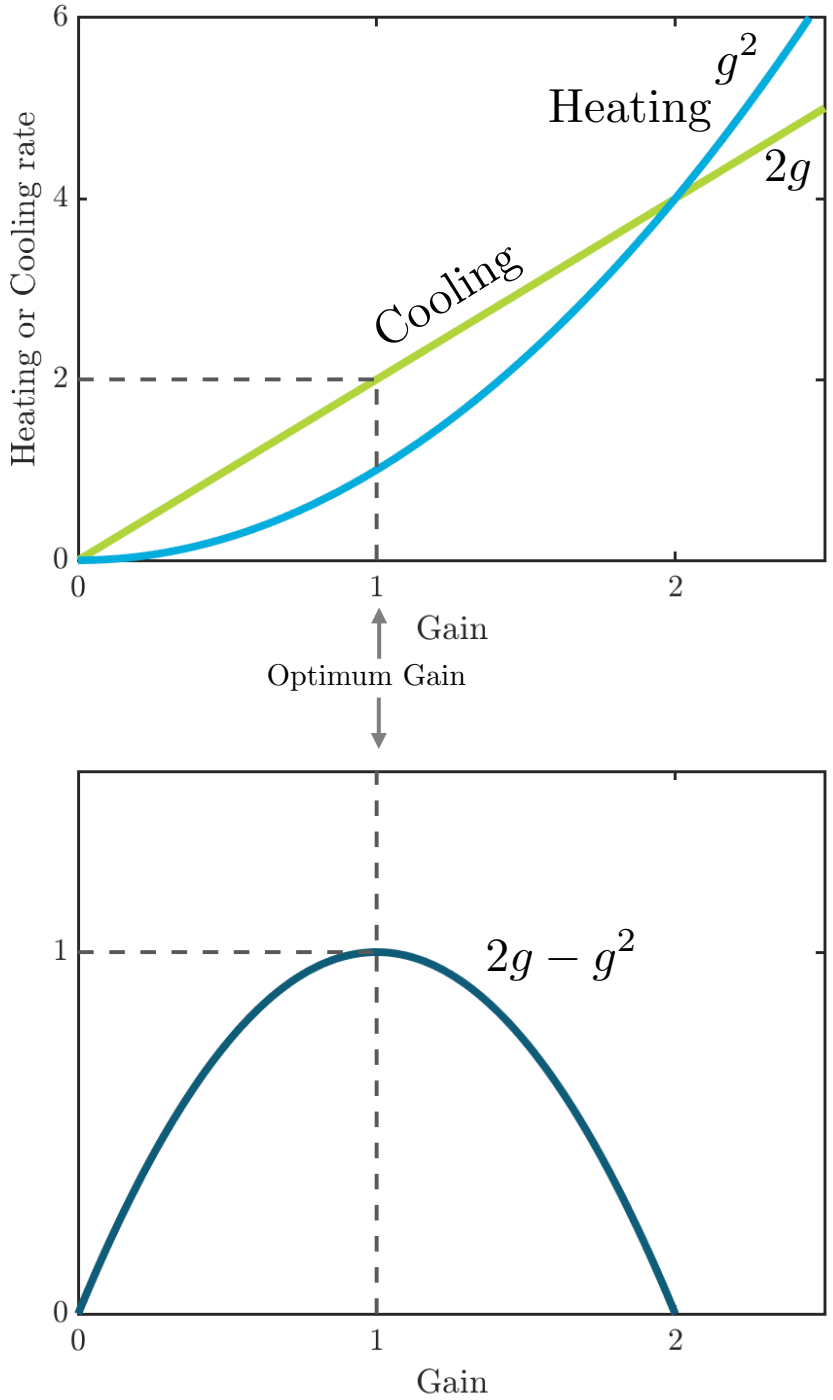
*Note*: $\frac{1}{\tau} = \frac{1}{2\tau_{x^2}}$

# How can we evaluate an incoherent term?

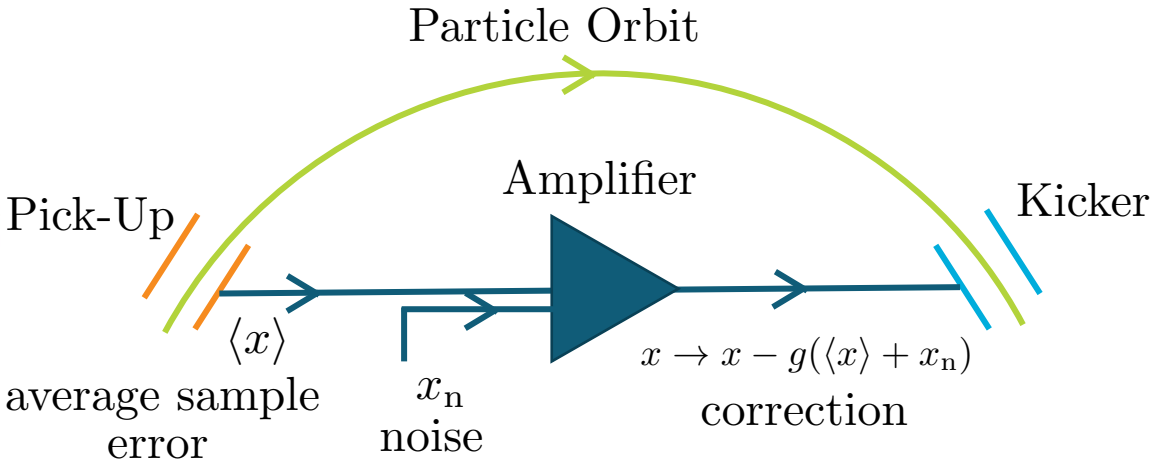
$$\frac{1}{\tau_{x^2}} = \frac{1}{T}\frac{\Delta(x^2_{\mathrm{rms}})}{x^2_{\mathrm{rms}}} = \frac{2W}{N}\left(2g - g^2\right)$$

*Important observations:*

- Gain shouldn't be too high!
- Optimum cooling is obtained with $g = 1$
- Anti-damping occurs when $g > 2$



# What about the noise in the system itself?




Including system noise: $x_c = x - g\langle x \rangle_s - g x_n$

Cooling rate: $\frac{1}{\tau_{x^2}} = \frac{2W}{N}\left[2g - g^2(1+U)\right]$

where $U = \frac{E(x_n^2)}{E(\langle x \rangle_s^2)}$ is noise-to-signal ratio

*Important observations:*

- *Noise increases the incoherent term by* $(1+U)$
- *Cooling remains possible despite very poor signal-to-noise ratio* $(1/U \ll 1)$ *. All we have to do is to choose small enough gain:* $g < g_0 = 1/(1+U) \approx 1/U$*. Be patient:* $\tau_{x^2} > NU/(2W)$

- $U$ *tends to increase as cooling proceeds: the electronic noise is the same, but the signal decreases.*
- *With changing* $U$ *, cooling is no longer exponential. Cooling stops completely when* $U$ *is such that* $1+U = 2/g$*.*
- *To avoid this saturation, decrease the gain during cooling to work close to the optimum* $g = g_0 = 1/(1+U)$

- *We want: Large number of pickups as close as possible to the beam. High quality, low-noise preamplifiers often working at cryo temperatures.*

# Mixing: Kicker-to-Pick-Up

So far, we have assumed random samples.
However, due to the momentum spread, particles go at slightly different speeds → sample population changes = MIXING.

Dispersion of the revolution time with momentum:

$$\frac{\Delta T_{\mathrm{rev}}}{T_{\mathrm{rev}}} = \eta \frac{\Delta p}{p}$$

with slip-factor $\eta = \gamma_{\mathrm{t}}^{-2} - \gamma^{-2}$ characterizing the distance between the working and transition energies.

