PHY542. Emittance Measurements using solenoid scan

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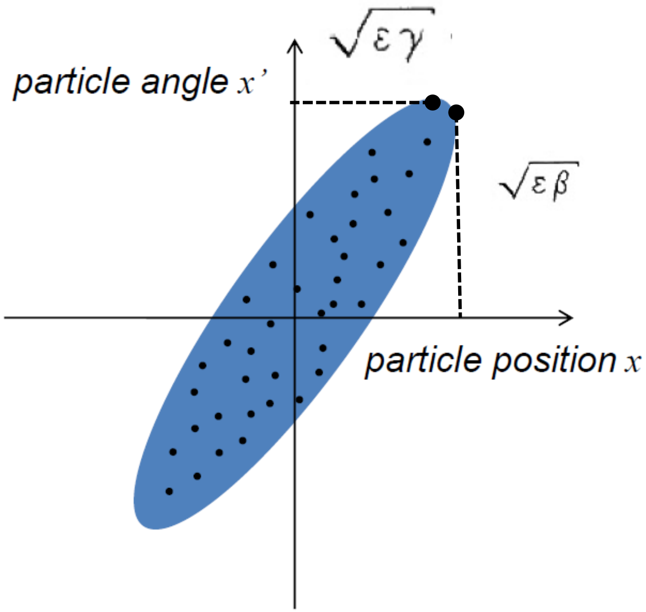
Emittance, what is it ?



