Presentation notes

Wideroe Criterion

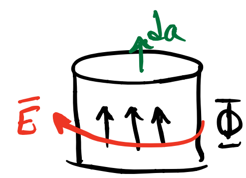
Suppose the is a uniformly distributed magnetic flux $\Phi = \int \bar{B}.d\bar{a}$

rate of change $= \dot{\Phi}$

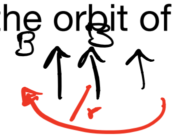
Relevant Maxwell's ~~equation~~ : $\oint \bar{E}.dl = -\dot{\Phi} = -\frac{\partial}{\partial t}\int \bar{B}.d\bar{a}$

If Φ is uniformly distributed,

$$E.\oint dl = -\dot{\Phi} \Rightarrow |E| = \frac{\dot{\Phi}}{2\pi r} \quad \dots (1)$$



Let's now investigate the magnetic field variation on the orbit of the electron:



$$\therefore \frac{p}{r} = eB \Rightarrow \frac{1}{r} = \frac{eB}{p}$$

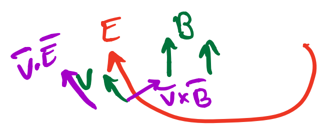
So the condition for constant radius through the changing magnetic field is

$$\frac{d}{dt}\left(\frac{1}{r}\right) = 0 \Rightarrow e\left[\frac{\dot{B}}{p} - \frac{B\dot{p}}{p^2}\right] = 0 \quad \dots (2)$$

$$\dot{p} = F_{\parallel} = q(\bar{E} + \cancel{\bar{v}\times\bar{B}}) = \frac{e\dot{\Phi}}{2\pi r}$$

longitudinal force term

acceleration term ⊥r to velocity in circular motion: does not lead to change in p



$$(2) \Rightarrow \frac{\dot{B}}{\cancel{p}} = \frac{B\dot{p}}{p^{\cancel{2}}} \Rightarrow \dot{B}p = B\dot{p}$$

$$\dot{B}\,(\cancel{4}\cancel{\pi} r) = \cancel{B}\,\frac{\cancel{4}\Phi}{2\pi r} \Rightarrow \dot{B}\,(2\pi r^2) = \Phi$$

$$\Rightarrow \dot{\Phi} = 2\,(\dot{B}\pi r^2)$$

So, the flux through the orbit must change at twice the rate of the magnetic field at the location of the orbit to maintain constant electron orbit.

Cyclotron

$$\omega = \frac{qB}{\gamma m}$$

$$r_c = \frac{p}{qB} \quad \text{or} \quad B\rho = \frac{p}{q}$$

Phase Stability

The particle accelerators have a large number of sections (or one section that is traversed over and over again). One method of achieving consistent acceleration throughout the device is by adhering to the so called "synchrotron condition", that is to assume that the particle will arrive at each accelerating station at the same phase of the RF voltage.

By implication, there is a perfect particle that adheres to this perfect plan for the accelerator system, I.e it has the right energy and traverse time through the structure to receive the right increment of energy. But we are also concerned with other particles that deviate slightly in energy and transit time from the perfect particle. Finding out whether these particles that start near the perfect particle in energy and transit time stay with the ideal particle in this "phase space" constitutes a Phase Stability problem.

We will find that there is a strong stability condition that dictates that particles near the ideal particle will remain nearby and oscillate about the ideal particle in energy-transit time space.

Since these oscillations were first analyzed for a device called synchrotron, they are called synchrotron oscillations.

The concern is that the particles that are different in energy from the perfect or synchronized particle arrive at a different time in the RF cycle, and will experience different acceleration force. The question is whether this difference in energy gain reinforces the disparity in transit time to the next station (unstable case) or reduces it (stable case).

transit time – equivalent to position differences along z,

i.e longitudinal degree of freedom

$\Downarrow$

Because freq. of longitudinal oscillations $\ll$ transverse oscillations, to a reasonable approximation, we can decouple this from the other two.

Synchrotron oscillations

We want to find the difference in the time it takes to traverse between two accelerating sections as a function of the difference between the momentum of a particle and the momentum of the synchronized particle. Suppose we have a number of accelerating stations.

→ one cavity or a system of cavities

→ assume a source of radiofrequency power ω_{rf}

→ assume ideal particle arrives at each station at the same phase (multiples of 2π) & recieves the same increment of energy at each station

The progress of the ideal particle through the accelerator is charted in the design of the device. In general however, a particle will deviate from the design motion, and we wish to develop equations of motion that treat those deviations:

Let

τ : time interval between passages of 2 successive stations for ideal particle

L : space between stations

v : particle speed,

$$\tau = \frac{L}{v}$$

fractional time due to deviations in L or v:

$$\frac{1}{\tau} \times \left[d\tau \right] = \left[\frac{dL}{v} - \frac{L}{v^2} dv \right] \times \frac{v}{L}$$

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$$\boxed{\frac{d\tau}{\tau} = \frac{dL}{L} - \frac{dv}{v}}$$

