Stony Brook University

CASE
Center for Accelerator Science and Education

PHY 564
Advanced Accelerator Physics
Lecture 9 and 10
1D, 2D and 3D cases for *exp*[D*s*]

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These lectures will be dedicated for explicit derivations of all necessary exponents for matrix **D***s*.

Sylvester formulae for normal and degenerated cases:

For case with distinct eigne values

$$f(\mathbf{D}) = \sum_{k=1}^{2n} f(\lambda_k) \cdot \prod_{j \neq k} \frac{\mathbf{D} - \lambda_j \cdot \mathbf{I}}{\lambda_k - \lambda_j}$$

and degenerated case

$$f(\mathbf{D}) = \sum_{i=1}^{m} \left[\left(\sum_{j=0}^{l_i - 1} \frac{\phi_i^{(j)}(\lambda_i)}{j!} (\mathbf{D} - \lambda_i I)^j \right) \prod_{k \neq i} (\mathbf{D} - \lambda_k \mathbf{I})^{l_k} \right];$$

where m is the number of distinct eigen values and l_i is maximum size of the Jordan blocks related to eigen value λ_i and

$$\phi_i(\lambda) = f(\lambda) / \prod_{j \neq i} (\lambda - \lambda_j)^{l_j};\ \phi_i^{(j)}(\lambda_i) = \left. \frac{d^j \phi_i(\lambda)}{d\lambda^j} \right|_{\lambda = \lambda_i}.$$

1D case – decouple motions. Let's start from simple case of n=2 – a one dimensional (or a 2D phase space component of decoupled multidimensional motion):

$$H=\frac{1}{2}X^T\cdot\mathbf{H}\cdot X;\ \mathbf{H}=\begin{bmatrix} h_{11} & h_{12}\\ h_{12} & h_{22}\end{bmatrix};\ \mathbf{S}=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix};\ \mathbf{D}=\mathbf{S}\cdot\mathbf{H}=\begin{bmatrix} h_{12} & h_{22}\\ -h_{11} & -h_{12}\end{bmatrix};$$

with h_{12} usually (i.e. xp term) is usually zero and h_{22}=1 or $h_{22}=1/p_o$. But independently of the coefficients, trace of **D** is zero and

$$det(\mathbf{D}-\boldsymbol{\lambda}\cdot\mathbf{I})=det\begin{bmatrix} \pm h_{12}-\lambda & h_{22}\\ -h_{11} & -h_{12}-\lambda\end{bmatrix}=\lambda^2+h_{11}\cdot h_{22}-h_{22}{}^2=0;\ \lambda_{1.2}=\sqrt{-det\mathbf{D}}\equiv\pm\sqrt{-det\mathbf{H}};$$

with three distinct cases:

1. $det\mathbf{H}>0;\ \lambda_{1,2}=\pm i\cdot\sqrt{det\mathbf{H}};\ \lambda_1\neq\lambda_2$: stable (oscillations), simple Sylvester formula

$$\lambda_1=-\lambda_2=i\omega;\ \omega=\sqrt{det\mathbf{H}};\ f(\mathbf{D})=\sum_{k=1}^{2n}f(\lambda_k)\cdot\prod_{j\neq k}\frac{\mathbf{D}-\lambda_j\cdot\mathbf{I}}{\lambda_k-\lambda_j}=f(\lambda_1)\cdot\frac{\mathbf{D}-\lambda_2\cdot\mathbf{I}}{\lambda_1-\lambda_2}+f(\lambda_2)\cdot\frac{\mathbf{D}-\lambda_1\cdot\mathbf{I}}{\lambda_2-\lambda_1};$$

$$f(\mathbf{D})=\mathbf{I}\cdot\left(f(i\omega)\cdot\frac{i\omega}{2i\omega}+f(-i\omega)\cdot\frac{-i\omega}{-2i\omega}\right)+\mathbf{D}\cdot\left(\frac{f(i\omega)}{2i\omega}+\frac{f(-i\omega)}{-2i\omega}\right)=\mathbf{I}\cdot\frac{f(i\omega)+f(-i\omega)}{2}+\mathbf{D}\cdot\frac{f(i\omega)-f(-i\omega)}{2i\omega}$$

2. $det\mathbf{H}<0;\ \lambda_{1,2}=\pm\sqrt{-det\mathbf{H}};\ \lambda_1\neq\lambda_2$: unstable (exp growth), simple Sylvester formula

$$\lambda_1=-\lambda_2=\omega;\ \omega=\sqrt{-det\mathbf{H}};\ f(\mathbf{D})=\sum_{k=1}^{2n}f(\lambda_k)\cdot\prod_{j\neq k}\frac{\mathbf{D}-\lambda_j\cdot\mathbf{I}}{\lambda_k-\lambda_j}=f(\lambda_1)\cdot\frac{\mathbf{D}-\lambda_2\cdot\mathbf{I}}{\lambda_1-\lambda_2}+f(\lambda_2)\cdot\frac{\mathbf{D}-\lambda_1\cdot\mathbf{I}}{\lambda_2-\lambda_1};$$

$$f(\mathbf{D})=\mathbf{I}\cdot\left(f(\omega)\cdot\frac{\omega}{2\omega}+f(-\omega)\cdot\frac{-\omega}{-2\omega}\right)+\mathbf{D}\cdot\left(\frac{f(\omega)}{2\omega}+\frac{f(-\omega)}{-2\omega}\right)=\mathbf{I}\cdot\frac{f(\omega)+f(-\omega)}{2}+\mathbf{D}\cdot\frac{f(\omega)-f(-\omega)}{2i\omega}$$

3. $det\mathbf{H}=0;\ \lambda_{1,2}=0;\ \lambda_1=\lambda_2$: indifferent, equal eigen vectors, general Sylvester formula with $m=1,\ l_1\leq 2$:

$$\lambda_1=0; m=1,\ l_1\leq 2;\ f(\mathbf{D})=\sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i-1}\frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{D}-\lambda_i\mathbf{I})^j\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right]=\phi_1(0)\cdot\mathbf{I}+\phi_1^{(1)}(0)\cdot(\mathbf{D}-0\cdot\mathbf{I});$$

$$\phi_1(\lambda)=f(\lambda)/\prod_{j\neq 1}(\lambda-\lambda_j)^{l_j};\ \phi_i(0)=f(0);\ \phi_i^{(1)}(0)=\left.\frac{df(\lambda)}{d\lambda}\right|_{\lambda=0_i}=f'(0);\ f(\mathbf{D})==f(0)\cdot\mathbf{I}+f'(0)\cdot\mathbf{D}.$$