ε = Area in *x, x'* plane occupied by beam particles divided by π

Beam ellipse and its orientation is described by 4 parameters

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

$\sqrt{\beta\varepsilon}$ Is the beam half width

$\sqrt{\gamma\varepsilon}$ Is the beam half divergence

α Describes how strong x and x' are correlated

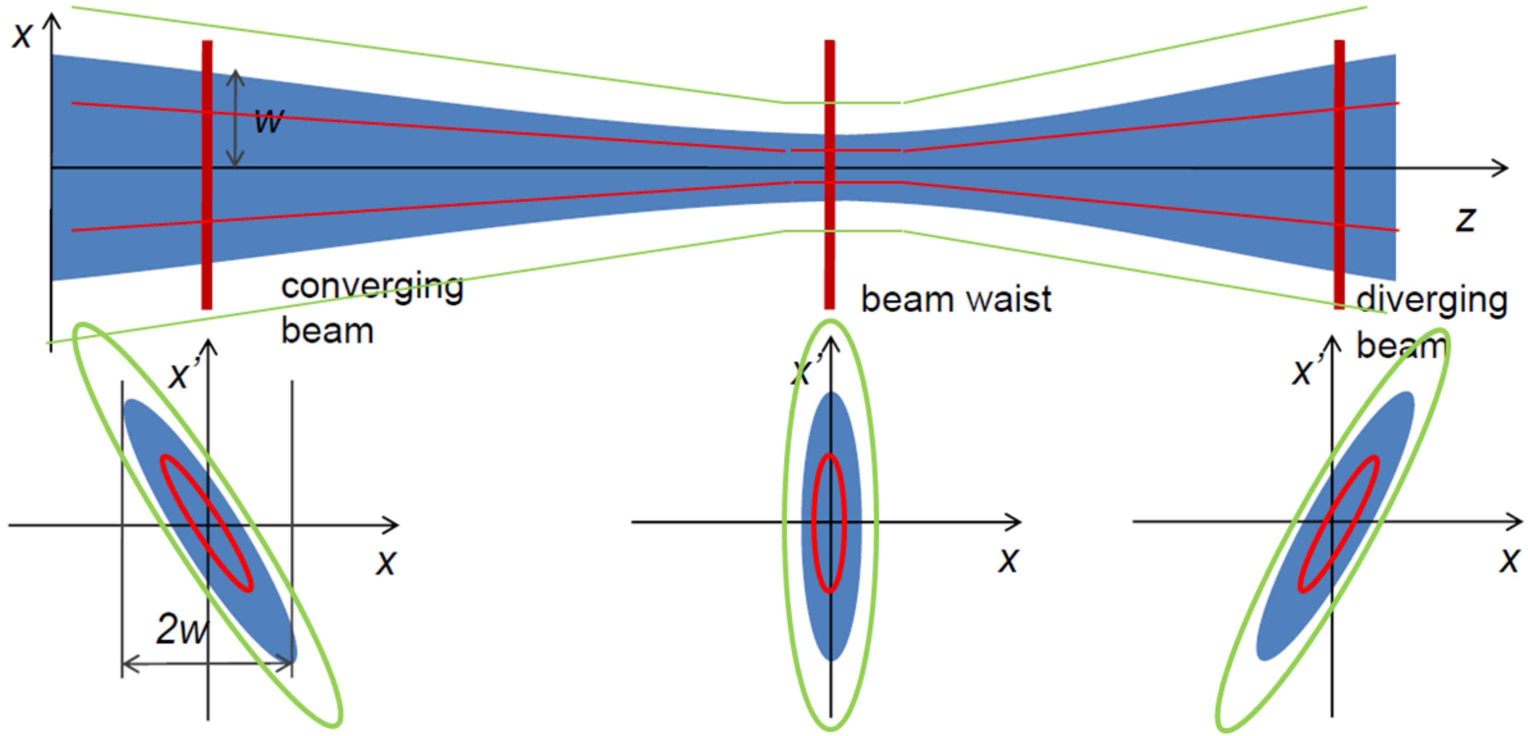
a<0 beam diverging

a>0 beam converging

a=0 beam size is maximum or minimum(waist)

The three orientation parameters are connected by the relation $\gamma = \dfrac{1+\alpha^2}{\beta}$

Beam envelope along a beamline.

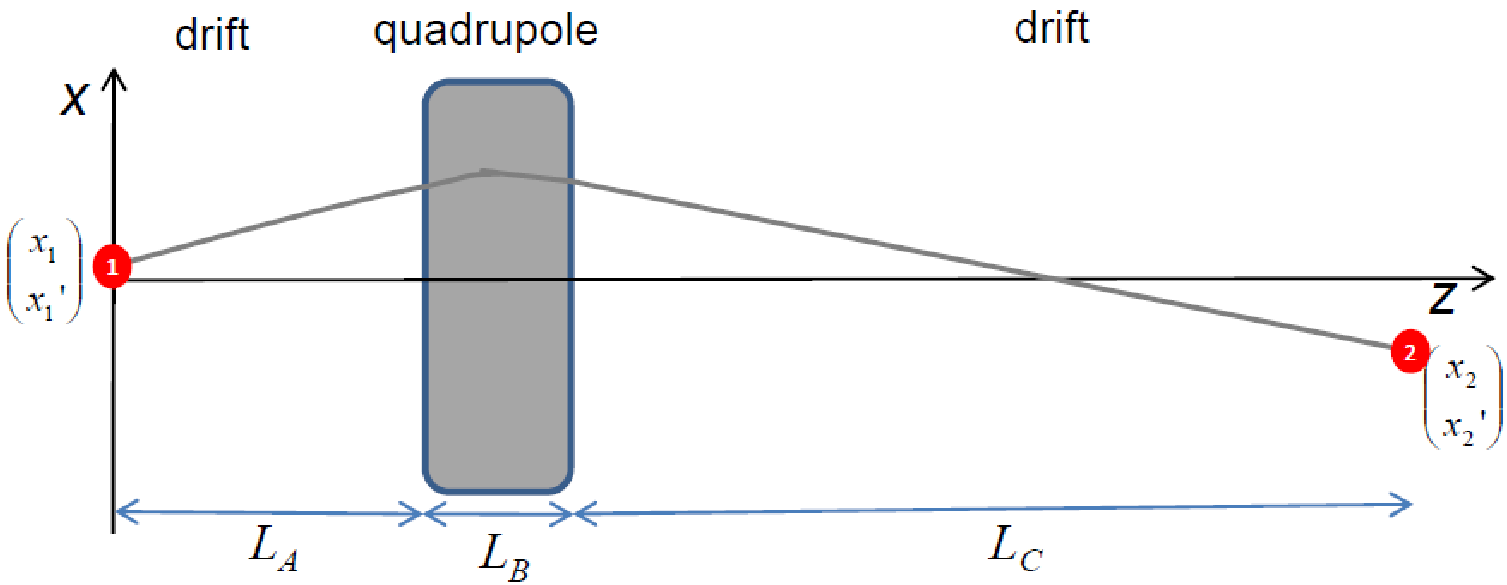


Along a beamline the orientation and aspect ratio of the beam ellipse in x, x' changes, but area (emittance) remains constant.
Alike initial beam distributions have similar phase space dynamics