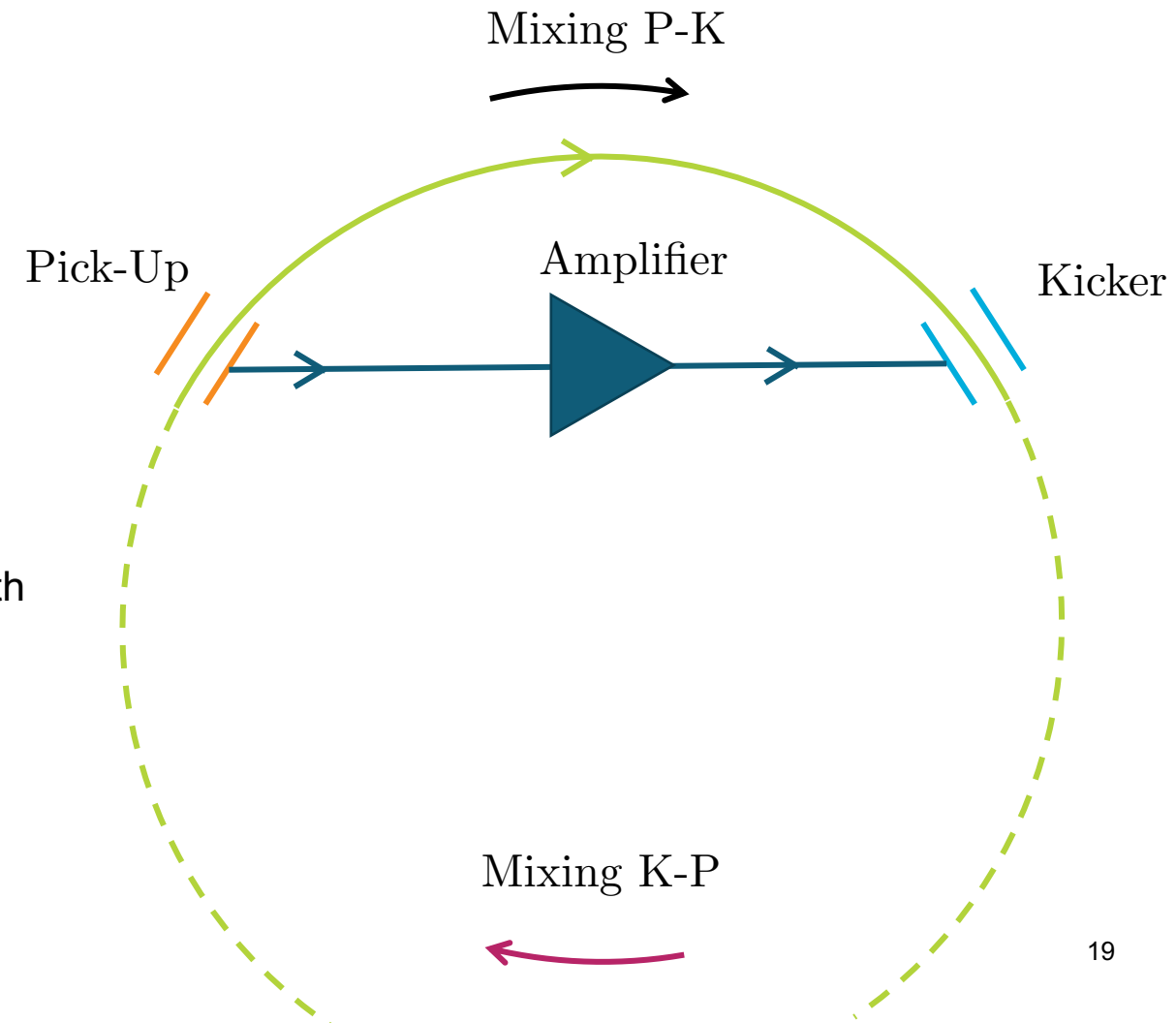
*Mixing K-P (Kicker-to-Pick-Up):*

- If **mixing is fast** → complete re-randomization → **random samples** approach is valid.
- If mixing is incomplete → slower cooling.
- **Bad mixing** = particle stays too long with the same noisy neighbors → **increased incoherent heating**.

$$\frac{1}{\tau_{x^2}} = \frac{2W}{N}\left[2g - g^2\left(M + U\right)\right]$$

$M \geq 1$ — mixing factor: the number of turns for a particle with one standard deviation in momentum to migrate by one sample length $T_{\mathrm{s}}$.

It's important to have good mixing $M \to 1$

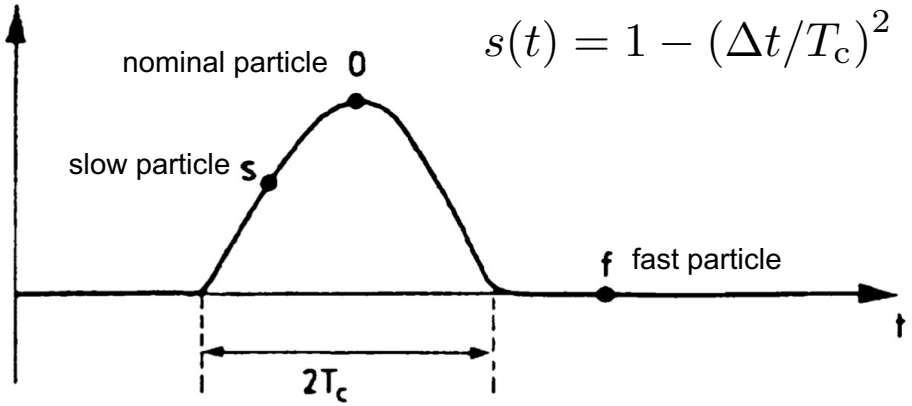


# Mixing: Pick-Up-to-Kicker

*Mixing P-K (Pick-Up-to-Kicker):*

- Now the sample observed is different from the sample to be corrected.
- **Incoherent effects won't change much** to first order.
- **Coherent effect will change** and will be zero if the particle slips by more than half the sample length.

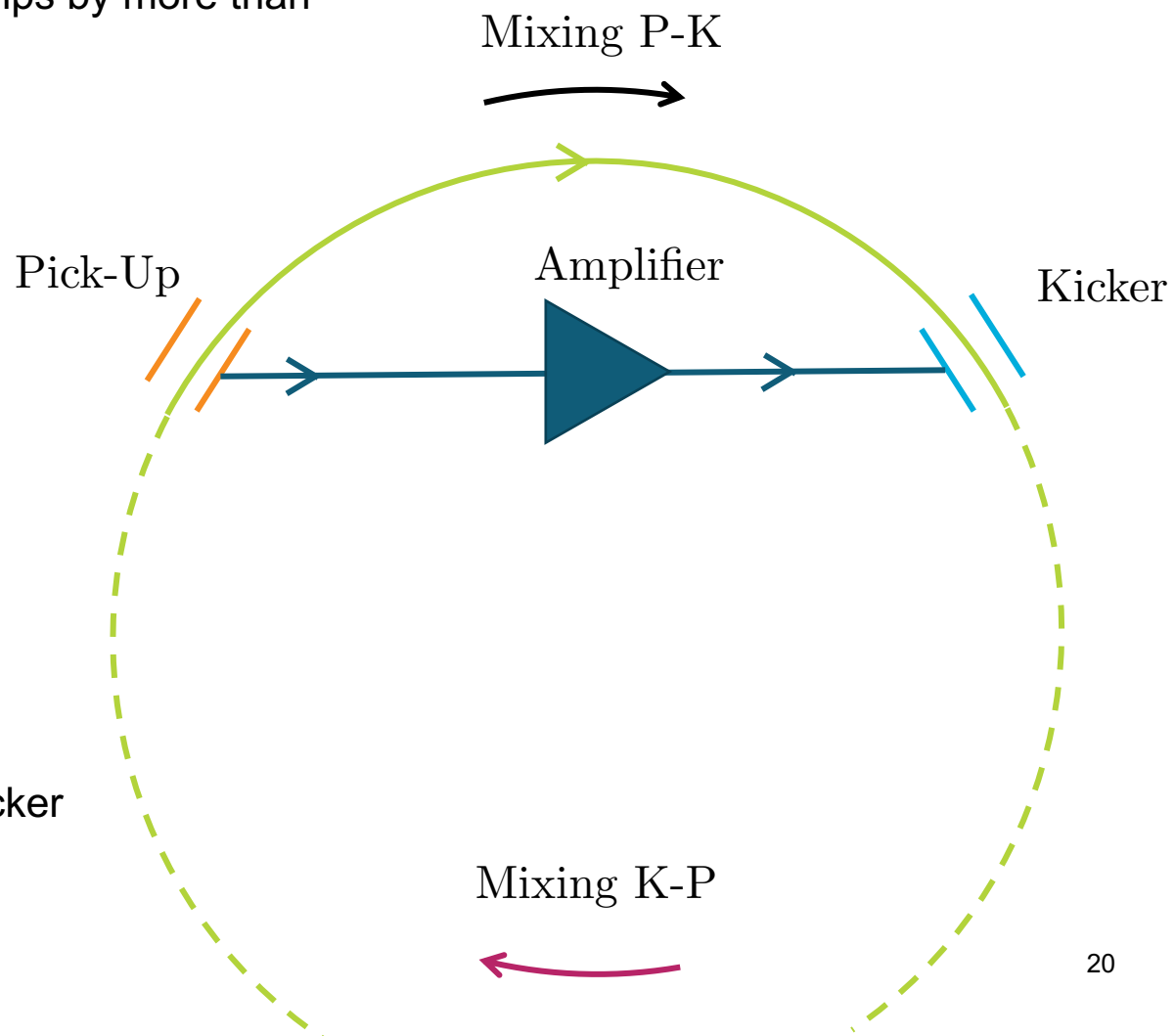
It is more suitable to use parabolic response rather than rectangular response used earlier:




$$s(t) = 1 - (\Delta t / T_c)^2$$

$T_c$ "useful" width of the correction pulse

$\Delta t$ time-of-flight error of the particle between pick-up and kicker



# Mixing: Pick-Up-to-Kicker

Let's account for unwanted P-K mixing:

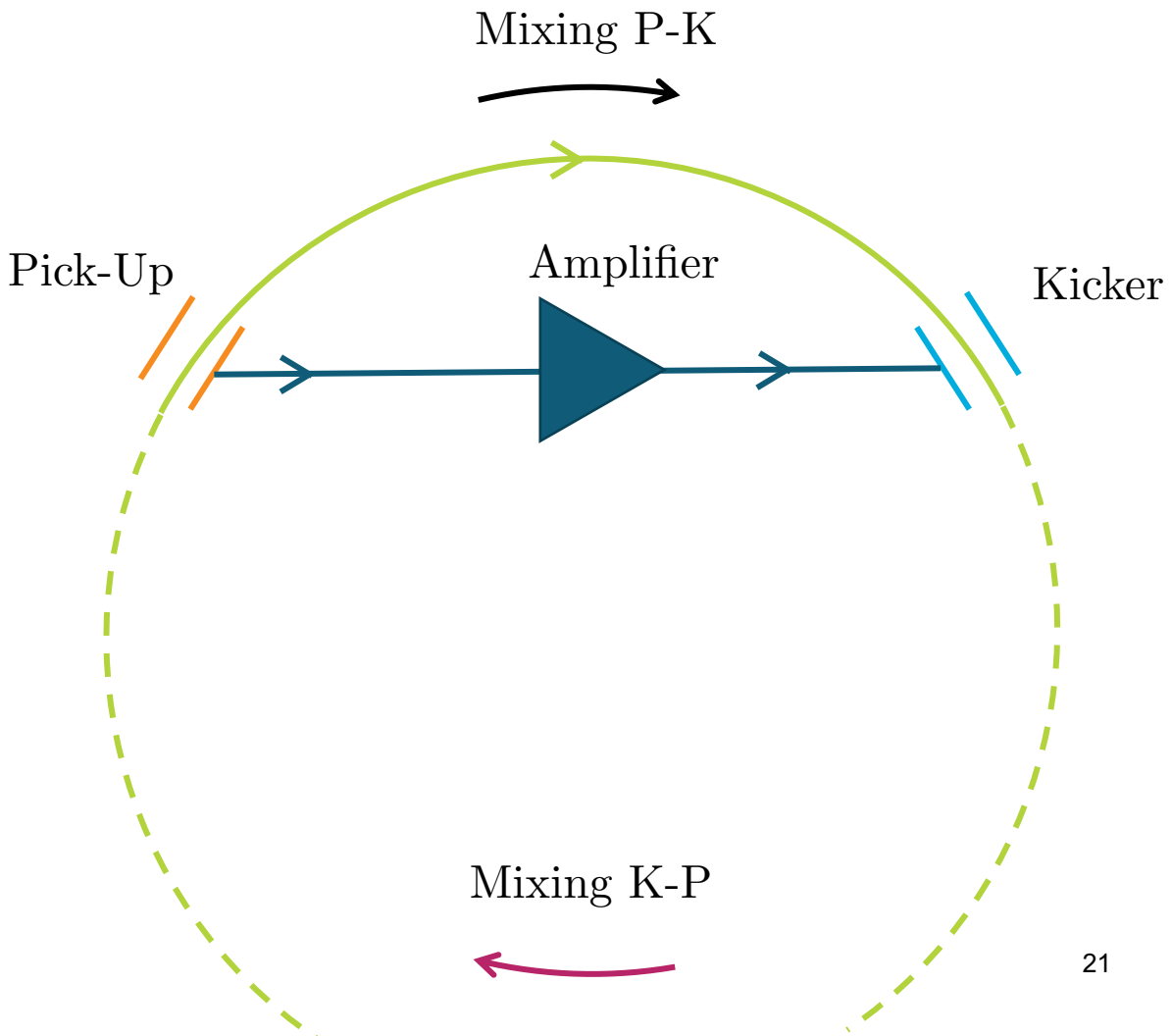
$$\Delta t_{\mathrm{PK}} = t_{\mathrm{PK}} \eta_{\mathrm{PK}} \Delta p/p \qquad \eta_{\mathrm{PK}} \text{ – local P-K slip factor}$$

We then define P-K mixing as:

$$\Delta t_{\mathrm{PK}}/T_{\mathrm{c}} = 1/\tilde{M}$$

r.m.s. cooling rate including mixing:

$$\frac{1}{\tau} = \frac{W}{N}\left[2g\left(1-\tilde{M}^{-2}\right) - g^2\left(M+U\right)\right]$$




*Important observations:*

- *By clever choice of the bending and focusing properties of the storage ring, one can make* $\eta_{\mathrm{PK}} = 0 \rightarrow \Delta t_{\mathrm{PK}} = 0$ *and* $\Delta t_{\mathrm{KP}}$ *large to achieve* $\tilde{M}^{-2} = 0$ *and* $M = 1$
  - *This will complicate the lattice → compromise: sacrifice re-randomization to avoid too much unwanted mixing.*

# Momentum cooling: Palmer Method

Before we were considering a system that corrects some error $x$ . Cooling methods are used to reduce the horizontal and/or vertical betatron oscillations and the momentum spread of the beam. Let's be more specific!

- A horizontal position pick-up is used to detect the horizontal orbit displacement of a test particle:

  $x_{\mathrm{p}} = D_{\mathrm{PU}}\,(\delta p/p)$ where $D_{\mathrm{PU}}$ is the value of the "orbit dispersion function" at the pick-up

- For now, we will assume that the pick-up is placed in a region of large dispersion so that $x_{\mathrm{p}}$ dominates over the betatron oscillations $x_\beta$ .

The RF kicker delivers a longitudinal kick → change of momentum proportional to the detected error.

$$x_{\mathrm{c}} = x - g\,(D\langle\delta p/p\rangle_{\mathrm{s}} + x_{\mathrm{n}})$$

Here we imply that at the kicker the dispersion function $D_{\mathrm{K}}$ as well as its derivative $D'_{\mathrm{K}}$ are zero. Otherwise, the momentum correction leads to an excitation of betatron oscillations.

The momentum kick introduces an abrupt change of the equilibrium orbit, and the particle starts to oscillate around this new displaced orbit — "mutual heating".

# Betatron Oscillation Cooling

- The single passage correction of the position of a test particle at the kicker:

position remains unchanged ⟶ $x_c = x$

$\beta_k x_c' = \beta_k x' - g\langle x_i \rangle_{s,\mathrm{PU}}$ ⟵ orbit angle is corrected in proportion to the sample error at the pick-up

- Let's calculate the long-term change of the square of the betatron amplitude normalized by the beta function = emittance.

$$x = (\varepsilon\beta)^{1/2}\cos\psi$$
$$x' = -(\varepsilon/\beta)^{1/2}\left(\alpha\cos\psi + \sin\psi\right)$$
$$\varepsilon = \beta^{-1}\left[x^2 + (x'\beta + \alpha x)^2\right]$$

where $\psi = \int \mathrm{d}s/\beta,\ \beta = \beta_x,\ \alpha = -\beta'/2$

- Corrected value after one passage: $\varepsilon_c = \beta_{\mathrm{K}}^{-1}\left[x^2 + (\beta_{\mathrm{K}}x' - gx_{s,\mathrm{PU}} + \alpha_{\mathrm{K}}x)^2\right]$

- The pick-up is upstream of the kicker with a phase advance $-\Delta\psi$. The test particle passes the kicker with phase $\psi$. Then the particle position at the pick-up is: $x_{\mathrm{PU}} = (\varepsilon\beta_{\mathrm{PU}})^{1/2}\cos(\psi - \Delta\psi)$

**Let's calculate the change in emittance** $\Delta\varepsilon = \varepsilon_c - \varepsilon$

# Betatron Oscillation Cooling

$$\Delta\varepsilon = \varepsilon_c - \varepsilon$$
$$\varepsilon = \beta^{-1}\left[x^2 + (x'\beta + \alpha x)^2\right]$$
$$\varepsilon_c = \beta_{\mathrm{K}}^{-1}\left[x^2 + (\beta_{\mathrm{K}}x' - gx_{s,\mathrm{PU}} + \alpha_{\mathrm{K}}x)^2\right]$$

1. Long-term expectation value of the linear term in $gx_{s,\mathrm{PU}}$:

$$-2gx_{s,\mathrm{PU}}\left(\beta_{\mathrm{K}}x' + \alpha_{\mathrm{K}}x\right) \to -2g\frac{x_{\mathrm{PU}}}{N_s}\left(\beta_{\mathrm{K}}x' + \alpha_{\mathrm{K}}x\right) \to -g\frac{\varepsilon}{N_s}\beta_{\mathrm{K}}^{1/2}\beta_{\mathrm{PU}}^{1/2}\sin(\Delta\psi)$$

2. To express the square term, use sampling theorem:

$$x_{s,\mathrm{PU}}^2 = \beta_{\mathrm{PU}}\left(\frac{1}{N_s}\sum \varepsilon_i^{1/2}\cos(\psi_i)\right)^2 \to \frac{\beta_{\mathrm{PU}}}{N_s}\frac{\varepsilon_{\mathrm{rms}}}{2}$$

3. Collecting terms:

$$\Delta\varepsilon = \frac{1}{N_s}\left[-g\sqrt{\frac{\beta_{\mathrm{PU}}}{\beta_{\mathrm{K}}}}\sin(\Delta\psi)\varepsilon + \frac{g^2}{2}\frac{\beta_{\mathrm{PU}}}{\beta_{\mathrm{K}}}\varepsilon_{\mathrm{rms}}\right]$$

4. The cooling rate for the amplitude $\sqrt{\varepsilon}$:

$$\frac{1}{\tau} = \frac{1}{2}\frac{W}{N}\left[2g_\beta\left(1 - \tilde{M}^{-2}\right)\sin(\Delta\psi) - g_\beta^2\left(M + U\right)\right] \qquad g_\beta = g\sqrt{\beta_{\mathrm{PU}}/\beta_{\mathrm{K}}}$$

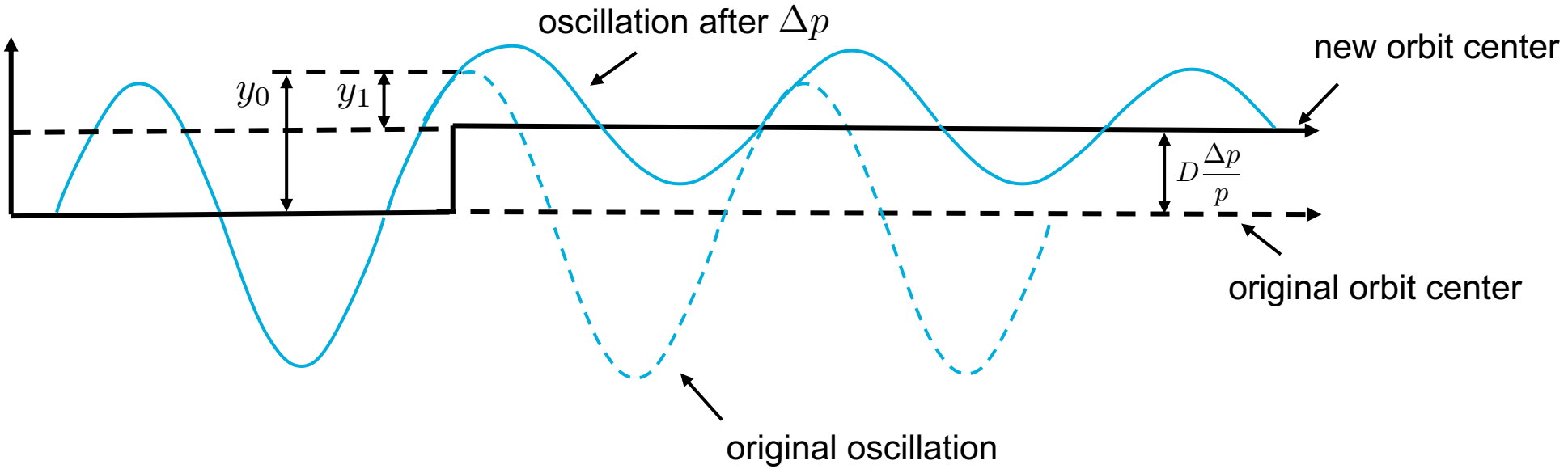
**Note from previous derivation:**

r.m.s. cooling rate including mixing:

$$\frac{1}{\tau} = \frac{W}{N}\left[2g\left(1 - \tilde{M}^{-2}\right) - g^2\left(M + U\right)\right]$$

