USPAS "Hadron Beam Cooling in Particle Accelerators"
HW1 - Monday, January 30, 2023
Problem 1: Reference particle and reference orbit. 10 points
Using accelerator Hamiltonian (M1.19), corresponding differential equations (M1.20), expansion of the vector and scalar potentials (M1.21), show that for a reference particle that is following a reference "trajectory":

$$
\vec{r}=\vec{r}_{o}(s) ; \quad t=t_{o}(s) ; H=H_{o}(s)=E_{o}(s)+\varphi_{o}\left(s, t_{o}(s)\right),
$$

with $x \equiv 0 ; y \equiv 0 ; p_{x} \equiv 0 ; p_{y} \equiv 0$ and $\left.h^{*}\right|_{\text {ref }}=-p_{o}(s)$ result in the following conditions:

$$
\begin{align*}
& K(s) \equiv \frac{1}{\rho(s)}=-\frac{e}{p_{o} c}\left(\left.B_{y}\right|_{r e f}+\left.\frac{E_{o}}{p_{o} c} E_{x}\right|_{r e f}\right) ;  \tag{1}\\
&\left.B_{x}\right|_{r e f}=\left.\frac{E_{o}}{p_{o} c} E_{y}\right|_{r e f} ;  \tag{2}\\
& \frac{d t_{o}(s)}{d s}=\frac{1}{\mathrm{v}_{o}(s)}  \tag{3}\\
& \frac{d E_{o}(s)}{d s}=-\left.e \frac{\partial \varphi}{\partial s}\right|_{r e f} \equiv e E_{2}\left(s, t_{o}(s)\right) . \tag{4}
\end{align*}
$$

Hints:

1. Use condition $\left.\vec{A}\right|_{\text {ref }}=0$ with

$$
\left.x\right|_{r e f}=0 ;\left.y\right|_{r e f}=0 ;\left.P_{1}\right|_{r e f}=\left.p_{x}\right|_{r e f}+\left.\frac{e}{c} A_{1}\right|_{r e f} \equiv 0 ;\left.\quad P_{3}\right|_{r e f}=\left.p_{y}\right|_{r e f}+\left.\frac{e}{c} A_{3}\right|_{r e f} \equiv 0 ;
$$

or in the differential form

$$
\begin{aligned}
& \left.\frac{d x}{d s}\right|_{r e f}=\left.\frac{\partial h^{*}}{d P_{1}}\right|_{r e f}=0 ;\left.\frac{d y}{d s}\right|_{r e f}=\left.\frac{\partial h^{*}}{d P_{3}}\right|_{r e f}=0 ; \\
& \left.\frac{d P_{1}}{d s}\right|_{r e f}=-\left.\frac{\partial h^{*}}{d x}\right|_{r e f}=0 ;\left.\frac{d P_{3}}{d s}\right|_{r e f}=-\left.\frac{\partial h^{*}}{d y}\right|_{r e f}=0 ;
\end{aligned}
$$

2. Keep only necessary (i.e. relatively low order) terms in expansion of vector potentials.

## Problem 2: Trace and determinant. 5 points

Solution of any linear n-dimensional differential equation

$$
\frac{d X}{d s}=\mathbf{D}(s) X
$$

can be expressed in a form of transport matrix

$$
X(s)=\mathbf{M}(s) X_{o} ; X_{o}=X(s=0)
$$

with

$$
\begin{equation*}
\frac{d \mathbf{M}(s)}{d s}=\mathbf{D}(s) \mathbf{M}(s) ; \mathbf{M}(s=0)=\mathbf{I} \tag{1}
\end{equation*}
$$

where $\mathbf{I}$ is unit $n x n$ matrix. Prove that

$$
\operatorname{det}(\mathbf{M}(s))=\exp \left(\int_{0}^{s} \operatorname{Trace}(\mathbf{D}(\zeta)) d \zeta\right)
$$

Hints:

1. Prove first that

$$
\frac{d}{d s} \operatorname{det} \mathbf{M}=\operatorname{Trace}(\mathbf{D}) \cdot \operatorname{det} \mathbf{M}
$$

2. Use infinitesimally small step in eq. (1) to conclude that

$$
\begin{gather*}
d \mathbf{M}(s)=\mathbf{D}(s) \mathbf{M}(s) d s+O\left(d s^{2}\right) \Rightarrow \mathbf{M}(s+d s)=(\mathbf{I}+\mathbf{D}(s) d s) \cdot \mathbf{M}(s)+O\left(d s^{2}\right) ; \\
\operatorname{det} \mathbf{M}(s+d s)=\operatorname{det}(\mathbf{I}+\mathbf{D}(s) d s) \cdot \operatorname{det} \mathbf{M}(s)+O\left(d s^{2}\right) \rightarrow  \tag{1}\\
\frac{1}{\operatorname{det} \mathbf{M}} \frac{d(\operatorname{det} \mathbf{M})}{d s}=\frac{\operatorname{det}(\mathbf{I}+\mathbf{D}(s) d s)-1}{d s}
\end{gather*}
$$