Beam width along Z is described $$w(z) = \sqrt{\beta(z)\,\varepsilon}$$

$\beta(z)$ describes the beam line, ε – describes beam quality

Transport of single particle described with matrix

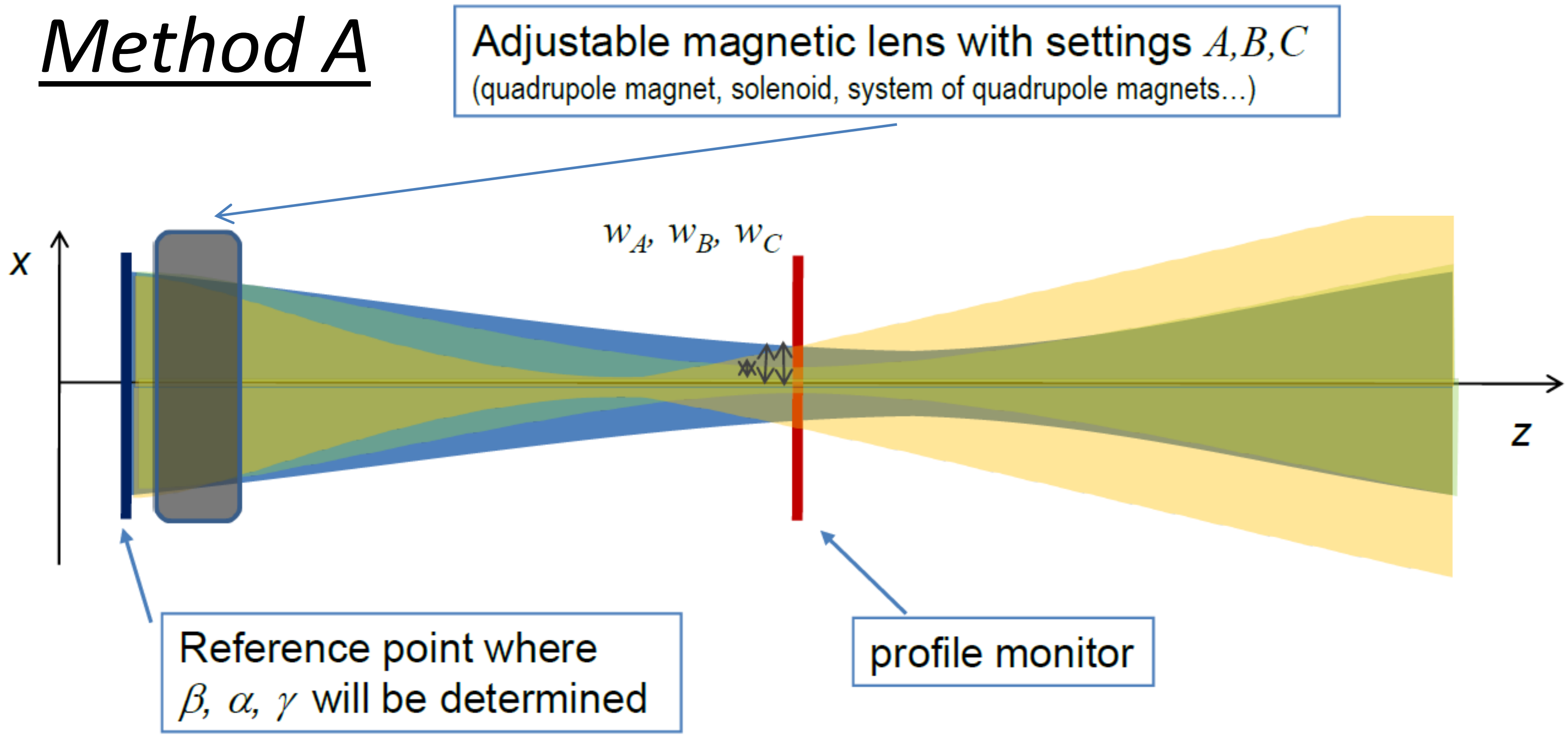


$$M_{Drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \qquad M_{Quadrupole} = \begin{pmatrix} \cos\left(\sqrt{k}L\right) & 1/\sqrt{k}\sin\left(\sqrt{k}L\right) \\ -\sqrt{k}\sin\left(\sqrt{k}L\right) & \cos\left(\sqrt{k}L\right) \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = M \cdot \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$ There $$M = \begin{pmatrix} 1 & L_C \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\sqrt{k}L_B\right) & 1/\sqrt{k}\sin\left(\sqrt{k}L_B\right) \\ -\sqrt{k}\sin\left(\sqrt{k}L_B\right) & \cos\left(\sqrt{k}L_B\right) \end{pmatrix} \cdot \begin{pmatrix} 1 & L_A \\ 0 & 1 \end{pmatrix}$$