With $f[\mathbf{D}]=exp(\mathbf{D}\cdot s)$ we have

$$det\mathbf{H}>0;\ f(\mathbf{D})=\mathbf{I}\cdot cos(\omega\cdot s)+\mathbf{D}\cdot\frac{sin(\omega\cdot s)}{\omega}=\begin{bmatrix} cos(\omega\cdot s)+h_{12}\frac{sin(\omega\cdot s)}{\omega} & h_{22}\frac{sin(\omega\cdot s)}{\omega} \\ -h_{11}\frac{sin(\omega\cdot s)}{\omega} & cos(\omega\cdot s)-h_{12}\frac{sin(\omega\cdot s)}{\omega}\end{bmatrix};\ \omega=\sqrt{det\mathbf{H}}$$

$$det\mathbf{H}<0;\ f(D)=\mathbf{I}\cdot cosh(\omega\cdot s)+\mathbf{D}\cdot\frac{sinh(\omega\cdot s)}{\omega}=\begin{bmatrix} cosh(\omega\cdot s)+h_{12}\frac{sinh(\omega\cdot s)}{\omega} & h_{22}\frac{sinh(\omega\cdot s)}{\omega} \\ -h_{11}\frac{sinh(\omega\cdot s)}{\omega} & cosh(\omega\cdot s)-h_{12}\frac{sinh(\omega\cdot s)}{\omega}\end{bmatrix};\ \omega=\sqrt{-detH}$$

$$det\mathbf{H}=0;\ f(D)=\mathbf{I}\cdot+\mathbf{D}\cdot s=\begin{bmatrix} 1+h_{12}s & h_{22}s \\ -h_{11}s & 1-h_{12}s\end{bmatrix};$$

In case of 1D accelerator Hamiltonians with do not *xp* term and $h_{12}=h_{21}=0$. Furthermore, h_{22} term is never equal zero, hence

$$det\mathbf{H}=h_{11}h_{22}>0;\ f(\mathbf{D})=\mathbf{I}\cdot cos(\omega\cdot s)+\mathbf{D}\cdot\frac{sin(\omega\cdot s)}{\omega}=\begin{bmatrix} cos\left(\sqrt{h_{11}h_{22}}\cdot s\right) & \sqrt{\frac{h_{22}}{h_{11}}}\cdot sin\left(\sqrt{h_{11}h_{22}}\cdot s\right) \\ -\sqrt{\frac{h_{11}}{h_{22}}}\cdot sin\left(\sqrt{h_{11}h_{22}}\cdot s\right) & cos\left(\sqrt{h_{11}h_{22}}\cdot s\right)\end{bmatrix};\ \omega=\sqrt{h_{11}h_{22}}$$

$$det\mathbf{H}=h_{11}h_{22}<0;\ f(\mathbf{D})=\mathbf{I}\cdot cosh(\omega\cdot s)+\mathbf{D}\cdot\frac{sinh(\omega\cdot s)}{\omega}=\begin{bmatrix} cosh\left(\sqrt{-h_{11}h_{22}}\cdot s\right) & \sqrt{-\frac{h_{22}}{h_{11}}}\cdot sinh\left(\sqrt{-h_{11}h_{22}}\cdot s\right) \\ \sqrt{-\frac{h_{11}}{h_{22}}}\cdot sin\left(\sqrt{-h_{11}h_{22}}\cdot s\right) & cosh\left(\sqrt{-h_{11}h_{22}}\cdot s\right)\end{bmatrix};\ \omega=\sqrt{-h_{11}h_{22}}$$

$$det\mathbf{H}=0,\ h_{11}=0;\ f(\mathbf{D})=\mathbf{I}\cdot+\mathbf{D}\cdot s=\begin{bmatrix} 1 & h_{22}\cdot s \\ 0 & 1\end{bmatrix}; \qquad (8.1)$$

It was worse noting that for 1D case, there is no difference between under and over-scored conventions:

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When equations of motion in N-deletional system are decoupled, i.e. the Hamiltonian of the system is a simple sum of the Hamiltonian for each degree of freedom:

$$H = \sum_{k=1}^{N} H_k; \; H_k = \frac{1}{2} X_k^T \cdot \mathbf{H}_k \cdot X_k; \; X_k = \begin{bmatrix} q_k \\ P_k \end{bmatrix}$$

Then matrix for the system in under-scored convection

$$\underline{X} = \begin{bmatrix} X_1 \\ \ldots \\ X_i \\ \ldots \\ X_N \end{bmatrix}$$

is direct diagonal sum of the individual 2x2 matrices for each degree of freedom:

$$\underline{\mathbf{M}} = exp\left(\underline{\mathbf{S}} \cdot \underline{\mathbf{H}} \cdot s \right) = \begin{bmatrix} exp\left(\sigma \cdot \mathbf{H}_1 \cdot s \right) & 0 & \ldots & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots\ldots & exp\left(\sigma \cdot \mathbf{H}_k \cdot s \right) & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 0 & exp\left(\sigma \cdot \mathbf{H}_N \cdot s \right) \end{bmatrix} = \oplus\, exp\left(\sigma \cdot \mathbf{H}_k \cdot s \right) \equiv \oplus\, \mathbf{M}_k \quad (8.2)$$

where we already know how to calculate 2x2 matrices using three equations in (8.1).

But matrix in over-scored convention is not looking anything like direct sum for 2x2 matrices for uncoupled motion: instead, it has four diagonal *NxN* matrices:

$$\overline{\mathbf{M}} = exp\left(\overline{\mathbf{S}} \cdot \overline{\mathbf{H}} \cdot s \right) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}; \; M_{ik} = \begin{bmatrix} \left[\mathbf{M}_1\right]_{ik} & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \left[\mathbf{M}_j\right]_{ik} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & \left[\mathbf{M}_N\right]_{ik} \end{bmatrix}$$

which is easy to prove noticing that in both conventions we have:

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which is easy to prove noticing that in both conventions we have:

$$q_j \Rightarrow [\mathbf{M}_j]_{11} \cdot q_j + [\mathbf{M}_j]_{21} \cdot P_j;\; P_j \Rightarrow [\mathbf{M}_j]_{21} \cdot q_j + [\mathbf{M}_j]_{22} \cdot P_j$$

2D case – coupled motion.

For a general linear 2D Hamiltonian we have possibility 10 coefficients, which in the under-scored look like:

$$H = \frac{1}{2}\underline{X}^T \cdot \underline{\mathbf{H}} \cdot \underline{X};\ \underline{\mathbf{H}} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{12} & h_{22} & h_{23} & h_{24} \\ h_{13} & h_{23} & h_{33} & h_{34} \\ h_{14} & h_{24} & h_{34} & h_{44} \end{bmatrix} \underline{\mathbf{S}} = \begin{bmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{bmatrix};\ \underline{\mathbf{D}} = \underline{\mathbf{S}} \cdot \underline{\mathbf{H}} = \begin{bmatrix} h_{12} & h_{22} & h_{23} & h_{24} \\ -h_{11} & -h_{12} & -h_{13} & -h_{14} \\ h_{14} & h_{24} & h_{34} & h_{44} \\ -h_{13} & -h_{23} & -h_{33} & -h_{34} \end{bmatrix};$$

and in the over-scored cases

$$H = \frac{1}{2}\overline{X^T} \cdot \overline{\mathbf{H}} \cdot \overline{X};\ \overline{\mathbf{H}} = \begin{bmatrix} H_{11} & H_{12} \\ H^T_{12} & H_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix};\ \overline{\mathbf{D}} = \overline{\mathbf{S}} \cdot \overline{\mathbf{H}} = \begin{bmatrix} H^T_{12} & H_{22} \\ -H_{11} & -H_{12} \end{bmatrix};$$

$$H_{11} = \begin{bmatrix} h_{11} & h_{13} \\ h_{13} & h_{33} \end{bmatrix};\ H_{22} = \begin{bmatrix} h_{22} & h_{24} \\ h_{24} & h_{44} \end{bmatrix};\ H_{12} = \begin{bmatrix} h_{12} & h_{14} \\ h_{32} & h_{34} \end{bmatrix}.$$