This is time difference between the sync particle and some other particle (it's not a change of time or velocity for the same particle)

higher speed particle takes less time

We want to convert $\frac{d\tau}{\tau}$ in terms of momentum:

One can show that

$$\frac{dv}{v} = \frac{1}{\gamma^2}\frac{dp}{p}$$

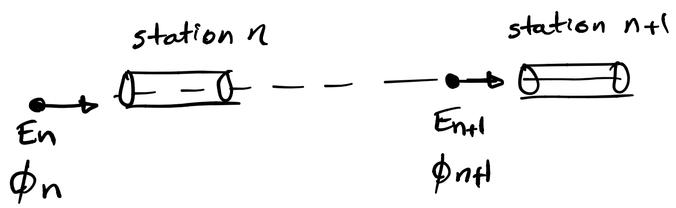
introduce

$$\frac{dL}{L} = \frac{1}{\gamma_t^2}\left(\frac{dp}{p}\right)$$

here, γ_t is determined by the time and device being studied as well as its particular design

$$\therefore \frac{d\tau}{\tau} = \underbrace{\left[\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right]}_{\equiv \eta \text{ the slip factor}} \frac{dp}{p}$$

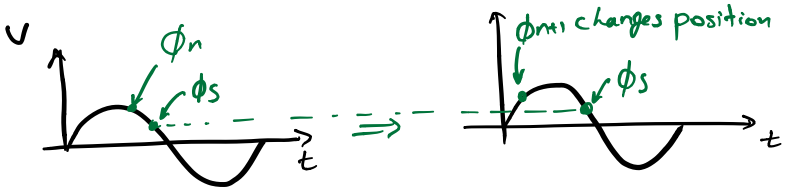
at $\gamma_t = \gamma$, η flips sign. This energy is called the transition energy, hence subscript t



E: energy

ϕ: phase with respect to the accelerating voltage.

e.g.



$$\phi_{n+1} = \phi_n + \omega_{rf}\,\Delta\tau_{n+1}$$

← ϕ_n changes from one cycle to the next because its transit time is different from the synchronous particle

$$= \phi_n + \omega_{rf}\,\tau_{n+1}\left(\frac{\Delta\tau_{n+1}}{\tau_{n+1}}\right)$$

$$\therefore \boxed{\phi_{n+1} = \phi_n + \eta\,\omega_{rf}\,\tau_{n+1}\left(\frac{\Delta p}{p}\right)_{n+1}}$$

← difference equation of motion.

(2.32 in E&S)

<u>Note</u>: To arrive at this condition, the book (S&E) starts from total phase

$$\varphi_n = \phi_n + \omega_{rf}\,\tau_n$$

$\tau_n = 0$ at the 1st station.

I found this notation to introduce annecessary complication and avoided it here, but the final results are the same

In a synchrotron situation, $\omega_{rf}\,\tau = 2h\pi$, and is independent of n. 'h' is called the harmonic number.

We now have phase difference as a function of momentum variation. We need a second equation to solve for ϕ & P (or E as it turns out):

$$(E_s)_{n+1} = \underbrace{(E_s)_n}_{\text{energy of synchrous particle at station } n} + eV \sin\phi_s \quad (2.33)$$

V: amplitude of emf across the cavity,
ϕ_s: phase of arrival for ideal particle (synchronous phase)

for any particle, $E_{n+1} = E_n + eV\underline{\sin\phi_n}$ (2.34)

so difference in acceleration & velocity must correspond to difference b/t phase of arrival

$\Delta E \equiv E - E_s$

$$\therefore \quad \Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s)$$ → subtract the two equations above.

$$\frac{dp}{p} = \frac{c^2}{v^2}\frac{dE}{E}$$ insert in 2.32

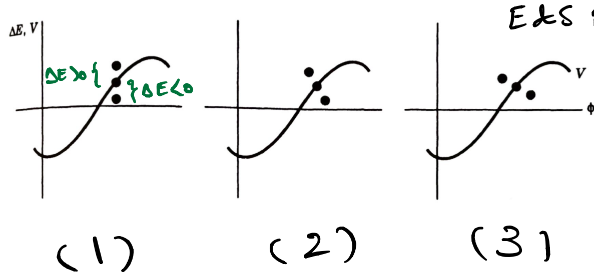
$$\begin{cases} \phi_{n+1} = \phi_n + \dfrac{\omega_{rf}\,T\,\eta\,c^2}{v^2 E_s}\Delta E_{n+1} \\ \Delta E_{n+1} = \Delta E_n + eV(\sin(\phi_n) - \sin(\phi_s)) \end{cases}$$

Difference Equations
(2.37) & (2.38) in E&S

So will particles close in phase space remain close? What follows applies to <u>circular accelerators</u>, but the argument is nearly the same for linear ones.