Emittancemeasurement in transfer line or linac, (count.)

- ***Method A***



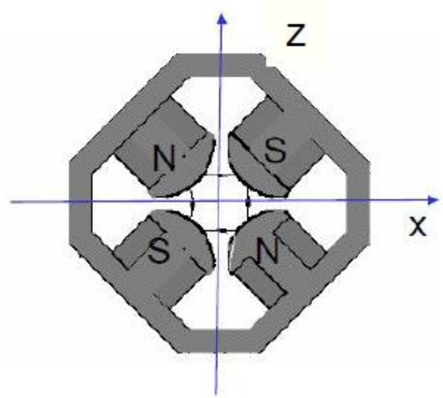
$$w^2 = c^2\,\beta\varepsilon - 2cs\,\alpha\,\varepsilon + s^2\,\gamma\,\varepsilon, \qquad \begin{pmatrix} c & s \\ c' & s' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} m_{11}(I_{mag}) & m_{12}(I_{mag}) \\ m_{21}(I_{mag}) & m_{22}(I_{mag}) \end{pmatrix}$$

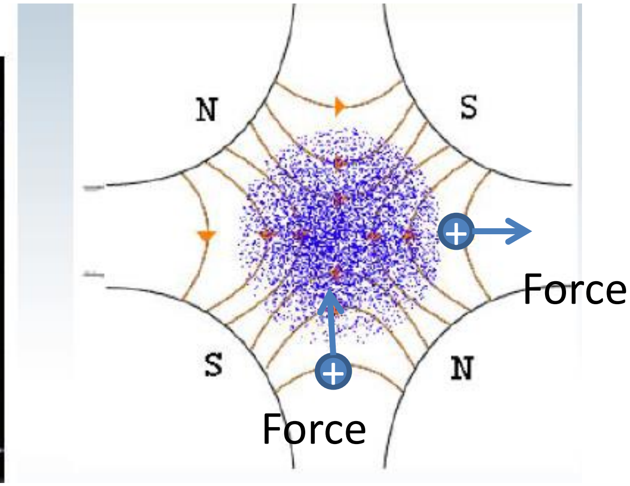
$$w_A^2 = c_A^2\beta\,\varepsilon - 2c_A s_A\,\alpha\,\varepsilon + s_A^2\,\gamma\,\varepsilon$$
$$w_B^2 = c_B^2\beta\,\varepsilon - 2c_B s_B\,\alpha\,\varepsilon + s_B^2\,\gamma\,\varepsilon$$
$$w_C^2 = c_C^2\beta\,\varepsilon - 2c_C s_C\,\alpha\,\varepsilon + s_C^2\,\gamma\,\varepsilon$$

3 linear equations, 3 independent variable
Solved by inverting matrix.