Because system is Hamiltonian, the eigen values of matrix **D** are paired as *(λ,-λ)* because

$$f(\lambda) = det(\mathbf{D} - \lambda\cdot\mathbf{I}) \equiv det(\mathbf{D} - \lambda\cdot\mathbf{I})^{T} = det(\mathbf{S}\cdot\mathbf{H} - \lambda\cdot\mathbf{I})^{T} = det\left(\mathbf{H}^{T}\cdot\mathbf{S}^{T} - \lambda\cdot\mathbf{I}^{T}\right) = det(-\mathbf{H}\cdot\mathbf{S} - \lambda\cdot\mathbf{I}) =$$
$$(-1)^{2N} det(\mathbf{H}\cdot\mathbf{S} + \lambda\cdot\mathbf{I}) = det(\mathbf{H}\cdot\mathbf{S} + \lambda\cdot\mathbf{I}) = det\left(\mathbf{S}(\mathbf{H}\cdot\mathbf{S} + \lambda\cdot\mathbf{I})\mathbf{S}^{-1}\right) = det(\mathbf{D} + \lambda\cdot\mathbf{I}) = f(-\lambda);$$

$$f(\lambda) = \sum_{k=0}^{2N} a_k\cdot\lambda^k = f(-\lambda) = \sum_{k=0}^{2N} a_k\cdot(-\lambda)^k \Rightarrow a_k = (-1)^k\cdot a_k \Rightarrow a_{2k+1} = 0;\ f(\lambda) = \sum_{k=0}^{N} a_{2k}\cdot\lambda^{2k}.$$

i.e. equation for eigen values is bi-quadratic in order of N – in it is quadratic equation for λ^2 with trivial $a_4 = 1$ and well known but not simple a_0 coefficients:

$$\lambda^4 + a_2\cdot\lambda^2 + a_o = 0;\quad a_o = det\mathbf{D} = det\mathbf{H}$$

Because over- and under-scored **D** matrices are similar, they have identical eigenvectors and it does not matter what form we are using to calculate less trivial coefficient a_2:

$$\underline{\mathbf{D}} - \lambda\cdot\mathbf{I} = \begin{bmatrix} h_{12}-\lambda & h_{22} & h_{23} & h_{24} \\ -h_{11} & -h_{12}-\lambda & -h_{13} & -h_{14} \\ h_{14} & h_{24} & h_{34}-\lambda & h_{44} \\ -h_{13} & -h_{23} & -h_{33} & -h_{34}-\lambda \end{bmatrix};\ \lambda^2{}_{(34)}\begin{vmatrix} h_{12} & h_{22} \\ -h_{11} & -h_{12} \end{vmatrix} + \lambda^2{}_{(13)}\begin{vmatrix} h_{34} & h_{44} \\ -h_{33} & -h_{34} \end{vmatrix}$$

$$-\lambda^2{}_{(13)}\begin{vmatrix} -h_{12} & -h_{14} \\ -h_{23} & -h_{34} \end{vmatrix} + \lambda^2{}_{(14)}\begin{vmatrix} -h_{12} & -h_{13} \\ h_{24} & h_{34} \end{vmatrix} + \lambda^2{}_{(23)}\begin{vmatrix} h_{12} & h_{24} \\ -h_{13} & -h_{34} \end{vmatrix} - \lambda^2{}_{(24)}\begin{vmatrix} h_{12} & h_{23} \\ h_{14} & h_{34} \end{vmatrix};$$

$$a_2 = h_{11}h_{22} - h_{12}^2 + h_{44}h_{33} - h_{34}^2 + 2h_{13}h_{24} - 2h_{14}h_{23}.$$

and we get to

$$\lambda^2 = -\frac{a_2}{2} \pm \sqrt{\left(\frac{a_2}{2}\right)^2 - a_o};$$

Expression for determinant is not short either

$$a_o = h_{14}^2 h_{23}^2 - 2h_{13} h_{14} h_{23} h_{24} + h_{13}^2 h_{24}^2 - h_{14}^2 h_{22} h_{33} + 2h_{12} h_{14} h_{24} h_{33} - h_{11} h_{24}^2 h_{33}$$
$$+ 2h_{13} h_{14} h_{22} h_{34} - 2h_{12} h_{14} h_{23} h_{34} - 2h_{12} h_{13} h_{24} h_{34} + 2h_{11} h_{23} h_{24} h_{34} + h_{12}^2 h_{34}^2$$
$$- h_{11} h_{22} h_{34}^2 - h_{13}^2 h_{22} h_{44} + 2h_{12} h_{13} h_{23} h_{44} - h_{11} h_{23}^2 h_{44} - h_{12}^2 h_{33} h_{44} + h_{11} h_{22} h_{33} h_{44}$$

Fortunately, in most of the case in accelerator problems many of coefficients are zeros:

It takes a bit of effort to calculate. In the most general cases of accelerator Hamiltonian,

$$\tilde{h} = \frac{P_1^2 + P_2^2}{2 \cdot p_o} + F \cdot \frac{x^2}{2} + N \cdot x \cdot y + G \cdot \frac{y^2}{2} + L\left(x \cdot P_y - y \cdot P_x\right) +$$
$$\frac{\delta^2}{2 \cdot p_o} \cdot \left(\frac{mc}{p_o}\right)^2 + U \cdot \frac{\tau^2}{2} + g_x \cdot x \cdot \delta + g_y \cdot y \cdot \delta + F_x \cdot x \cdot \tau + F_y \cdot y \cdot \tau$$

any of 2D Hamiltonians has zero h_{12} , h_{34} and h_{24} components in making expression a little bit more compact:

$$a_o = h_{14}^2 h_{23}^2 + h_{13}^2 h_{24}^2 - h_{14}^2 h_{22} h_{33} - h_{13}^2 h_{22} h_{44} - h_{11} h_{23}^2 h_{44} + h_{11} h_{22} h_{33} h_{44} =$$
$$h_{14}^2 (h_{23}^2 - h_{22} h_{33}) + h_{11} h_{44} (h_{22} h_{33} - h_{23}^2) + h_{13}^2 (h_{24}^2 - h_{22} h_{44})..$$

$$a_2 = h_{11} h_{22} + h_{44} h_{33} - 2h_{14} h_{23}; \;\; a_2(x,y) = \frac{F+G}{p_o} + L^2; \; a_2(x,\tau) = \frac{F}{p_o} + \frac{U}{p_o} \cdot \left(\frac{mc}{p_o}\right)^2.$$

Let's discuss options for two pairs of eigen values $\lambda^2 = -\frac{a_2}{2} \pm \sqrt{\left(\frac{a_2}{2}\right)^2 - a_o}$;