3. What remained is to prove us that

$$
\operatorname{det}(\mathbf{I}+\varepsilon \mathbf{D})=1+\varepsilon \cdot \operatorname{Trace}[\mathbf{D}]+O\left(\varepsilon^{2}\right)
$$

where $\varepsilon$ is infinitesimally small real number and term $O\left(\varepsilon^{2}\right)$ contains second and higher orders of $\varepsilon$.
4. First, fist look on the product of diagonal elements $\prod_{m=1}^{n}\left(1+\varepsilon a_{m m}\right)$ in $\operatorname{det}[I+\varepsilon A]$ in the first order of $\varepsilon$. Then prove that contributions to determinant from non-diagonal terms $a_{k m} ; k \neq m$ is $O\left(\varepsilon^{2}\right)$ or higher order of $\varepsilon$. It is possible to do it directly for an arbitrary $n x n$ matrix, or start from $n=1$ and use induction from $n$ to $n+1$.

## By doing this you also prove the sum of decrements theorem!

P.S. Any elegant and unexpected solution will have result in quadrupled points.

## Problem 3: Distribution of decrements/increments. 10 points

Let's consider a storage ring with plane orbit (no torsion) and absence of elements coupling horizontal and vertical degrees of freedom. In this case, transvers components of the vector potential can be set to zero with Canonical momenta coinciding with mechanical momenta:

$$
\pi_{x}=\frac{P_{x}}{p_{o}}=x^{\prime} .
$$

Horizontal and longitudinal oscillations in a storage ring remain coupled in all places where dispersion is non-zero. As we discuss in class, in the absence of dissipative processes,

$$
\frac{d X}{d s}=\mathbf{D}(s) X ; X=\left[\begin{array}{c}
x \\
x^{\prime} \\
\tau \\
\delta
\end{array}\right] ; \mathbf{D}(s)=\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
-K_{1}(s) & 0 & 0 & K_{o}(s) \\
-K_{o}(s) & 0 & 0 & 1 /(\gamma \beta)^{2} \\
0 & 0 & -U(s) & 0
\end{array}\right]
$$

and slow synchrotron oscillations $\left(Q_{s} \ll 1\right)$ it is described by two actions and two phases with associated periodic eigen vectors (M3.33):

$$
\begin{gathered}
X=\operatorname{Re}\left(a_{x} Y_{x}(s) e^{i\left(\psi_{x}+\varphi_{x}\right)}+a_{s} Y_{s}(s) e^{i\left(\psi_{s}+\varphi_{s}\right)}\right) ; \psi_{x}^{\prime}=\frac{1}{\mathrm{w}_{x}(s)^{2}} \equiv \frac{1}{\beta_{x}(s)} ; \psi_{s}^{\prime} \cong 2 \pi Q_{s} \frac{s}{C} ; \\
Y_{x}=\left[\begin{array}{c}
\mathrm{w}_{x} \\
\mathrm{w}_{x}^{\prime}+\frac{i}{\mathrm{w}_{x}} \\
\eta\left(\mathrm{w}_{x}^{\prime}+\frac{i}{\mathrm{w}_{x}}\right)-\eta^{\prime} \mathrm{w}_{x} \\
0
\end{array}\right] ; Y_{s}=\left[\begin{array}{c}
\frac{i}{\mathrm{w}_{s}} \eta \\
\frac{i}{\mathrm{w}_{s}} \eta^{\prime} \\
\mathrm{w}_{s} \\
\frac{i}{\mathrm{w}_{s}}
\end{array}\right] ; \mathrm{w}_{s}=\sqrt{\left|\eta_{\tau} / \mu_{3}\right|}
\end{gathered}
$$

Let's add a distributed weak cooling process with linear drag forces:

$$
\begin{gathered}
\frac{d X}{d s}=(\mathbf{D}(s)+\delta \mathbf{D}(s)) X ; \delta \mathbf{D}(s)=\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & -\chi_{x}(s) & 0 & 0 \\
0 & 0 & 0 & 0 \\
g_{x s}(s) & 0 & 0 & -\chi_{s}(s)
\end{array}\right] ; \\
\operatorname{Trace}(\delta \mathbf{D})=-\left(\chi_{x}(s)+\chi_{s}(s)\right)<0 .
\end{gathered}
$$

Assuming that this is a weak perturbation, calculate one turn decrements. Assuming that both friction forces are dissipative $\chi_{x, y}(s)>0$ find if it is possible that one degree of freedom experiences growth instead of damping. In other words what an off-diagonal tear $g_{x s}(s)$ (asymmetric, i.e. non-Hamiltonian, $\mathrm{x}-\delta$ coupling) contributes to distribution of decrements?