$$\beta\varepsilon\cdot\gamma\varepsilon - (\alpha\varepsilon)^2 = \varepsilon^2\left(\beta\cdot\gamma - \alpha^2\right) = \varepsilon^2 \quad \Rightarrow \quad \sqrt{\beta\varepsilon\cdot\gamma\varepsilon - (\alpha\varepsilon)^2} = \varepsilon, \quad \beta = \frac{\beta\varepsilon}{\varepsilon}, \quad \alpha = \frac{\alpha\varepsilon}{\varepsilon}$$

Quadrupoles

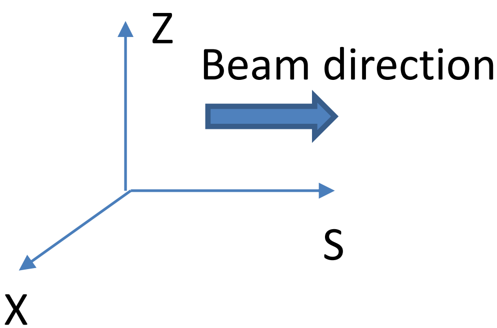




$$B = B_1\left(z\hat{x} + x\hat{z}\right)$$

- Due to special field symmetry focus beam in one direction but defocus in other.
- Particles moving at axis are not experienced any force.

Sometimes in accel. physics especially in circular accel. (X,Z, S) coordinates are used



Quadrupoles(cont.)

- Particle displaced by (x,z) from the center

$$\boxed{\mathrm{B} = \mathrm{B}_1\left(\mathrm{z}\hat{\mathrm{x}} + \mathrm{x}\hat{\mathrm{z}}\right)}$$

$$\vec{\mathrm{F}} = \mathrm{evB}_1\hat{\mathrm{s}} \times \left(\mathrm{z}\hat{\mathrm{x}} + \mathrm{x}\hat{\mathrm{z}}\right) = -\mathrm{evB}_1\mathrm{z}\hat{\mathrm{z}} + \mathrm{evB}_1\mathrm{x}\hat{\mathrm{x}}$$

the equations of motion become:

$$\frac{1}{\mathrm{v}^2}\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{dt}^2} = \frac{\mathrm{eB}_1}{\gamma\mathrm{mv}}\mathrm{x}, \quad \frac{1}{\mathrm{v}^2}\frac{\mathrm{d}^2\mathrm{z}}{\mathrm{dt}^2} = -\frac{\mathrm{eB}_1}{\gamma\mathrm{mv}}\mathrm{z}$$

or $\frac{d^2x}{ds^2} = x'' = \kappa x \quad \frac{d^2z}{ds^2} = -\kappa z$ where $\kappa = \frac{eB_1}{\gamma m v}$

When matrix transformation from entrance to exit of quadrupole :

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos\sqrt{\kappa}L & \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}L \\ -\sqrt{\kappa}\sin\sqrt{\kappa}L & \cos\sqrt{\kappa}L \end{pmatrix} \begin{pmatrix} x_0 \\ x0' \end{pmatrix} \qquad \begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{\kappa}L & \frac{1}{\sqrt{\kappa}}\sinh\sqrt{\kappa}L \\ \sqrt{\kappa}\sinh\sqrt{\kappa}L & \cosh\sqrt{\kappa}L \end{pmatrix} \begin{pmatrix} z_0 \\ z0' \end{pmatrix}$$

Thin lens approximation

- For thin lens when K<<1/L^2 $\kappa = \frac{eB_1}{\gamma m v}$

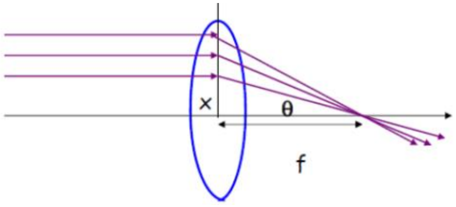
In horizontal plane

$$\begin{pmatrix} \cos\sqrt{\kappa}L & \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}L \\ -\sqrt{\kappa}\sin\sqrt{\kappa}L & \cos\sqrt{\kappa}L \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

In vertical plane

$$\begin{pmatrix} \cosh\sqrt{\kappa}L & \frac{1}{\sqrt{\kappa}}\sinh\sqrt{\kappa}L \\ \sqrt{\kappa}\sinh\sqrt{\kappa}L & \cosh\sqrt{\kappa}L \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{F} & 1 \end{pmatrix}$$