1. $det\mathbf{H} = a_o \neq \left(\frac{a_2}{2}\right)^2$ and $det\mathbf{H} = a_o \neq 0$ - four distinct eigen values

$$\lambda_{1,2} = \pm\sqrt{-\frac{a_2}{2} + \sqrt{\left(\frac{a_2}{2}\right)^2 - a_o}};\ \lambda_{3,4} = \pm\sqrt{-\frac{a_2}{2} - \sqrt{\left(\frac{a_2}{2}\right)^2 - a_o}};\ \frac{\mathbf{D}-\lambda_3\mathbf{I}}{\lambda-\lambda_3}\cdot\frac{\mathbf{D}+\lambda_3\mathbf{I}}{\lambda+\lambda_3} = \frac{\mathbf{D}^2-\lambda_3{}^2\mathbf{I}}{\lambda^2-\lambda_3{}^2};\ \frac{\mathbf{D}-\lambda_2\mathbf{I}}{\lambda_1-\lambda_2} = \frac{\mathbf{D}+\lambda_1\mathbf{I}}{2\lambda_1};$$

$$exp[\mathbf{D}\cdot s] = e^{\lambda_1 s}\cdot\frac{\mathbf{D}+\lambda_1\mathbf{I}}{2\lambda_1}\cdot\frac{\mathbf{D}^2-\lambda_3{}^2\mathbf{I}}{\lambda_1{}^2-\lambda_3{}^2} - e^{-\lambda_1 s}\cdot\frac{\mathbf{D}-\lambda_1\mathbf{I}}{2\lambda_1}\cdot\frac{\mathbf{D}^2-\lambda_3{}^2\mathbf{I}}{\lambda_1{}^2-\lambda_3{}^2} + e^{\lambda_3 s}\cdot\frac{\mathbf{D}+\lambda_3\mathbf{I}}{2\lambda_3}\cdot\frac{\mathbf{D}^2-\lambda_1{}^2\mathbf{I}}{\lambda_3{}^2-\lambda_1{}^2} - e^{-\lambda_3 s}\cdot\frac{\mathbf{D}-\lambda_3\mathbf{I}}{2\lambda_3}\cdot\frac{\mathbf{D}^2-\lambda_3{}^2\mathbf{I}}{\lambda_3{}^2-\lambda_1{}^2} =$$

$$\frac{1}{\lambda_1{}^2-\lambda_3{}^2}\cdot\left(\left(\mathbf{D}\cdot\frac{e^{\lambda_1 s}-e^{-\lambda_1 s}}{2\lambda_1} + \mathbf{I}\cdot\frac{e^{\lambda_1 s}+e^{-\lambda_1 s}}{2}\right)\cdot\left(\mathbf{D}^2-\lambda_3{}^2\mathbf{I}\right) - \left(\mathbf{D}\cdot\frac{e^{\lambda_3 s}-e^{-\lambda_3 s}}{2\lambda_3} + \mathbf{I}\cdot\frac{e^{\lambda_3 s}+e^{-\lambda_3 s}}{2}\right)\cdot\left(\mathbf{D}^2-\lambda_1{}^2\mathbf{I}\right)\right)$$

2. $det\mathbf{H} = a_o = 0;\ a_2 \neq 0$, one degenerated eigen value, two distinct non-zero values

$$\lambda_1 = 0;\ \lambda_{2,3} = \pm\sqrt{-a_2};\ m = 3;\ l_1 = 2; l_{2,3} = 1;$$

$$\lambda_1 = 0;\ \lambda_{2,3} = \pm\lambda_o \neq 0; m = 3,\ l_1 \leq 2; l_2 = 1;\ \prod_{k\neq 1}(\mathbf{D}-\lambda_k I)^{l_k} = (\mathbf{D}-\lambda_0 I)^{l_2}(\mathbf{D}+\lambda_0 I)^{l_3} = (\mathbf{D^2}-\boldsymbol{\lambda_o}^2\cdot\mathbf{I})$$

$$exp(\mathbf{D}s) = \sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i-1}\frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{D}-\lambda_i\mathbf{I})^j\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right] = \phi_1(0)\cdot\mathbf{I}\cdot(\mathbf{D^2}-\boldsymbol{\lambda_o}^2\cdot\mathbf{I}) + \phi_1^{(1)}(0)\cdot(\mathbf{D}-0\cdot\mathbf{I})\cdot(\mathbf{D^2}-\boldsymbol{\lambda_o}^2\cdot\mathbf{I}) +$$

$$\phi_2(\lambda_o)\cdot(\mathbf{D}-0\cdot\mathbf{I})^2\cdot(\mathbf{D}+\lambda_o\cdot\mathbf{I}) + \phi_3(-\lambda_o)\cdot(\mathbf{D}-0\cdot\mathbf{I})^2\cdot(\mathbf{D}-\lambda_o\cdot\mathbf{I}) =$$

$$\phi_1(0)\cdot(\mathbf{D^2}-\boldsymbol{\lambda_o}^2\cdot\mathbf{I}) + \phi_1^{(1)}(0)\cdot\mathbf{D}\cdot(\mathbf{D^2}-\boldsymbol{\lambda_o}^2\cdot\mathbf{I}) + \mathbf{D^2}\cdot\left(\mathbf{D}\cdot(\boldsymbol{\phi_2}(\boldsymbol{\lambda_o}) + \boldsymbol{\phi_3}(-\boldsymbol{\lambda_o})) + \boldsymbol{\lambda_o}\cdot\mathbf{I}\cdot(\boldsymbol{\phi_2}(\boldsymbol{\lambda_o}) - \boldsymbol{\phi_3}(-\boldsymbol{\lambda_o}))\right);$$

$$\phi_1(\lambda) = \frac{e^{\lambda s}}{\lambda^2-\lambda_o^2}; \phi_1(0) = -\frac{1}{\lambda_o^2} \quad \phi_1^{(1)}(\lambda) = s\cdot\frac{e^{\lambda s}}{\lambda^2-\lambda_o^2} - 2s\cdot\lambda\frac{e^{\lambda s}}{(\lambda^2-\lambda_o^2)^2};\ \phi_1^{(1)}(0) = = -\frac{s}{\lambda_o^2};$$

$$\phi_2(\lambda) = \frac{e^{\lambda s}}{(\lambda-\lambda_3)(\lambda-\lambda_1)}; \phi_2(\lambda_o) = \frac{e^{\lambda_o s}}{2\lambda_o^2};\ \phi_3(\lambda) = \frac{e^{\lambda s}}{(\lambda-\lambda_2)(\lambda-\lambda_1)}; \phi_3(-\lambda_o) = \frac{e^{-\lambda_o s}}{-2\lambda_o^2};$$

$$\exp(\mathbf{D}s) = -\frac{\mathbf{D^2}-\boldsymbol{\lambda_o}^2\cdot\mathbf{I}}{\lambda_o^2} - s\cdot\frac{\mathbf{D}\cdot(\mathbf{D^2}-\boldsymbol{\lambda_o}^2\cdot\mathbf{I})}{\lambda_o^2} + \frac{\mathbf{D^2}}{2\lambda_o^2}\left(\mathbf{D}\cdot(e^{\lambda_o s} - e^{-\lambda_o s}) + \lambda_o\cdot\mathbf{I}\cdot(e^{\lambda_o s} + e^{-\lambda_o s})\right)$$