# Hereward Cooling

- Let's consider both, momentum and betatron contributions. For simplicity, we'll assume that at the kicker $\beta' = 0,\ D' = 0$
- Reduction of betatron oscillations by a momentum jump:




- The particle continues betatron oscillation around a new orbit center, which is displaced by $D\dfrac{\Delta p}{p}$ and initial conditions $x_1 = x_0 - D\Delta p/p,\ x_1' = x_0'$

$$\frac{1}{D_{\mathrm{K}}}\Delta x_{p,\mathrm{K}} = -g\frac{1}{D_{\mathrm{PU}}}\langle x\rangle_{s,\mathrm{PU}} \quad \leftarrow \text{change in the momentum displacement}$$

$$\frac{1}{D_{\mathrm{K}}}\Delta x_{\beta,\mathrm{K}} = g\frac{1}{D_{\mathrm{PU}}}\langle x\rangle_{s,\mathrm{PU}} \quad \leftarrow \text{change in the betatron displacement}$$

$$\Delta x'_{p,\mathrm{K}} = \Delta x'_{p,\mathrm{PU}} = 0$$

# Hereward Cooling

- Cooling rate for the mean square momentum deviation including noise and mixing:

$$\frac{1}{\tau_p} = \frac{W}{N}\left[2g\left(1-\tilde{M}^{-2}\right) - g^2\left(M + \frac{x^2_{\beta,\mathrm{rms,PU}} + x^2_{n,\mathrm{PU}}}{D^2_{\mathrm{PU}}(\Delta p/p)^2_{\mathrm{rms}}}\right)\right]$$

momentum heating by the betatron oscillations*

- Betatron part at the location of the kicker: $\frac{1}{D_{\mathrm{K}}}\Delta x_{\beta,\mathrm{K}} = g\frac{1}{D_{\mathrm{PU}}}\sqrt{\frac{\beta_{\mathrm{PU}}}{\beta_{\mathrm{K}}}}\langle x_\beta\rangle_{s,\mathrm{K}}\cos(\Delta\psi)$

- The uncorrelated momentum part acts as noise.

- One can find the cooling rate for the r.m.s betatron amplitude:

$$\frac{1}{\tau_\beta} = \frac{1}{2}\frac{W}{N}\left[-2g_h\left(1-\tilde{M}^{-2}\right)\cos(\Delta\psi) - g_h^2\left(M + \frac{D^2_{\mathrm{PU}}(\Delta p/p)^2_{\mathrm{rms}} + x^2_{n,\mathrm{PU}}}{x^2_{\beta,\mathrm{rms,PU}}}\right)\right],\ g_h = gD_{\mathrm{K}}\sqrt{\beta_{\mathrm{PU}}}/(D_{\mathrm{PU}}\sqrt{\beta_{\mathrm{K}}})$$

Simultaneous cooling is possible if the P-to-K distance is such that cosine is negative

Heating of the betatron motion by the momentum oscillations

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*here we assume no correlation between betatron and momentum oscillations

# Hereward Cooling

$$\frac{1}{\tau_\beta} = \frac{1}{2}\frac{W}{N}\left[-2g_h\left(1-\tilde{M}^{-2}\right)\cos(\Delta\psi) - g_h^2\left(M + \frac{D_{\mathrm{PU}}^2(\Delta p/p)_{\mathrm{rms}}^2 + x_{n,\mathrm{PU}}^2}{x_{\beta,\mathrm{rms,PU}}^2}\right)\right], \quad g_h = gD_{\mathrm{K}}\sqrt{\beta_{\mathrm{PU}}}/(D_{\mathrm{PU}}\sqrt{\beta_{\mathrm{K}}})$$

Simultaneous cooling is possible if the P-to-K distance is such that cosine is negative

Heating of the betatron motion by the momentum oscillations

- If the P-to-K distance can't be chosen such that $\cos(\Delta\psi) < 0$, then $D_{\mathrm{K}} = 0$ avoids heating of betatron oscillations by momentum cooling.
- The condition $x_{\beta,\mathrm{rms,PU}} \gg D_{\mathrm{PU}}\,(\Delta p/p)_{\mathrm{rms}}$ ensures good betatron cooling.
- For methods using horizontal position pick-up $D_{\mathrm{PU}} = 0, D'_{\mathrm{PU}} = 0$ is aimed for at the position of the horizontal pick-up for transverse cooling.
- For Palmer momentum cooling, the inverse $x_{\beta,\mathrm{rms,PU}} \ll D_{\mathrm{PU}}\,(\Delta p/p)_{\mathrm{rms}}$ is favored to minimize the betatron noise.
- For Hereward cooling a compromise $x_{\beta,\mathrm{rms,\ PU}} \approx x_{p,\mathrm{rms,\ PU}}$ has to be found.
- For avoiding horizontal heating by momentum cooling:
  1. Split the momentum kicker in two units spaced by $\Delta\psi = \pi$ , or
  2. Use horizontal kick proportional to the momentum kick.

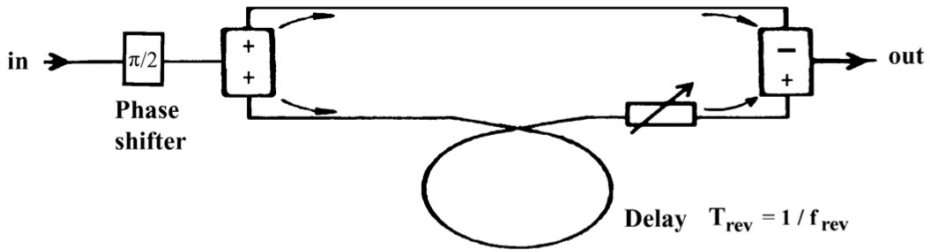
# The filter method

Longitudinal Pick-Up → Pre-Amplifier → Filter (Analyzer) → Amplifier → Momentum Kicker

- The signal from the pick-up is pre-amplified and compared in the analyzer with the signal from the previous turn.
- The correction signal is sent through amplifier to the kicker.
- A particle with nominal momentum is not affected by its own signal.
- Particles which are too slow/fast will be accelerated/decelerated.

**Filter—a periodic notch filter with ideally zero transmission at the harmonics of the revolution frequency in the pass-band:**

- *Example 1:* TEM line of length $l = \frac{1}{2}C/\beta$ , signals travel to trough in half of the nominal revolution time. The line is shorted at the far end, its open end is connected to the cooling path. *For the synchronous particle:* the reflected signal from the previous turn annihilates the direct signal. *For particles with finite* $\Delta p$*:* the cancellation is only partial, and acceleration/deceleration occurs.
- *Example 2:* correlators—TEM or optical lines (high quality, sometimes cryo-cooled). Correlator is preceded by a 90° phase shifter. The "cancellation" leads to transmission characteristic that is linear for small momentum deviation.

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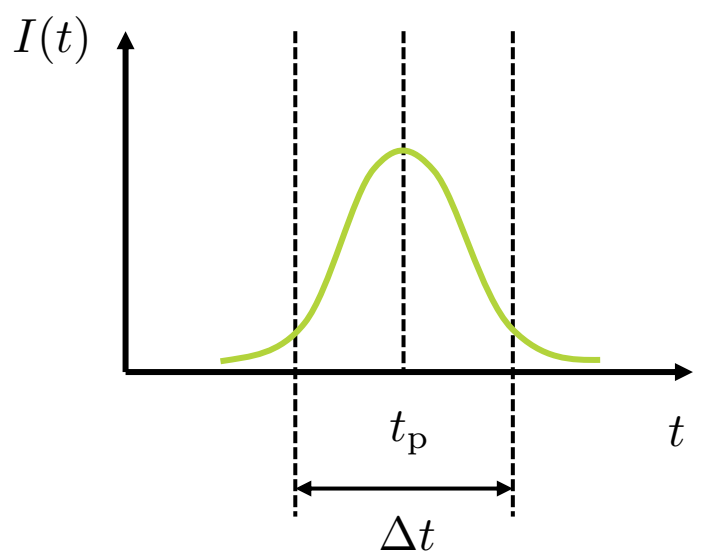
# Frequency-Domain Picture

- Consider a single particle coasting in a storage ring with some revolution frequency $\omega_{\mathrm{rev}} = 2\pi f_{\mathrm{rev}}$.
- The current detected by a longitudinal pickup = sum of a short pulse at each traversal.