- In accelerator physics $\mathrm{B}\rho = \gamma m v/e$ calls beam rigidity and measured in kG*cm or T*m $\mathrm{B}\rho[kG * cm] = pc[MeV]/0.3$
- If the quadrupole is thin enough, the particles coordinate doesn't change while momentum change. The quad works almost as a optical lens…



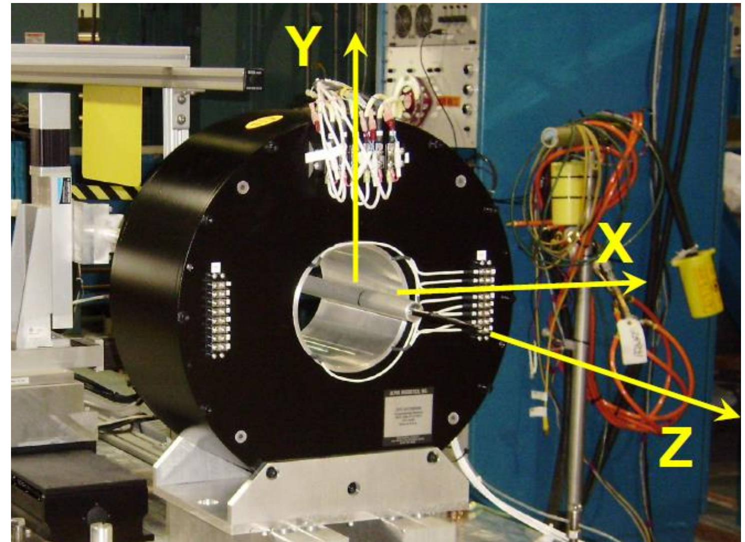
$$\Delta x' = \frac{x}{f}$$

- With only one difference: It Focus in the one plane and defocus in the other plane

Thin Solenoid

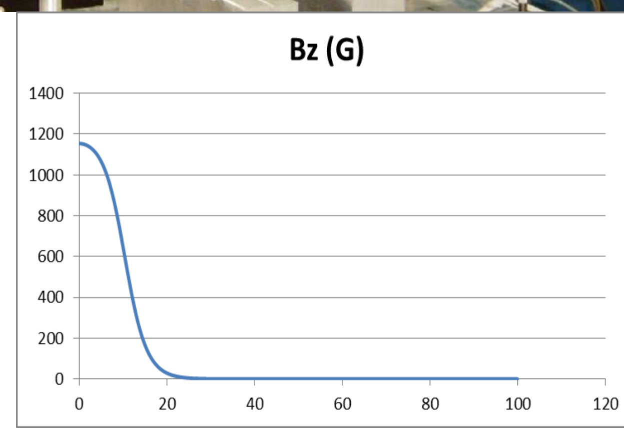
- A solenoid is a set of helical coils.
- Typically, solenoid radius is smaller than the its length.
- Magnetic field is generated along the axis line.
- Solenoid couples X and Y motion.
- Solenoid produced focusing in both direction