Consider 3 particles that start out at the same phase, but have different energy. (e.g. because the beam has an energy spread)

E&S pg35



$\Delta\phi_1 = 0,\ \Delta E_1 \neq 0$

Also, assume $\eta < 0$,

Note: if $\eta < 0$, that means speed changes effects dominate path length differences

$$\Delta E_2 = \Delta E_1 + eV(\underbrace{\sin(\phi_1) - \sin\phi_s}_{\Delta\phi = 0}) = \Delta E_1$$

$$\phi_2 = \phi_1 + \frac{(\omega\tau)\eta}{\beta^2}\frac{\Delta E_2}{E_s}$$

$$\phi_2 = \phi_s + \frac{(\omega\tau)\eta}{\beta^2}\frac{\Delta E_1}{E_s} \begin{cases} < \phi_s \text{ for } \Delta E_1 > 0 \\ > \phi_s \text{ for } \Delta E_1 < 0 \end{cases}$$

($\eta < 0$ case)

because the particle w/ $E > E_s$ arrives early ($\phi_2 < \phi_s$), it sees a smaller voltage & recieves less energy

$$\Delta E_3 = \Delta E_2 + eV(\underbrace{\sin(\phi_2) - \sin\phi_s}_{< 0})$$

$$= \Delta E_1 - eV\delta \rightarrow \text{energy gap is reduced}$$

$\rightarrow$ leads to stable oscillations

$$\phi_3 = \phi_2 + \frac{(\omega\tau)\eta}{\beta^2}\frac{\Delta E_3}{E_s}$$

$\Delta E_3 < \Delta E_2 \rightarrow \phi_3$ is closer to ϕ_2 than ϕ_2 was to ϕ_1

$\rightarrow$ stable oscillation

In this case we looked at three particles that start with different energy and the same phase. You can see that the difference between energies gets reduced and so the does the phase slippage on the next station. This reduction in both phase change and energy slippage from turn to turn is the seed of stable oscillations.

Features of $\phi - \Delta E$ space

→ There is a well defined boundary between the stable & non-stable motion. This boundary is called separatrix.

1. There are two points in the phase space at which the particle undergoes no phase motion:

- Ideal particle with $\phi = \phi_s$ & $\Delta E = 0$
- Point on the separatrix, called the unstable fixed point. The closer point separation on the unstable fixed point indicates that the particle moves very slowly in this region. In particular, a particle on the separatrix moving towards the fixed point would require infinite many turns to reach it.

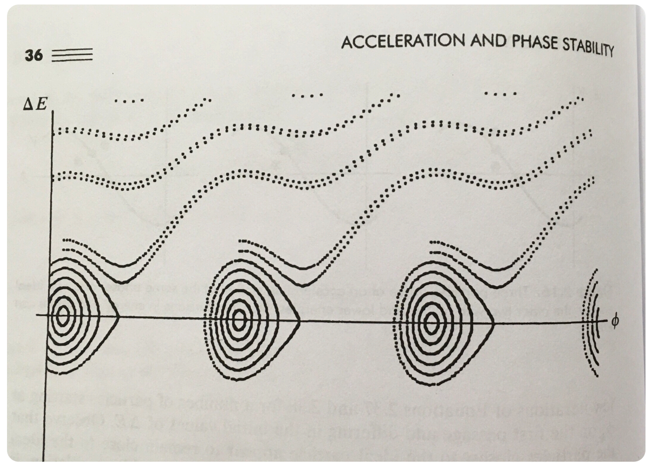


Figure shows 3 stable fixed points

2. In the case of a circular accelerator, where the harmonic number is generally greater 1, there could be many stable points distributed and spaced by 2π (see figure below)

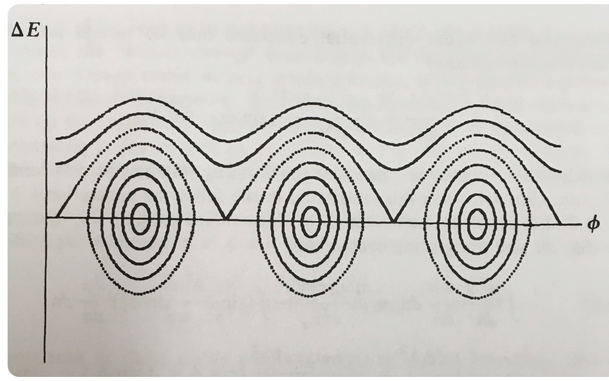
3. The area within the separatrix is called a bucket in accelerator jargon, so this figure depicts three stable buckets.

At LEP, ONLY 4 buckets are occupied. The length of LEP is 27 km and the buckets are about a meter about. So just over a 0.1% of buckets are occupied. Note that each bucket basically means a 2pi shift in the radial frequency angle of the RF pulse.

Electrons in each bucket are called a bunch.

4. If the synchronous phase is zero or PI, the ideal particle is unaccelerated and the phase stable region is the entire 2π
these are called stationary buckets. For this case, a particle outside the separatrix will simply oscillate in energy and may never leave the

accelerator. However, it is an unstable particle because it will continue drifting in phase.



Case of $\phi_s = 0$ or π