Signal induced on a pick-up at a passage time $t_{\mathrm{p}}$




- The signal can be approximated by a Dirac distribution.
- Then after many revolutions:

$$I_{\mathrm{s}}(t) = e \sum_{k} \delta\left(t - t_{\mathrm{p}} + kT_{\mathrm{rev}}\right)$$

- Using Fourier expansion:

$$I_{\mathrm{s}}(t) = e f_{\mathrm{rev}} \sum_{m=-\infty}^{\infty} e^{im\omega_{\mathrm{rev}}(t-t_{\mathrm{p}})} = e f_{\mathrm{rev}} + \sum_{n=1}^{\infty} 2e f_{\mathrm{rev}} \cos\left[n\omega_{\mathrm{rev}}(t - t_{\mathrm{p}})\right]$$

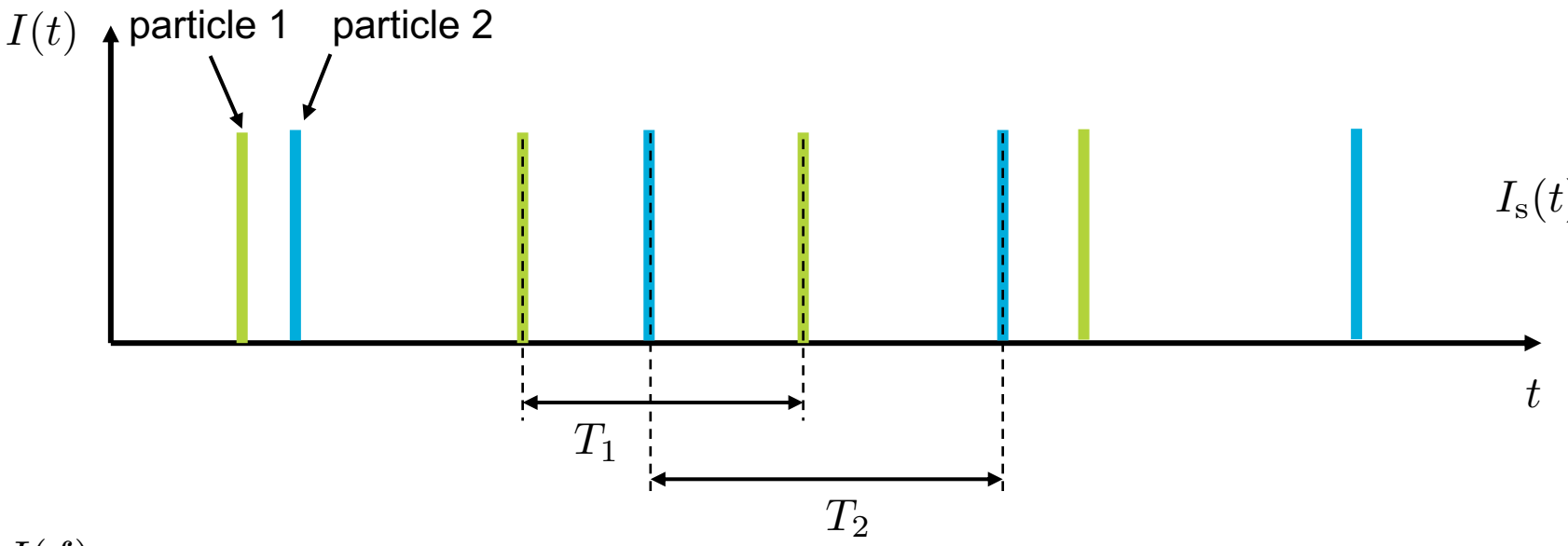
DC part of the beam (single particle)

**Single particle current:**

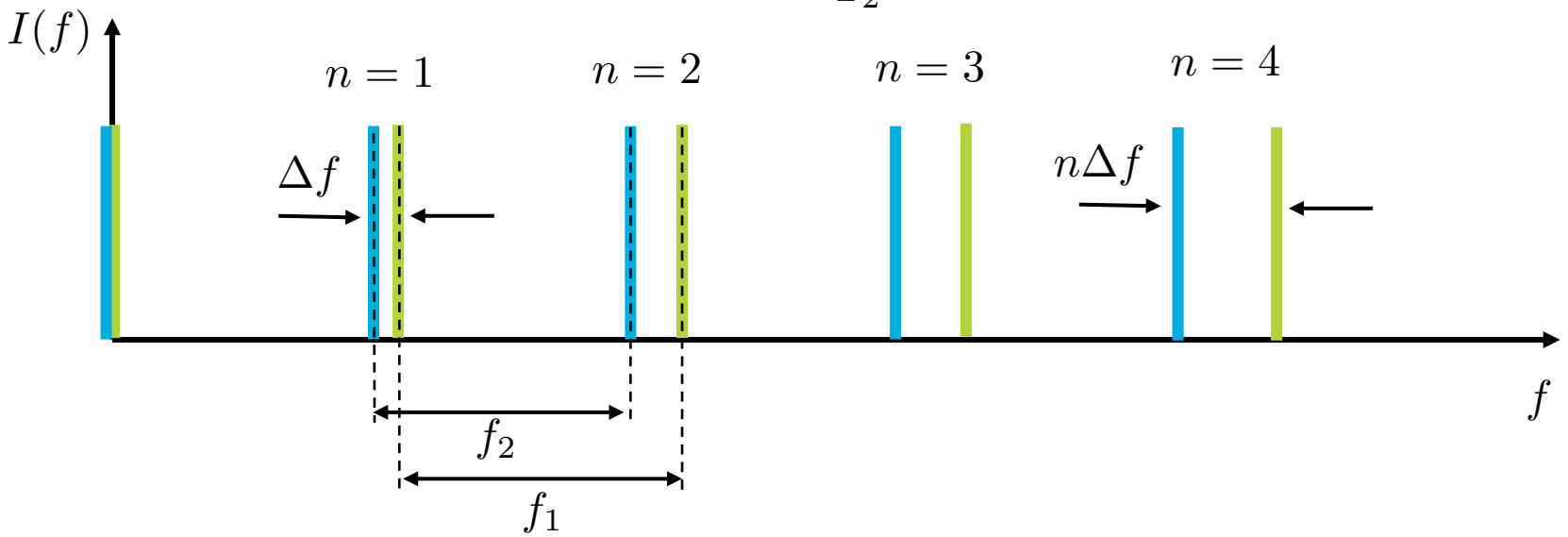
$$I_{\mathrm{s}}(t) = e f_{\mathrm{rev}} + \sum_{n=1}^{\infty} a_n \cos(n\omega_{\mathrm{rev}} t) + b_n \sin(n\omega_{\mathrm{rev}} t) \quad \text{with} \quad a_n = 2e f_{\mathrm{rev}} \cos(n\omega_{\mathrm{rev}} t_p),\ b_n = 2e f_{\mathrm{rev}} \sin(n\omega_{\mathrm{rev}} t_p)$$

# Time and Frequency Domain

- Assume a second particle circulating with a slightly different revolution frequency $\omega_{\mathrm{rev1}}$.



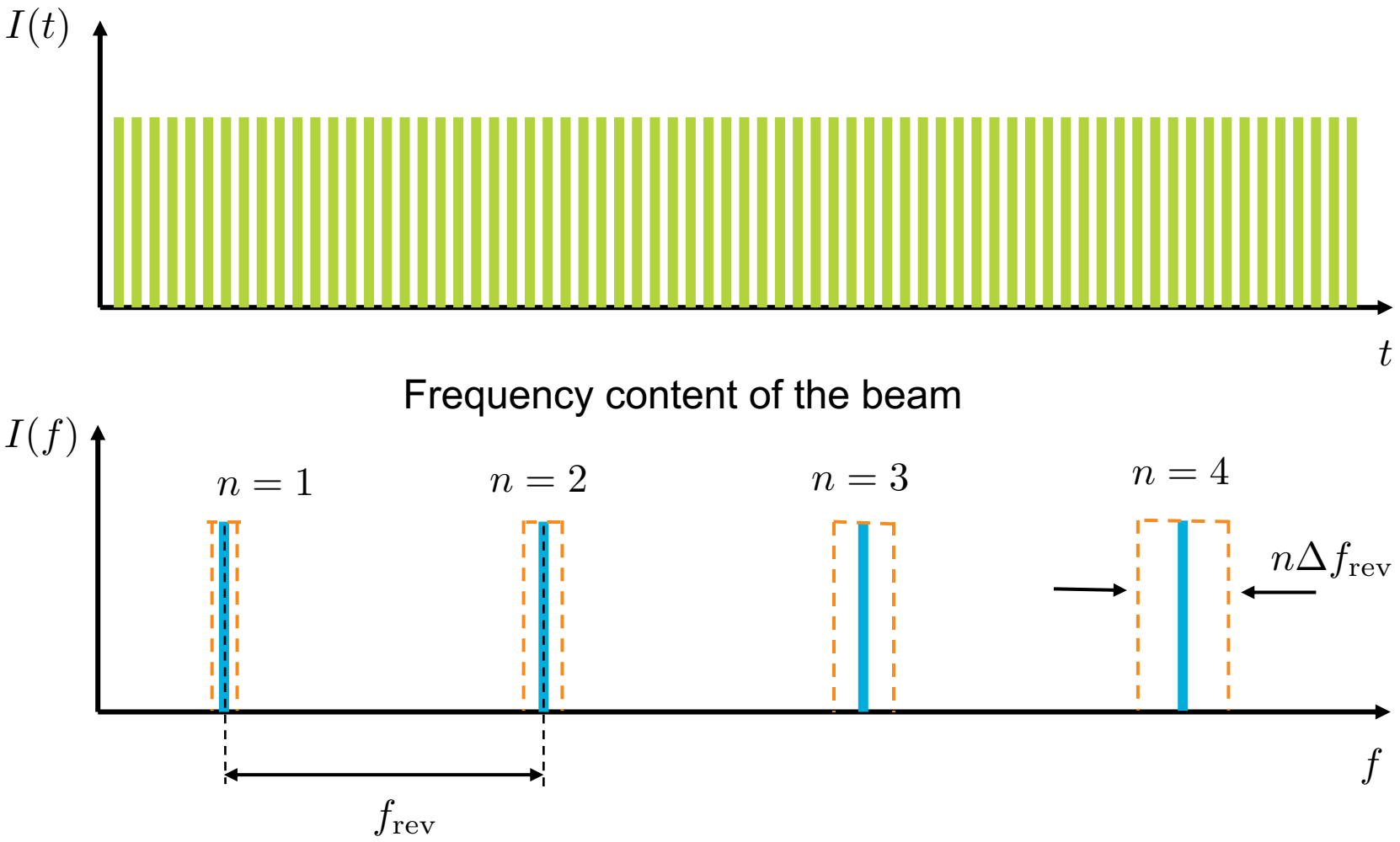

$$I_{\mathrm{s}}(t) = e \sum_k \delta\left(t - t_{\mathrm{p}} + kT_{\mathrm{rev}}\right)$$




$$\Delta f = f_1 \eta \frac{\Delta p}{p}$$

# Time and Frequency Domain

- For a beam with a distribution of revolution frequencies between $\omega_1 = \omega_{\mathrm{rev}} - \Delta\omega_{\mathrm{rev}}/2$ and $\omega_2 = \omega_{\mathrm{rev}} + \Delta\omega_{\mathrm{rev}}/2$, we expect a spectrum with bands around each harmonic $n\omega_{\mathrm{rev}}$ .