$$1/f = \int Bz^2 dz/(2pc/e)^2$$



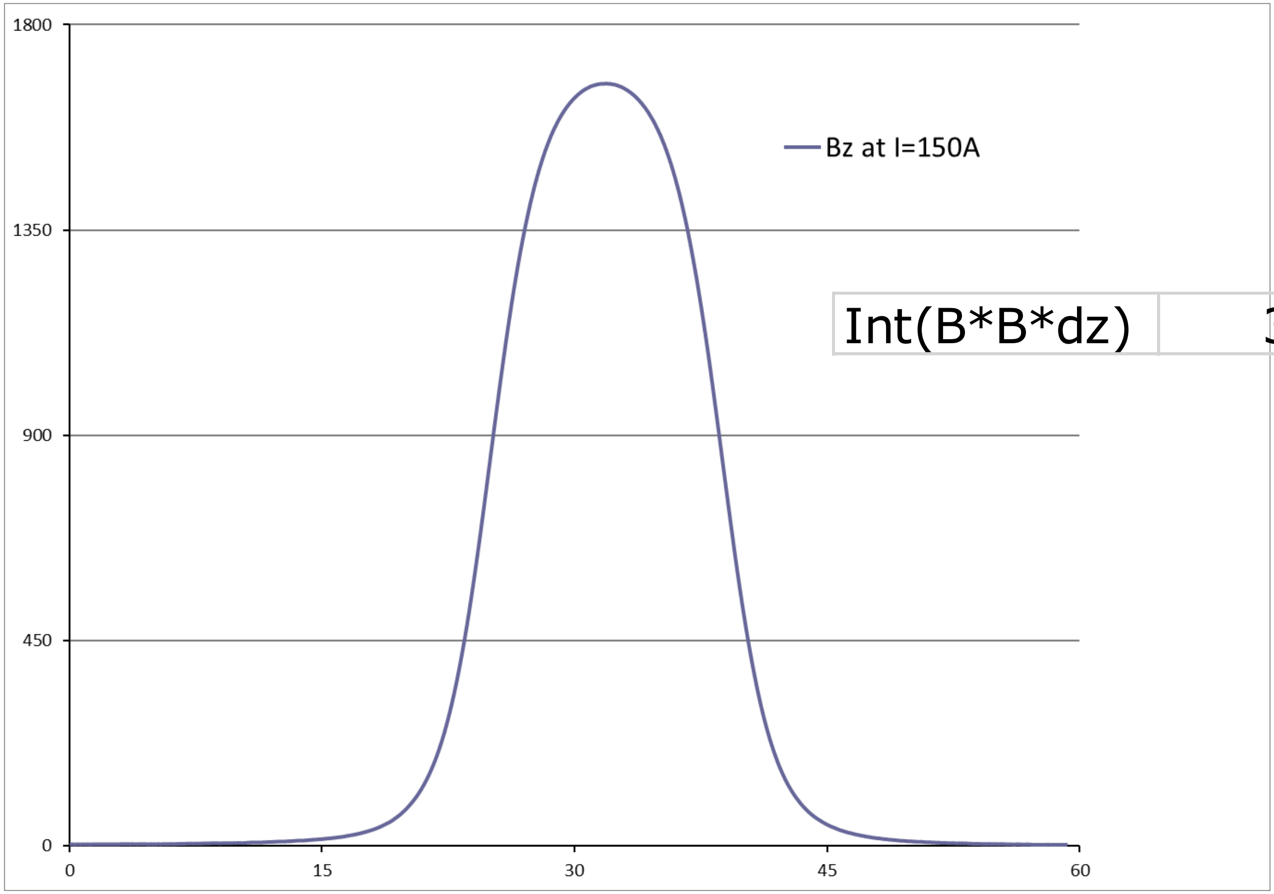
In both directions

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$



- Solenoids are preferable at low energy

SDL solenoid field measurements at I=150A



Int(B*B*dz)	31.20307	kG^2*cm

For example:
for beam energy 5 MeV
Brho=166.7 kG*cm
Focal length f(at I=150A) =
4*Brho^2/INT(B**dz)
f(I=150A)=35 cm

Summary beam profile technics

- To determine ε, β, α at a reference point in a beamline one needs at least 3 *w* measurements with different transfer matrices between the reference point and the *w* measurements location.
- Different transfer matrices can be achieved with different profile monitor locations, different focusing magnet settings or combinations of both.
- Once β, α at one reference point is determined the values of β, α at every point in the beamline can be calculated.

Measurements for more different matrix helps to reconstruct emittance better

Quadrupole or solenoid scan

The quadrupole scan technique is a standard technique used in accelerator facilities to measure the transverse emittance. It is based on the fact that the squared rms beam radius (x_{rms}^2) is proportional to the quadrupole "strength" or inverse focal-length f squared, so

$$x_{rms}^2 =< x^2 >= A\left(\frac{1}{f^2}\right) - 2AB\left(\frac{1}{f}\right) + (C + AB^2) \qquad (1)$$

where A, B, C are constants and f is the focal length defined as

$$\frac{1}{f} = \kappa l, \qquad (2)$$

For quadrupole:

$$k\left[\frac{1}{cm^2}\right] = \frac{G\left[\frac{kGauss}{cm}\right]}{Brho[kGauss*cm]}$$

For solenoid:

$$k\left[\frac{1}{cm^2}\right] = \left[\frac{B_z[kGauss]}{Brho[kGauss*cm]}\right]^2$$

here κ is the magnet focusing strength in units of 1 over length squared and l is the effective length of the magnet.