3. $det\mathbf{H} = a_o = \left(\frac{a_2}{2}\right)^2$ we have two pairs of degenerated eigen values

$$\lambda_{1,2} = \pm\lambda_o; \lambda_o = \sqrt{-\frac{a_2}{2}};\ m=2;\ l_{1,2} \le 2;\ exp(\mathbf{D}s) = \sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i-1}\frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{D}-\lambda_i\mathbf{I})^j\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right] =$$

$$\left(\phi_1(\lambda_1)\cdot\mathbf{I} + \phi_1^{(1)}(\lambda_1)(\mathbf{D}-\lambda_1\mathbf{I})\right)\cdot(\mathbf{D}-\lambda_2 I)^2 + \left(\phi_2(\lambda_2)\cdot\mathbf{I} + \phi_2^{(1)}(\lambda_2)(\mathbf{D}-\lambda_2\mathbf{I})\right)\cdot(\mathbf{D}-\lambda_1 I)^2;.$$

$$\phi_1(\lambda) = \frac{e^{\lambda s}}{(\lambda-\lambda_2)^2};\ \frac{d\phi_1(\lambda)}{d\lambda} = s\frac{e^{\lambda s}}{(\lambda-\lambda_2)^2} - 2\frac{e^{\lambda s}}{(\lambda-\lambda_2)^3}; \phi_2(\lambda) = \frac{e^{\lambda s}}{(\lambda-\lambda_1)^2};\ \frac{d\phi_2(\lambda)}{d\lambda} = s\frac{e^{\lambda s}}{(\lambda-\lambda_1)^2} - 2\frac{e^{\lambda s}}{(\lambda-\lambda_1)^3};$$

$$\phi_1(\lambda_1) = \frac{e^{\lambda_o s}}{4\lambda_o^2};\ \phi_1^{(1)}(\lambda_1) = \frac{e^{\lambda_o s}}{4\lambda_o^3}(\lambda_o s - 1); \phi_2(\lambda_2) = \frac{e^{-\lambda_o s}}{4\lambda_o^2};\ \phi_1^{(1)}(\lambda_1) = \frac{e^{-\lambda_o s}}{4\lambda_o^3}(\lambda_o s + 1);$$

$$exp(\mathbf{D}s) = \frac{e^{\lambda_o s}}{4\lambda_o^3}\left(\lambda_o\cdot\mathbf{I} + (\lambda_o s - 1)(\mathbf{D}-\lambda_o\mathbf{I})\right)\cdot(\mathbf{D}+\lambda_o\mathbf{I})^2 + \frac{e^{-\lambda_o s}}{4\lambda_o^3}\left(\lambda_o\cdot\mathbf{I} + (\lambda_o s + 1)(\mathbf{D}+\lambda_o\mathbf{I})\right)\cdot(\mathbf{D}-\lambda_o\mathbf{I})^2$$

Collecting the terms:

$$exp(\mathbf{D}s) = \frac{e^{\lambda_o s}}{4\lambda_o^3}\left(\lambda_o\cdot\mathbf{I} + (\lambda_o s - 1)(\mathbf{D}-\lambda_o\mathbf{I})\right)\cdot(\mathbf{D}+\lambda_o\mathbf{I})^2 + \frac{e^{-\lambda_o s}}{4\lambda_o^3}\left(\lambda_o\cdot\mathbf{I} + (\lambda_o s + 1)(\mathbf{D}+\lambda_o\mathbf{I})\right)\cdot(\mathbf{D}-\lambda_o\mathbf{I})^2 =$$

$$= \frac{e^{\lambda_o s}}{4}\left(\mathbf{I}\cdot(2-\lambda_o s) + \mathbf{D}\cdot\frac{3-\lambda_o s}{\lambda_o} + \mathbf{D}^2\cdot\frac{\mathbf{s}}{\boldsymbol{\lambda}_\mathbf{o}} + \mathbf{D}^3\cdot\frac{\lambda_o s - 1}{\lambda_o^3}\right) + \frac{e^{-\lambda_o s}}{4}\left(\mathbf{I}\cdot(2+\lambda_o s) - \mathbf{D}\cdot\frac{3+\lambda_o s}{\lambda_o} - \mathbf{D}^2\cdot\frac{s}{\lambda_o} + \mathbf{D}^3\cdot\frac{\lambda_o s + 1}{\lambda_o^3}\right);$$

$$exp(\mathbf{D}s) = \frac{e^{\lambda_o s} + e^{-\lambda_o s}}{2}\cdot\left(2\cdot\mathbf{I} - \mathbf{D}\cdot s + \mathbf{D}^3\cdot\frac{s}{\lambda_o^2}\right) + \frac{e^{\lambda_o s} - e^{-\lambda_o s}}{2}\cdot\left(-\mathbf{I}\cdot\lambda_o s + \frac{3\mathbf{D}}{\lambda_o} + \mathbf{D}^2\cdot\frac{s}{\lambda_o} - \mathbf{D}^3\right)$$

1. Fully degenerated case: $\lambda_1 = 0;\ m = 1;\ l_1 = 4$ is very easy:

$$\lambda_1 = 0;\ m = 1;\ l_1 \le 4;\ \ exp(\mathbf{D}\cdot \mathrm{s}) = \sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i - 1} \frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{D}-\lambda_i \mathbf{I})^j\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right] =$$

$$\phi_1^{(0)}(0)\cdot \mathbf{I} + \frac{\phi_1^{(1)}(0)}{1!}\cdot(\mathbf{D}-0\cdot\mathbf{I}) + \frac{\phi_1^{(2)}(0)}{2!}\cdot(\mathbf{D}-0\cdot\mathbf{I})^2 + \frac{\phi_1^{(3)}(0)}{3!}\cdot(\mathbf{D}-0\cdot\mathbf{I})^3;$$

$$\phi_1(\lambda) = e^{\lambda\cdot s} \to \phi_1^{(k)}(\lambda) = s^k \cdot e^{\lambda\cdot s} \Rightarrow exp(\mathbf{D}\mathrm{s}) = \mathbf{I} + \mathbf{D}\cdot\mathbf{s} + \frac{(\mathbf{D}\cdot \mathrm{s})^2}{2!} + \frac{(\mathbf{D}\cdot \mathrm{s})^3}{3!}.$$

The later result can be anticipated because all eigne values are zero with maximum Jordan block in **D** of 4. and

$$\mathbf{D} = \mathbf{U}\cdot\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\cdot \mathbf{U}^{-1};\ \mathbf{D}^4 = \mathbf{0};\ exp[\mathbf{D}\cdot s] = \mathbf{I} + \sum_{k=0}^{\infty}\mathbf{D}^k\cdot\frac{s^k}{k!} = \mathbf{I} + \mathbf{D}\cdot s + \mathbf{D}^2\cdot\frac{s^2}{2!} + \mathbf{D}^k\cdot\frac{s^3}{3!}.$$

If **D** with all zero eigen values has Jordan maximum with size *m*, then $\mathbf{D}^{m}=\mathbf{0}$ and then there is *m* terms in the Taylor expansion.