# Evaluating the bands: mean-squared fluctuations

- Assume an unbunched beam: passage times are distributed between 0 and $T_{\mathrm{rev}}$.
- In the time domain it looks like a direct current with superimposed high-frequency current with a wavelength spectrum given by the particle spacing.
- Consider a group of particles with the same revolution frequency:

$$I'(t) = N' e f_{\mathrm{rev}} + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + B_n \sin(n\omega t) \qquad A_n = \sum_{k=1}^{N'} 2 e f_{\mathrm{rev}} \cos(\omega t_{\mathrm{k}}),\ \ B_n = \sum_{k=1}^{N'} 2 e f_{\mathrm{rev}} \sin(\omega t_{\mathrm{k}})$$

- We will focus on the average current $I'_{\mathrm{av}}$ and the mean squared fluctuation $(\Delta I')^2 = \left(I'^2 - I'^2_{\mathrm{av}}\right)$
- This fluctuating noise current leads to heating of a test particle during stochastic cooling.
- The mean over the revolution period: $\langle (\Delta I')^2 \rangle_T = \frac{1}{T} \int_0^T (\Delta I')^2 \, \mathrm{d}t = \sum_n \left(A_n^2/2\right) + \left(B_n^2/2\right)$
- Assuming a random distribution of the initial phases $\omega_{\mathrm{rev}} t_{\mathrm{k}} \equiv \psi_{\mathrm{k}}$ , such that the sums over cross-terms give zero:

$$A_n^2 = (2 e f_{\mathrm{rev}})^2 \left( \sum_{k=1}^{N'} \cos(n\psi_{\mathrm{k}}) \right)^2 = (2 e f_{\mathrm{rev}})^2 \sum_{k=1}^{N'} (\cos(n\psi_{\mathrm{k}}))^2 = (2 e f_{\mathrm{rev}})^2 N'/2$$

$$B_n^2 = (2 e f_{\mathrm{rev}})^2 \left( \sum_{k=1}^{N'} \sin(n\psi_{\mathrm{k}}) \right)^2 = (2 e f_{\mathrm{rev}})^2 N'/2$$

$$\langle (\Delta I')^2 \rangle_n = 2 (e f_{\mathrm{rev}})^2 N'$$

# Evaluating the bands: mean-squared fluctuations

$$\langle (\Delta I')^2 \rangle_n = 2(e f_{\mathrm{rev}})^2 N'$$

- Specifies the intensity of the n[th] spectral line of the mean-squared noise current for a group of mono-energetic particles with random initial phases.

- **In reality:** distribution of revolution frequencies between $\omega_{\mathrm{rev}} \pm \Delta\omega_{\mathrm{rev}}/2$ . Different components contribute current in the range $n(\omega_{\mathrm{rev}} \pm \Delta\omega_{\mathrm{rev}}/2)$ around each harmonic.

## *Schottky noise spectrum*

- For a subgroup with $\mathrm{d}\omega_{\mathrm{rev}}$ , fraction of particles with a revolution frequency in the range $\omega_{\mathrm{rev}} \pm \mathrm{d}\omega_{\mathrm{rev}}/2$ is $(dN/\mathrm{d}\omega_{\mathrm{rev}})\,\mathrm{d}\omega_{\mathrm{rev}}$

$$\mathrm{d}(I)_n^2 = 2(e f_{\mathrm{rev}})^2 \,(\mathrm{d}N/\mathrm{d}\omega_{\mathrm{rev}})\,\mathrm{d}\omega_{\mathrm{rev}},$$

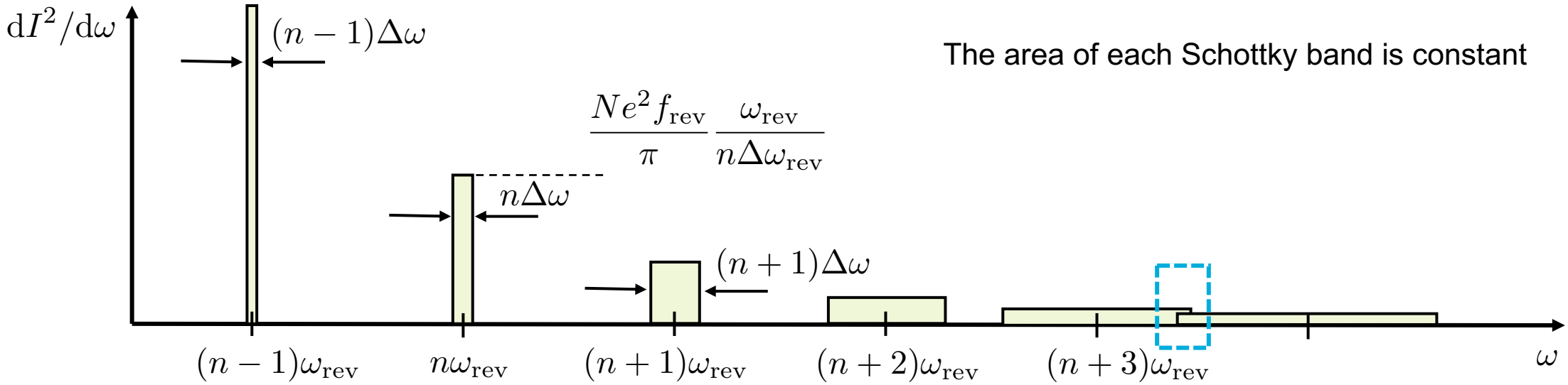
$$\mathrm{d}(I)_n^2/\mathrm{d}\omega_{\mathrm{rev}} = 2(e f_{\mathrm{rev}})^2 \,(\mathrm{d}N/\mathrm{d}\omega_{\mathrm{rev}})\,,$$

$$\mathrm{d}(I)_n^2/\mathrm{d}\omega_n = 2(e f_{\mathrm{rev}})^2 \,(\mathrm{d}N/\mathrm{d}n\omega_{\mathrm{rev}}) = 2(e f_{\mathrm{rev}})^2 \,(\mathrm{d}N/\mathrm{d}\omega_n)$$ ⟵ spectral power density of the noise in band n

- Integrating over a band: $(I)_n^2 = 2(e f_{\mathrm{rev}})^2 N$

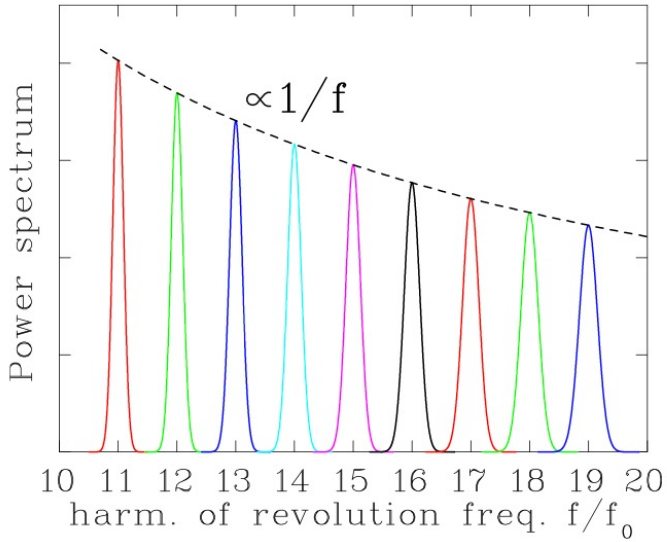
The area of each Schottky noise band in a $\mathrm{d}I^2/\mathrm{d}\omega$ vs. $\omega$ diagram is constant