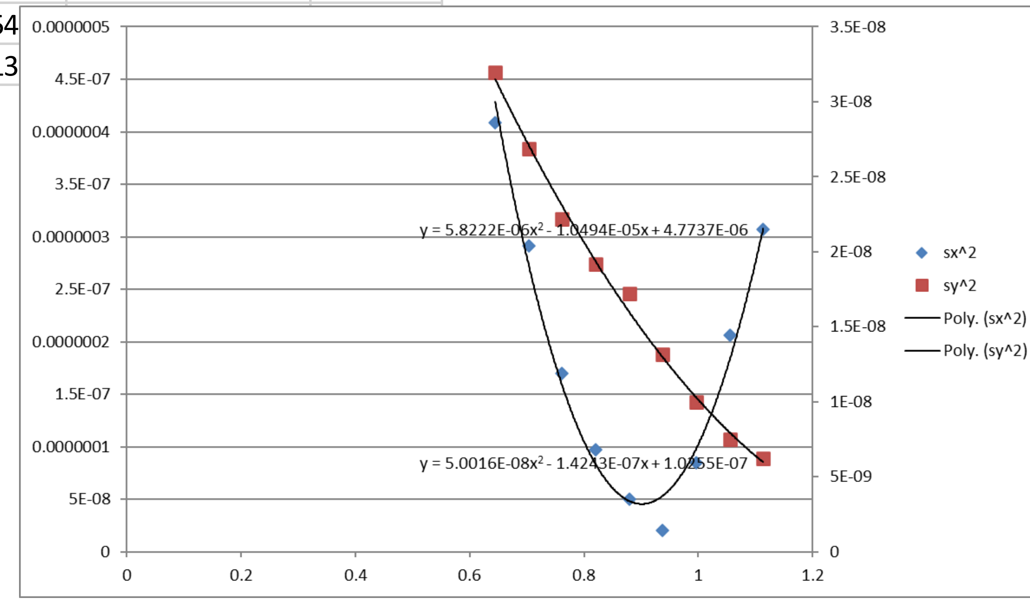
The emittance can be estimated according to

$$\varepsilon = \frac{\sqrt{AC}}{d^2} \qquad (3)$$

where d is the distance from the magnet you scan to the point you calculate the beam rms radius.

Example of Quadrupole scan data analysis from previous year

I, A	sx, pix	sy, pix	sx, m	sy, m	p, 1/m	sx^2	sy^2
5.5	45	12	0.000639	0.0001788	0.644596	4.08321E-07	3.2E-08
6	38	11	0.0005396	0.0001639	0.703195	2.91168E-07	2.69E-08
6.5	29	10	0.0004118	0.000149	0.761795	1.69579E-07	2.22E-08
7	22	9.3	0.0003124	0.0001386	0.820395	9.75938E-08	1.92E-08
7.5	15.7	8.8	0.0002229	0.0001311	0.878994	4.97022E-08	1.72E-08
8	10	7.7	0.000142	0.0001147	0.937594	2.0164E-08	1.32E-08
8.5	20.5	6.7	0.0002911	9.983E-05	0.996193	8.47392E-08	9.97E-09
9	32	5.8	0.0004544	8.642E-05	1.054		
9.5	39	5.3	0.0005538	7.897E-05	1.113		



	a	b	c	A	B	C	e	en
y	**5.002E-08**	**-1.424E-07**	**1.026E-06**	5.002E-08	1.424E+00	9.241E-07	9.216E-09	1.039E-06
x	**5.822E-06**	**-1.049E-05**	**4.774E-06**	5.822E-06	9.012E-01	4.507E-08	2.196E-08	2.475E-06