PHY 554. Homework 3.

(a) The closed orbit of a three-bump system is

$$y(s) = \frac{\sqrt{\beta}}{2\sin\pi\nu}\sum_{i=1}^{3}\sqrt{\beta_i}\theta_i\cos(\pi\nu - |\psi - \psi_i|).$$

Using the condition $y(s_3) = y'(s_3) = 0$, Then

$$\begin{cases} \sqrt{\beta_1}\theta_1\cos(\pi\nu + \psi_{13}) + \sqrt{\beta_2}\theta_2\cos(\pi\nu + \psi_{23}) + \sqrt{\beta_3}\theta_3\cos\pi\nu = 0 \\ \sqrt{\beta_1}\theta_1\sin(\pi\nu + \psi_{13}) + \sqrt{\beta_2}\theta_2\sin(\pi\nu + \psi_{23}) + \sqrt{\beta_3}\theta_3\sin\pi\nu = 0 \end{cases}$$

where $\psi_{13} = \psi_1 - \psi_3$ and $\psi_{23} = \psi_2 - \psi_3$, we find

$$\theta_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}, \qquad \theta_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\frac{\sin\psi_{12}}{\sin\psi_{23}}.$$

(b) When $\psi_{31} = n\pi$, we find $\theta_2 = 0$, i.e. only two steering dipoles are needed for a local bump. Since $\psi_{32} = \psi_{31} - \psi_{21} = n\pi - \psi_{21}$, we have $\sin\psi_{32} = (-1)^{n-1}\sin\psi_{31}$, and $\theta_3 = (-1)^{n-1}\sqrt{\beta_1/\beta_2}\,\theta_1$.

#1 the transfer matrix is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -1/2f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1+L & L\theta/2 + L_1\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & L_1+L & L\theta/2 + L_1\theta \\ -1/2f & -\frac{L_1}{2f} - L/2f + 1 & -1/2f\left(L\theta/2 + L_1\theta\right) + \theta \\ 0 & 0 & 1 \end{pmatrix}$$

initial $D_0 = D_0' = 0 \Rightarrow$

$$\begin{pmatrix} D_c \\ D_c' \\ 1 \end{pmatrix} = M \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} L\theta/2 + L_1\theta \\ -1/2f\,(L\theta/2 + L_1\theta) + \theta \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{D_c = (L/2 + L_1)\theta} \quad D_c' = \theta\left(1 - \frac{1}{2f}\,(L/2 + L_1)\right) = 0$$

$$\Rightarrow \boxed{f = L_1/2 + L/4}$$

#2.

a) $D = a\cos\sqrt{k}s + b\sin\sqrt{k}s + 1/\rho k$

$\Rightarrow D'' = -ak\cos\sqrt{k}s - bk\sin\sqrt{k}s$

so $D'' + kD = 1/\rho k \cdot k = 1/\rho$. satisfies the condition.

at $s=0$. $D_0 = a + 1/\rho k$ $\quad D_0' = b\sqrt{k}$

thus $a = D_0 - \frac{1}{\rho k}$ $\quad b = \frac{D_0'}{\sqrt{k}}$

$$\Rightarrow D = \left(D_0 - \frac{1}{\rho k}\right)\cos\sqrt{k}s + \frac{D_0'}{\sqrt{k}}\sin\sqrt{k}s + \frac{1}{\rho k}$$

$$= D_0\cos\sqrt{k}s + \frac{D_0'}{\sqrt{k}}\sin\sqrt{k}s + \frac{1}{\rho k}\left(1 - \cos\sqrt{k}s\right)$$

$$D' = -D_0\sqrt{k}\sin\sqrt{k}s + D_0'\cos\sqrt{k}s + \frac{1}{\rho\sqrt{k}}\sin\sqrt{k}s$$

thus M can be expressed as

$$M = \begin{pmatrix} \cos\sqrt{k}s & \frac{1}{\sqrt{k}}\sin\sqrt{k}s & \frac{1}{\rho k}(1-\cos\sqrt{k}s) \\ -\sqrt{k}\sin\sqrt{k}s & \cos\sqrt{k}s & \frac{1}{\rho\sqrt{k}}\sin\sqrt{k}s \\ 0 & 0 & 1 \end{pmatrix}$$

b) for $k < 0$. substitute $k \to -k$ in above matrix

and use $\cos iA = \cosh A$ $\quad \sin iA = i\sinh A$

can easily show that

$$M = \begin{pmatrix} \cosh\sqrt{|k|}s & \frac{1}{\sqrt{|k|}}\sinh\sqrt{|k|}s & \frac{1}{\rho|k|}(-1 + \cosh\sqrt{k}s) \\ \sqrt{|k|}\sinh\sqrt{|k|}s & \cosh\sqrt{|k|}s & \frac{1}{\rho\sqrt{|k|}}\sinh\sqrt{|k|}s \\ 0 & 0 & 1 \end{pmatrix}$$

proof. $\cos iA = \cosh A \qquad \sin iA = i \sinh A$.

we know $\qquad \cos x = \dfrac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$$\Rightarrow \cos iA = \frac{e^{-A} + e^{A}}{2} = \cosh A.$$

$$\sin iA = \frac{e^{-A} - e^{A}}{2i} = -i\,\frac{e^{-A} - e^{A}}{2} = i\,\frac{e^{A} - e^{-A}}{2} = i \sinh A.$$

q.e.d.

c) for pure dipole $\quad K = \frac{1}{\rho^2} \Rightarrow D'' + \frac{1}{\rho^2} D = \frac{1}{\rho}$.

thus we can write

$$D = a\cos\theta + b\sin\theta + \rho(1-\cos\theta) \qquad \text{where } \theta = S/\rho.$$

$$D' = -\frac{a}{\rho}\sin\theta + b/\rho\,\cos\theta + \sin\theta$$

$$D'' = -\frac{a}{\rho^2}\cos\theta \mp b/\rho^2\,\sin\theta + \frac{1}{\rho}\cos\theta$$

$$D'' + \frac{1}{\rho^2} D = \frac{1}{\rho}\cos\theta + \frac{1}{\rho}(1-\cos\theta) = \frac{1}{\rho}.$$

when $\quad S = \theta = 0 \qquad D = D_0 = a$

$$D' = D_0' = b/\rho \quad \Rightarrow \begin{cases} a = D_0 \\ b = \rho \cdot D_0' \end{cases}$$

$$\Rightarrow D = D_0\cos\theta + \rho \cdot D_0'\sin\theta + \rho(1-\cos\theta)$$

$$D' = -\frac{D_0}{\rho}\sin\theta + D_0'\cos\theta + \sin\theta$$

$$\Rightarrow M = \begin{pmatrix} \cos\theta & \rho\cdot\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

only dipoles contribute to $D \cdot D'$

d) using $S \to 0$ and $KS = \frac{1}{f}$ we can show

$$M_{quad} = \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\rho\theta = L \quad \Rightarrow \quad M_{dip} = \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$