# Schottky bands



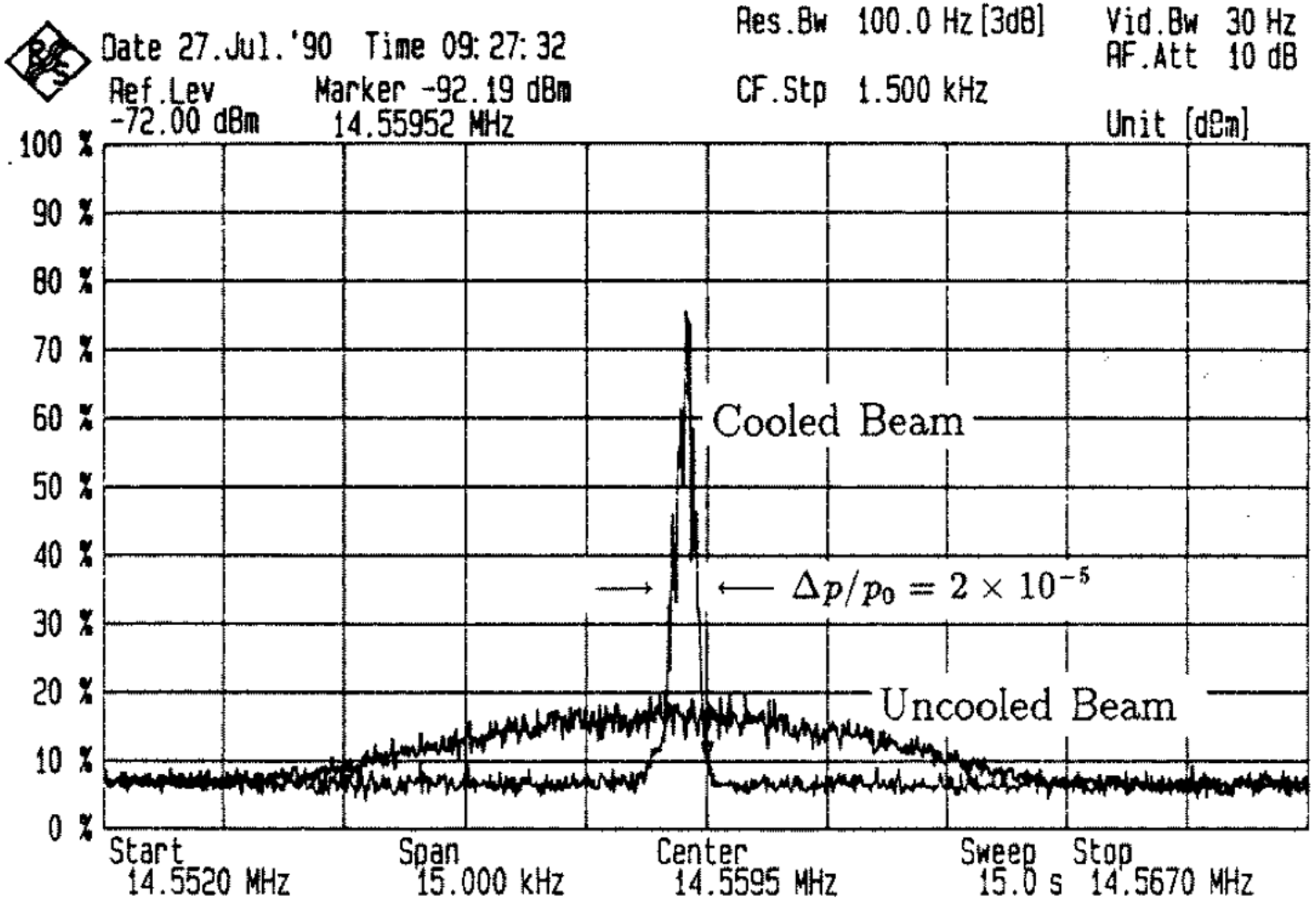

The area of each Schottky band is constant

- At high harmonics bands overlap and the height of the noise power is $\frac{Ne^2 f_{\mathrm{rev}}}{\pi}$ as for classical shot noise.
- Each longitudinal Schottky band has the same area – integrated power.
- The spectral density decreases with $1/f$

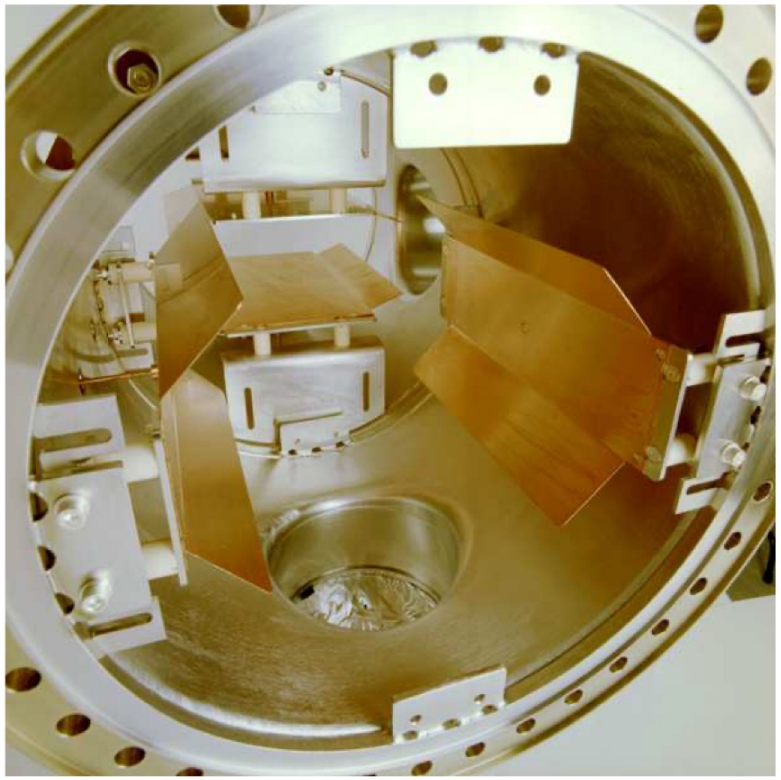


# Longitudinal Schottky scan: example

Longitudinal Schottky scan can be used to monitor cooling.
Example below shows a result of electron cooling of 300 MeV/u $Ar^{18+}$: decrease of $\Delta p/p$ from $10^{-3}$ to $2x10^{-5}$.




Schottky pick-up—similar to BPM.



# What about transverse motion?

- For a particle performing betatron oscillations:

initial betatron amplitude — initial betatron phase

$$x = \tilde{x} e^{i(Q\omega_{\mathrm{rev}} t + \mu_0)}$$

betatron tune

- Signal in a position sensitive pick-up:

Transfer impedance of the pick-up

$$V_{\mathrm{s}} = \tilde{x} e f_{\mathrm{rev}} \sum_{m=0}^{\infty} Z'_{\mathrm{PU}\perp}(\omega) e^{i(Q \pm m)\omega_{\mathrm{rev}} t - i(\pm m \omega_{\mathrm{rev}} t_{\mathrm{p}} + \mu_0)}, \; m > 0, \; Z_{\mathrm{PU}\perp} = Z'_{\mathrm{PU}\perp} x$$

- The particle induces signals at sidebands:

$$\omega = \omega_m = (m \pm ||Q|| \pm q)\, \omega_{\mathrm{rev}} = (n \pm q)\omega_{\mathrm{rev}}, m, n, ||Q|| \in \mathbb{Z}$$

- Using cosine functions, the transverse “dipole moment” of the beam current:

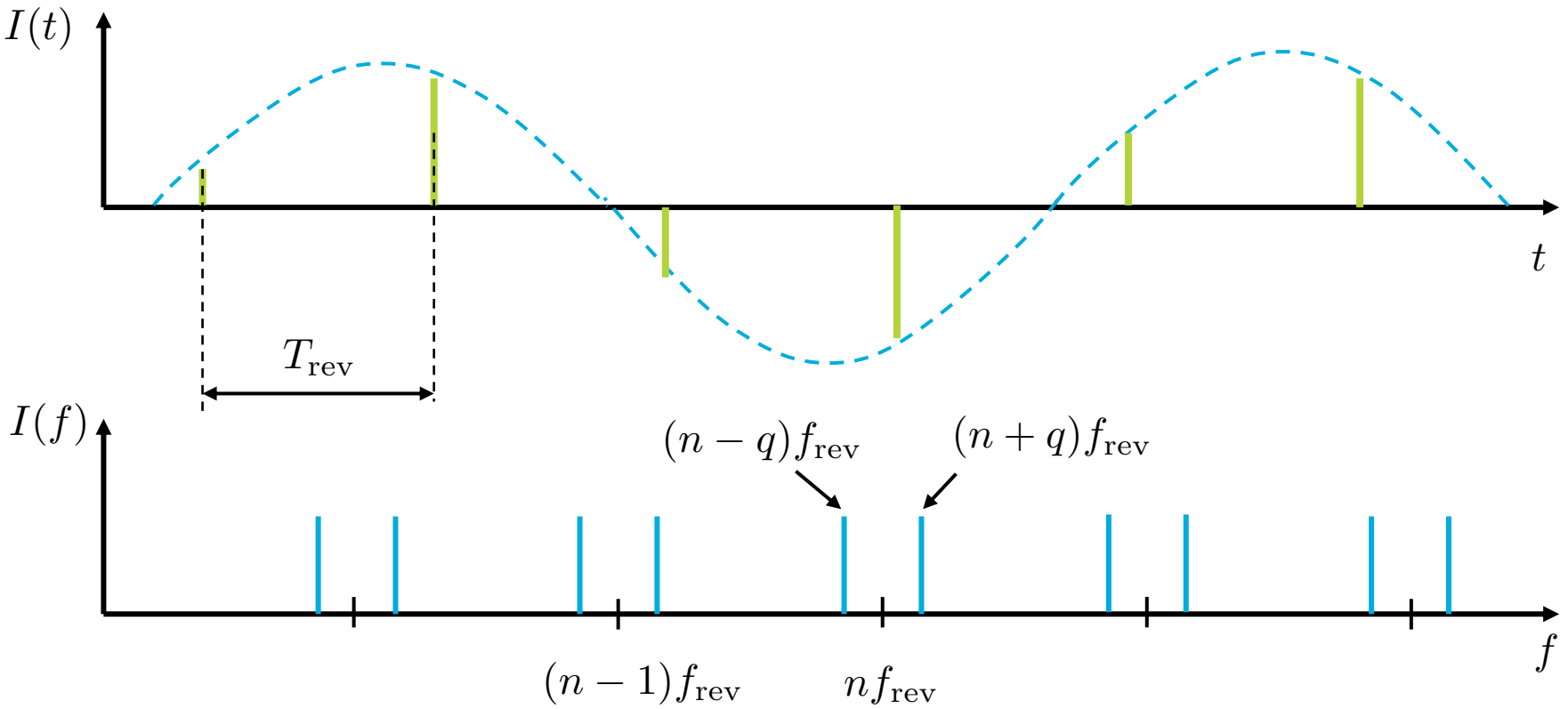
$$\begin{aligned} I_s x &= \left( e f_{\mathrm{rev}} + 2 e f_{\mathrm{rev}_n} \sum_{m=1}^{\infty} \cos(m\omega_{\mathrm{rev}}(t - t_p)) \right) \tilde{x} \cos(\omega_{\mathrm{rev}} Q t + \mu_0) \\ &= e f_{\mathrm{rev}} \tilde{x} \left( \sum_{m=0}^{\infty} \cos(\omega_{\mathrm{rev}}(m + Q)t - m\omega_{\mathrm{rev}} t_p + \mu_0) + \sum_{m=1}^{\infty} \cos(\omega_{\mathrm{rev}}(m - Q)t - m\omega_{\mathrm{rev}} t_p - \mu_0) \right) \end{aligned}$$