3D case – coupled motion. A generic symmetric Hamiltonian 6x6 matrix has 21 coefficients and there are simply too many sixth order products of these 21 coefficients in the expression for characteristic equation:

$$f(\lambda) = det(\mathbf{D} - \lambda \cdot \mathbf{I}) = det(\mathbf{S} \cdot \mathbf{H} - \lambda \cdot \mathbf{I}) = \lambda^6 + a_4\lambda^4 + a_2\lambda^2 + det\mathbf{H} = 0 \tag{9.1}$$

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{-h16^2 h25^2 h34^2 + 2 h15 h16 h25 h26 h34^2 - h15^2 h26^2 h34^2 + 2 h16^2 h24 h25 h34 h35 - 2 h15 h16 h24 h26 h34 h35 - 2 h14 h16 h25 h26 h34 h35 + 2 h14 h15 h26^2 h34 h35 - h16^2 h24^2 h35^2 + 2 h14 h16 h24 h26 h35^2 - h14^2 h26^2 h35^2 - 2 h15 h16 h24 h25 h34 h36 + 2 h14 h16 h25^2 h34 h36 +
 2 h15^2 h24 h26 h34 h36 - 2 h14 h15 h25 h26 h34 h36 + 2 h15 h16 h24^2 h35 h36 - 2 h14 h16 h24 h25 h35 h36 - 2 h14 h15 h24 h26 h35 h36 + 2 h14^2 h25 h26 h35 h36 - h15^2 h24^2 h36^2 + 2 h14 h15 h24 h25 h36^2 - h14^2 h25^2 h36^2 + h16^2 h25^2 h33 h44 - 2 h15 h16 h25 h26 h33 h44 +
 h15^2 h26^2 h33 h44 - 2 h16^2 h23 h25 h35 h44 + 2 h15 h16 h23 h26 h35 h44 + 2 h13 h16 h25 h26 h35 h44 - 2 h13 h15 h26^2 h35 h44 + h16^2 h22 h35^2 h44 - 2 h12 h16 h26 h35^2 h44 + h11 h26^2 h35^2 h44 + 2 h15 h16 h23 h25 h36 h44 - 2 h13 h16 h25^2 h36 h44 -
 2 h15^2 h23 h26 h36 h44 + 2 h13 h15 h25 h26 h36 h44 - 2 h15 h16 h22 h35 h36 h44 + 2 h12 h16 h25 h35 h36 h44 + 2 h12 h15 h26 h35 h36 h44 - 2 h11 h25 h26 h35 h36 h44 + h15^2 h22 h36^2 h44 - 2 h12 h15 h25 h36^2 h44 + h11 h25^2 h36^2 h44 - 2 h16^2 h24 h25 h33 h45 +
 2 h15 h16 h24 h26 h33 h45 + 2 h14 h16 h25 h26 h33 h45 - 2 h14 h15 h26^2 h33 h45 + 2 h16^2 h23 h25 h34 h45 - 2 h15 h16 h23 h26 h34 h45 - 2 h13 h16 h25 h26 h34 h45 + 2 h13 h15 h26^2 h34 h45 + 2 h16^2 h23 h24 h35 h45 - 2 h14 h16 h23 h26 h35 h45 -
 2 h13 h16 h24 h26 h35 h45 + 2 h13 h14 h26^2 h35 h45 - 2 h16^2 h22 h34 h35 h45 + 4 h12 h16 h26 h34 h35 h45 - 2 h11 h26^2 h34 h35 h45 - 2 h15 h16 h23 h24 h36 h45 - 2 h14 h16 h23 h25 h36 h45 + 4 h13 h16 h24 h25 h36 h45 + 4 h14 h15 h23 h26 h36 h45 -
 2 h13 h15 h24 h26 h36 h45 - 2 h13 h14 h25 h26 h36 h45 + 2 h15 h16 h22 h34 h36 h45 - 2 h12 h16 h25 h34 h36 h45 - 2 h12 h15 h26 h34 h36 h45 + 2 h11 h25 h26 h34 h36 h45 + 2 h14 h16 h22 h35 h36 h45 - 2 h12 h16 h24 h35 h36 h45 - 2 h12 h14 h26 h35 h36 h45 +
 2 h11 h24 h26 h35 h36 h45 - 2 h14 h15 h22 h36^2 h45 + 2 h12 h15 h24 h36^2 h45 + 2 h12 h14 h25 h36^2 h45 - 2 h11 h24 h25 h36^2 h45 - h16^2 h23^2 h45^2 + 2 h13 h16 h23 h26 h45^2 - h13^2 h26^2 h45^2 + h16^2 h22 h33 h45^2 - 2 h12 h16 h26 h33 h45^2 + h11 h26^2 h33 h45^2 -
 2 h13 h16 h22 h36 h45^2 + 2 h12 h16 h23 h36 h45^2 + 2 h12 h13 h26 h36 h45^2 - 2 h11 h23 h26 h36 h45^2 - h12^2 h36^2 h45^2 + h11 h22 h36^2 h45^2 + 2 h15 h16 h24 h25 h33 h46 - 2 h14 h16 h25^2 h33 h46 - 2 h15^2 h24 h26 h33 h46 + 2 h14 h15 h25 h26 h33 h46 -
 2 h15 h16 h23 h25 h34 h46 + 2 h13 h16 h25^2 h34 h46 + 2 h15^2 h23 h26 h34 h46 - 2 h13 h15 h25 h26 h34 h46 - 2 h15 h16 h23 h24 h35 h46 + 4 h14 h16 h23 h25 h35 h46 - 2 h13 h16 h24 h25 h35 h46 - 2 h14 h15 h23 h26 h35 h46 + 4 h13 h15 h24 h26 h35 h46 -
 2 h13 h14 h25 h26 h35 h46 + 2 h15 h16 h22 h34 h35 h46 - 2 h12 h16 h25 h34 h35 h46 - 2 h12 h15 h26 h34 h35 h46 + 2 h11 h25 h26 h34 h35 h46 - 2 h14 h16 h22 h35^2 h46 + 2 h12 h16 h24 h35^2 h46 + 2 h12 h14 h26 h35^2 h46 - 2 h11 h24 h26 h35^2 h46 +
 2 h15^2 h23 h24 h36 h46 - 2 h14 h15 h23 h25 h36 h46 - 2 h13 h15 h24 h25 h36 h46 + 2 h13 h14 h25^2 h36 h46 - 2 h15^2 h22 h34 h36 h46 + 4 h12 h15 h25 h34 h36 h46 - 2 h11 h25^2 h34 h36 h46 + 2 h14 h15 h22 h35 h36 h46 - 2 h12 h15 h24 h35 h36 h46 -
 2 h12 h14 h25 h35 h36 h46 + 2 h11 h24 h25 h35 h36 h46 + 2 h15 h16 h23^2 h45 h46 - 2 h13 h16 h23 h25 h45 h46 - 2 h13 h15 h23 h26 h45 h46 + 2 h13^2 h25 h26 h45 h46 - 2 h15 h16 h22 h33 h45 h46 + 2 h12 h16 h25 h33 h45 h46 + 2 h12 h15 h26 h33 h45 h46 -
 2 h11 h25 h26 h33 h45 h46 + 2 h13 h16 h22 h35 h45 h46 - 2 h12 h16 h23 h35 h45 h46 - 2 h12 h13 h26 h35 h45 h46 + 2 h11 h23 h26 h35 h45 h46 + 2 h13 h15 h22 h36 h45 h46 - 2 h12 h15 h23 h36 h45 h46 - 2 h12 h13 h25 h36 h45 h46 + 2 h11 h23 h25 h36 h45 h46 +