# Transverse motion: single particle

- A single particle passing through a position sensitive pick-up generates a periodic series of delta functions:

$$I(t) = \frac{e}{T}\sum_{n}^{\infty} \underbrace{\delta\left(t - nT + \varphi_k\right)}_{\text{longitudinal signal}} \times \underbrace{a_k \cos\left(q\omega t + \phi_k\right)}_{\text{betatron oscillations}}$$

Amplitude modulation of the **longitudinal signal** by the **betatron oscillations**



# Transverse motion: N particles

- Assume a beam with random longitudinal particle positions $0 \leq \omega_{\mathrm{rev}} t_p \leq 2\pi$ and betatron phases $0 \leq \mu_0 \leq 2\pi$
- Let's evaluate the spectral density of the mean squared dipole moment in the bands $D^2 = (Ix)^2$

$$\frac{\mathrm{d}D^2_{m\pm q}}{\mathrm{d}\omega} = \frac{\mathrm{d}N}{\mathrm{d}\omega}\frac{e^2 f_{\mathrm{rev}}^2}{2}\tilde{x}^2_{\mathrm{rms}}$$

from averaging $\tilde{x}^2$

from averaging the $\cos^2$

- Integrating over one separated band and approximating the band by a rectangle, the noise density:

$$\frac{\mathrm{d}D^2_{n\pm q}}{\mathrm{d}\omega} = N\frac{e^2 f_{\mathrm{rev}}}{4\pi}\tilde{x}^2_{\mathrm{rms}}\frac{\omega_{\mathrm{rev}}}{\Delta\omega_n}$$

- At high frequencies, where bands overlap:

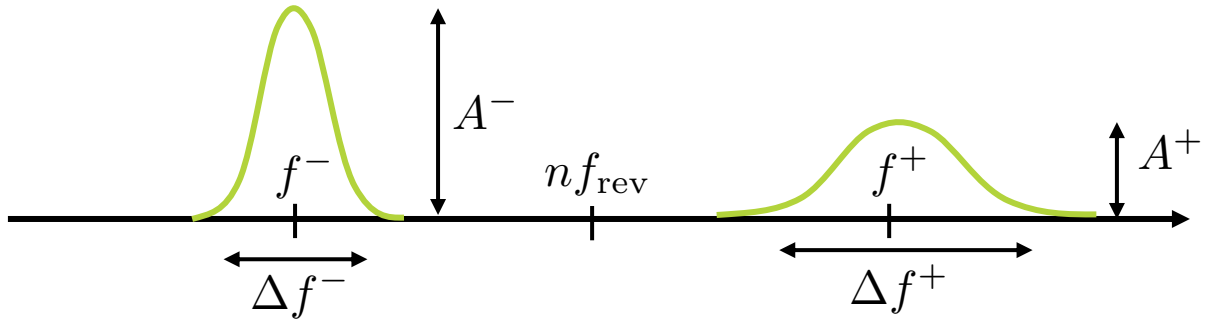
$$\frac{\mathrm{d}D^2}{\mathrm{d}\omega} = N\frac{e^2 f_{\mathrm{rev}}}{2\pi}\tilde{x}^2_{\mathrm{rms}}$$

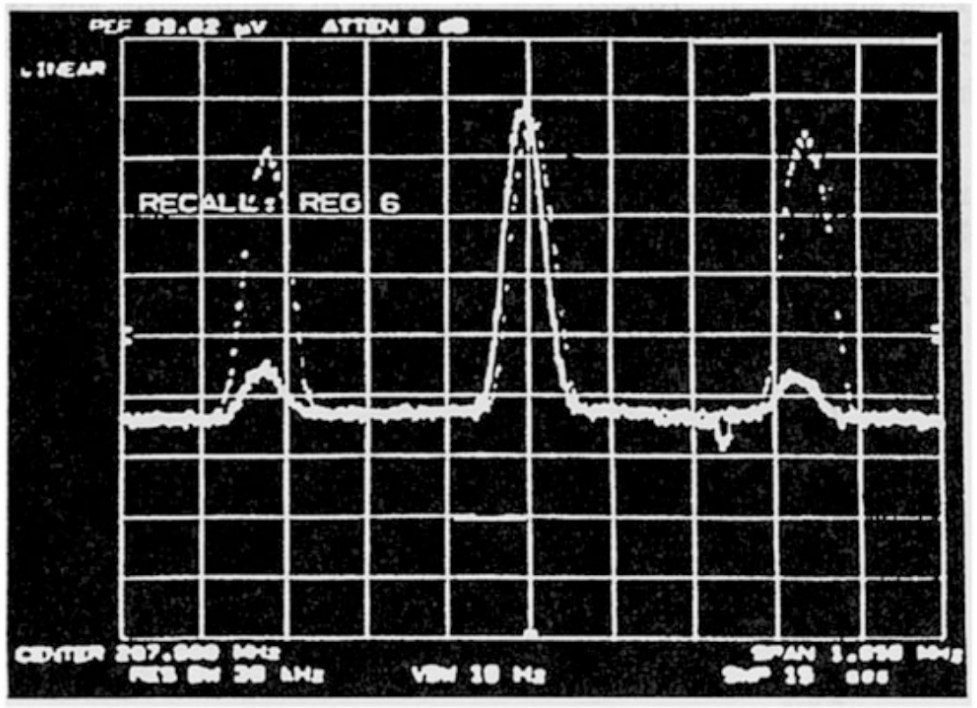
## Why is this important?

- A spectrum analyzer branched on a transverse pick-up records the square root of the signal $\frac{\mathrm{d}D^2_{m\pm q}}{\mathrm{d}\omega} = \frac{\mathrm{d}N}{\mathrm{d}\omega}\frac{e^2 f_{\mathrm{rev}}^2}{2}\tilde{x}^2_{\mathrm{rms}}$ multiplied by the pick-up impedance → **can be used for diagnostics!**

# Transverse motion: using spectrum analyzer as diagnostics

- The frequency band is centered around a revolution harmonic $nf_{\mathrm{rev}}$ and covers a width such that the sidebands $(n \pm q)f_{\mathrm{rev}}$ are visible.
- The height of sidebands is proportional to the r.m.s. betatron amplitude $\tilde{x}_{\mathrm{rms}}$ and can be used to observe transverse cooling process.
- A non-integer part of the tune can be determined from the distance of the sidebands.
- From a scan at lower harmonics $n\Delta\omega_{\mathrm{rev}} \leq \omega_{\mathrm{rev}}\Delta q$ , the chromaticity can be deduced by analyzing the width of the adjacent $n \pm q$ bands.




The central band, n = 100, is visible as the beam is not completely centered at the pick-up. The right and left bands (98+Q) and (102-Q) with Q=2.3. During cooling the height of the sidebands decreases.

Fractional part of the tune:

$$q = \frac{1}{2} + \frac{f^+ - f^-}{2f_{\mathrm{rev}}}$$

Momentum spread:

$$\frac{\mathrm{d}p}{p} = \frac{1}{\eta}\frac{\Delta f^+ + \Delta f^-}{2nf_{\mathrm{rev}}}$$

Emittance:

$$\varepsilon \propto A^-\Delta f^- + A^+\Delta f^+$$

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# Conclusions

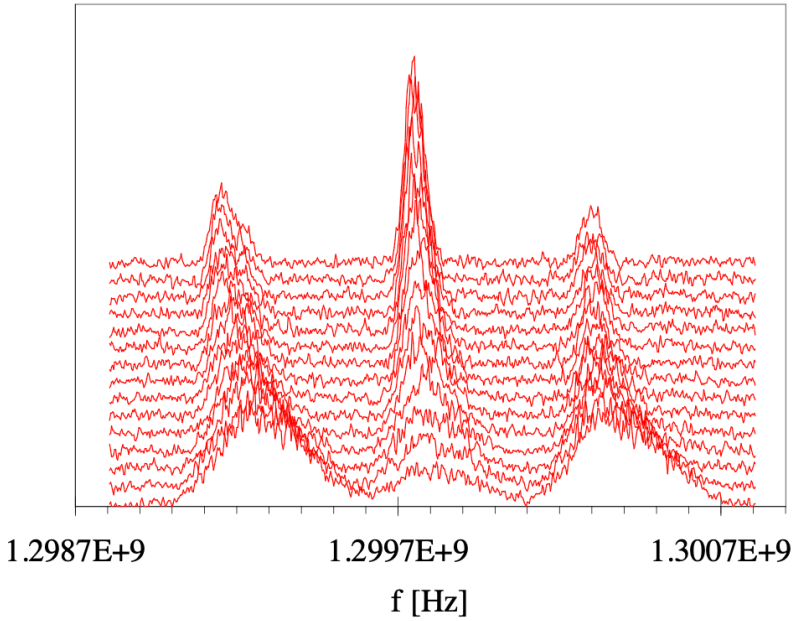
- In an SC system, **statistical fluctuations in the beam are sampled** using pick-ups (antennas) operating in the microwave regime. The **resulting signals are then amplified and applied** to electromagnetic kickers in a negative-feedback system, resulting in cooling of the circulating beam.

- As SC is based on the **sampling of random fluctuations**, the **bandwidth** of the integrated system and the **particle density** in the beam determine the number of passes required for cooling, and thus the achievable cooling rate.

- **Momentum cooling** can be achieved via **Palmer method**: signal is proportional to particle momentum and is measured by a pick-up at high dispersion location.

- **Simultaneous momentum and betatron cooling** is possible via Hereward's method

- **Filter cooling**: the signal proportional to particle momentum is obtained as difference of particle signals for two successive turns.




Transverse Schottky spectra recorded every 80 ms during stochastic cooling. Decrease of sidebands=transverse cooling.