 2 h12^2 h35 h36 h45 h46 - 2 h11 h22 h35 h36 h45 h46 - h15^2 h23^2 h46^2 + 2 h13 h15 h23 h25 h46^2 - h13^2 h25^2 h46^2 + h15^2 h22 h33 h46^2 - 2 h12 h15 h25 h33 h46^2 + h11 h25^2 h33 h46^2 - 2 h13 h15 h22 h35 h46^2 + 2 h12 h15 h23 h35 h46^2 + 2 h12 h13 h25 h35 h46^2 -
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 2 h13^2 h24 h26 h46 h55 + 2 h14 h16 h22 h33 h46 h55 - 2 h12 h16 h24 h33 h46 h55 - 2 h12 h14 h26 h33 h46 h55 + 2 h11 h24 h26 h33 h46 h55 - 2 h13 h16 h22 h34 h46 h55 + 2 h12 h16 h23 h34 h46 h55 + 2 h12 h13 h26 h34 h46 h55 - 2 h11 h23 h26 h34 h46 h55 -
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 2 h13 h15 h24 h26 h34 h56 + 4 h13 h14 h25 h26 h34 h56 - 2 h15 h16 h22 h34^2 h56 + 2 h12 h16 h25 h34^2 h56 + 2 h12 h15 h26 h34^2 h56 - 2 h11 h25 h26 h34^2 h56 - 2 h14 h16 h23 h24 h35 h56 + 2 h13 h16 h24^2 h35 h56 + 2 h14^2 h23 h26 h35 h56 - 2 h13 h14 h24 h26 h35 h56 +
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 2 h12 h15 h24 h34 h36 h56 - 2 h12 h14 h25 h34 h36 h56 + 2 h11 h24 h25 h34 h36 h56 - 2 h14^2 h22 h35 h36 h56 + 4 h12 h14 h24 h35 h36 h56 - 2 h11 h24^2 h35 h36 h56 - 2 h15 h16 h23^2 h44 h56 + 2 h13 h16 h23 h25 h44 h56 + 2 h13 h15 h23 h26 h44 h56 -
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0, h14^2 h23^2 - 2 h13 h14 h23 h24 + h13^2 h24^2 + 2 h14 h16 h23 h25 - 2 h13 h16 h24 h25 + h16^2 h25^2 - 2 h14 h15 h23 h26 + 2 h13 h15 h24 h26 - 2 h15 h16 h25 h26 + h15^2 h26^2 - h14^2 h22 h33 + 2 h12 h14 h24 h33 - h11 h24^2 h33 + 2 h13 h14 h22 h34 -
 2 h12 h14 h23 h34 - 2 h12 h13 h24 h34 + 2 h11 h23 h24 h34 + h12^2 h34^2 - h11 h22 h34^2 + 2 h16 h25 h34^2 - 2 h15 h26 h34^2 - 2 h14 h16 h22 h35 + 2 h12 h16 h24 h35 + 2 h12 h14 h26 h35 - 2 h11 h24 h26 h35 - 2 h16 h24 h34 h35 + 2 h14 h26 h34 h35 + 2 h14 h15 h22 h36 -
 2 h12 h15 h24 h36 - 2 h12 h14 h25 h36 + 2 h11 h24 h25 h36 + 2 h15 h24 h34 h36 - 2 h14 h25 h34 h36 - h13^2 h22 h44 + 2 h12 h13 h23 h44 - h11 h23^2 h44 - h12^2 h33 h44 + h11 h22 h33 h44 - 2 h16 h25 h33 h44 + 2 h15 h26 h33 h44 + 2 h16 h23 h35 h44 -
 2 h13 h26 h35 h44 - 2 h15 h23 h36 h44 + 2 h13 h25 h36 h44 + 2 h13 h16 h22 h45 - 2 h12 h16 h23 h45 - 2 h12 h13 h26 h45 + 2 h11 h23 h26 h45 + 2 h16 h24 h33 h45 - 2 h14 h26 h33 h45 - 2 h16 h23 h34 h45 + 2 h13 h26 h34 h45 + 2 h12^2 h36 h45 - 2 h11 h22 h36 h45 +
 2 h14 h23 h36 h45 - 2 h13 h24 h36 h45 + 2 h16 h25 h36 h45 - 2 h15 h26 h36 h45 + h36^2 h45^2 - 2 h13 h15 h22 h46 + 2 h12 h15 h23 h46 + 2 h12 h13 h25 h46 - 2 h11 h23 h25 h46 - 2 h15 h24 h33 h46 + 2 h14 h25 h33 h46 + 2 h15 h23 h34 h46 - 2 h13 h25 h34 h46 -
 2 h12^2 h35 h46 + 2 h11 h22 h35 h46 - 2 h14 h23 h35 h46 + 2 h13 h24 h35 h46 - 2 h16 h25 h35 h46 + 2 h15 h26 h35 h46 - 2 h35 h36 h45 h46 + h35^2 h46^2 - h16^2 h22 h55 + 2 h12 h16 h26 h55 - h11 h26^2 h55 - 2 h16 h24 h36 h55 + 2 h14 h26 h36 h55 - h36^2 h44 h55 +
 2 h16 h23 h46 h55 - 2 h13 h26 h46 h55 + 2 h34 h36 h46 h55 - h33 h46^2 h55 + 2 h15 h16 h22 h56 - 2 h12 h16 h25 h56 - 2 h12 h15 h26 h56 + 2 h11 h25 h26 h56 + 2 h16 h24 h35 h56 - 2 h14 h26 h35 h56 + 2 h15 h24 h36 h56 - 2 h14 h25 h36 h56 + 2 h35 h36 h44 h56 -
 2 h16 h23 h45 h56 + 2 h13 h26 h45 h56 - 2 h34 h36 h45 h56 - 2 h15 h23 h46 h56 + 2 h13 h25 h46 h56 - 2 h34 h35 h46 h56 + 2 h33 h45 h46 h56 + h12^2 h56^2 - h11 h22 h56^2 + 2 h14 h23 h56^2 - 2 h13 h24 h56^2 + h34^2 h56^2 - h33 h44 h56^2 - h15^2 h22 h66 + 2 h12 h15 h25 h66 -
 h11 h25^2 h66 - 2 h15 h24 h35 h66 + 2 h14 h25 h35 h66 - h35^2 h44 h66 + 2 h15 h23 h45 h66 - 2 h13 h25 h45 h66 + 2 h34 h35 h45 h66 - h33 h45^2 h66 - h12^2 h55 h66 + h11 h22 h55 h66 - 2 h14 h23 h55 h66 + 2 h13 h24 h55 h66 - h34^2 h55 h66 + h33 h44 h55 h66,
0, -h12^2 + h11 h22 - 2 h14 h23 + 2 h13 h24 - 2 h16 h25 + 2 h15 h26 - h34^2 + h33 h44 - 2 h36 h45 + 2 h35 h46 - h56^2 + h55 h66, 0, 1}
```

that it is nowadays is for AI to find any combinations except 1 and $a_o = det\mathbf{H}$, which make sense. Hence, we can assume that a_o, a_2, a_4 are arbitrary real numbers.

In the case of our form of accelerator Hamiltonian, we have 11, generally speaking, independent coefficients (i.e. two, h_{22} and h_{44}, are identical, two are connected. $h_{14}=-h_{23}$, and 8 are simply zeros. Even in this case coefficients are still rather long:

$$a_0 = \det H = h_{16}^2 h_{22}^2 h_{35}^2 - 2h_{15} h_{16} h_{22}^2 h_{35} h_{36} + h_{15}^2 h_{22}^2 h_{36}^2 + h_{14}^2 h_{16}^2 h_{22} h_{55} - h_{16}^2 h_{22}^2 h_{33} h_{55} + 2h_{13} h_{16} h_{22}^2 h_{36} h_{55} +$$
$$h_{14}^2 h_{22} h_{36}^2 h_{55} - h_{11} h_{22}^2 h_{36}^2 h_{55} + h_{14}^2 h_{15}^2 h_{22} h_{66} - h_{15}^2 h_{22}^2 h_{33} h_{66} + 2h_{13} h_{15} h_{22}^2 h_{35} h_{66} + h_{14}^2 h_{22} h_{35}^2 h_{66} - h_{11} h_{22}^2 h_{35}^2 h_{66} +$$
$$h_{14}^4 h_{55} h_{66} - h_{11} h_{14}^2 h_{22} h_{55} h_{66} - h_{13}^2 h_{22}^2 h_{55} h_{66} - h_{14}^2 h_{22} h_{33} h_{55} h_{66} + h_{11} h_{22}^2 h_{33} h_{55} h_{66};$$
$$a_2 = h_{14}^4 - h_{11} h_{14}^2 h_{22} - h_{13}^2 h22^2 - h_{14}^2 h_{22} h_{33} + h_{11} h_{22}^2 h_{33} - 4h_{14} h_{16} h_{22} h_{35} + 4h_{14} h_{15} h_{22} h_{36} -$$
$$h_{16}^2 h_{22} h_{55} - h_{22} h_{36}^2 h_{55} - h_{15}^2 h_{22} h_{66} - h_{22} h_{35}^2 h_{66} + 2h_{14}^2 h_{55} h_{66} + h_{11} h_{22};$$
$$a_4 = 2h_{14}^2 + h_{22}\left(h_{11} + h_{33}\right) + h_{55} h_{66};\ a_6 = 1.$$

With exception of a_6 and a_4 the expressions are still too generic to make assumptions and conclusions about their values. Equation for eigen values can be solved analytically:

$$\lambda_1^2 = -\frac{1}{6}\left(2a_4 + \frac{2^{4/3}\left(3a_2 - a_4^2\right)}{q} - 2^{2/3} q\right);$$

$$\lambda_{2,3}^2 = -\frac{1}{6}\left(2a_4 - \frac{2^{1/3}\left(1 \pm i\sqrt{3}\right)\left(3a_2 - a_4^2\right)}{q} \mp i \cdot 2^{-1/3}\left(i + \sqrt{3}\right) q\right);$$

$$q = \left(3\sqrt{3}\sqrt{27a_0^2 + 4a_2^3 - 18a_0 a_2 a_4 - a_2^2 a_4^2 + 4a_0 a_4^3} - 27a_0 + 9a_2 a_4 - 2a_4^3\right)^{1/3}.$$

https://en.wikipedia.org/wiki/Cubic_function

Number or real and complex λ_k^2 is determined by the discriminant of the cubic equation

$$\Delta = 18a_o a_2 a_4 - 4a_o a_4^3 + a_2^2 a_4^2 - 4a_2^3 - 27a_o^2$$

Two other combinations play important role in defining branches of roots of cubic equation:

$$\Delta_o = a_4^2 - 3a_2;\ \Delta_1 = 2a_4^3 - 9a_4 a_2 + 27a_0.$$

If $\Delta > 0$, then the cubit equation has three distinct real roots:

This corresponds to six distinct eigen values comprised of pairs $\lambda_k, -\lambda_k$ when none of λ_k^2 is zero. λ_k is real or purely imaginary depending on the sign of λ_k^2.

If $\Delta = 0$ then the cubic equation has a multiple root and all of its roots are real.

If $\Delta_o = 0$ than all λ_k^2 *are identical*

$$\lambda_k^2 = -\frac{a_4}{3}, k = 1,2,3$$

with sign of a_4 defining if λ_k is real or purely imaginary.

When $a_4 \neq 0$ the level of degeneration (maximum height of the eigen vectors) is 3.

When $a_4 = 0$, all eigen vectors are zero and level of denervation can be 6. But requirement of $a_4 = 0$, $\Delta_o = 0$ and $\Delta = 0$ mean that $a_2 = 0$ and $a_0 = 0$, e.g. the characteristic equation is

$$p(\lambda) = \det[\mathbf{D} - \lambda\mathbf{I}] = \lambda^6; \rightarrow p(\mathbf{D}) = \mathbf{D}^6 = 0 \rightarrow \exp(\mathbf{D}s) = \mathbf{I} + \sum_{n=1}^{5} \frac{\mathbf{D}^n s^n}{n!}$$

If $\Delta_o \neq 0$ than all λ_k^2 *are identical there is a double root*

$$\lambda_k^2 = \frac{9a_o - a_2 a_4}{2\Delta_o}; k = 1,2$$

and one unique

$$\lambda_3^2 = \frac{4a_2 a_4 - 9a_o - a_4^2}{\Delta_o}$$

and we have degeneration of at least second of second order.

If $\Delta < 0$, then the cubic equation has one real root and two non-real complex conjugate roots. Generally, this would correspond to a non-degenerated case.

Again, we know that we have 3 pairs of eigen valises $(\lambda_k, -\lambda_k)$, $k=1,2,3$ and the simples case is when all three pairs are distinct (i.e. they are also non-zero, $a_o = det\mathbf{H} \neq 0$):

1. $det\mathbf{H} = a_o \neq 0$ - six distinct eigen values $(\lambda_k, -\lambda_k)$, $k=1,2,3$

$$exp[\mathbf{D}\cdot s] = \left(\mathbf{I}\frac{e^{\lambda_1 s} + e^{-\lambda_1 s}}{2} + \mathbf{D}\frac{e^{\lambda_1 s} - e^{-\lambda_1 s}}{2\lambda_1}\right)\cdot\frac{\mathbf{D}^2 - \lambda_2^2\mathbf{I}}{\lambda_1^2 - \lambda_2^2}\cdot\frac{\mathbf{D}^2 - \lambda_3^2\mathbf{I}}{\lambda_1^2 - \lambda_3^2} +$$

$$\left(\mathbf{I}\frac{e^{\lambda_2 s} + e^{-\lambda_2 s}}{2} + \mathbf{D}\frac{e^{\lambda_2 s} - e^{-\lambda_2 s}}{2\lambda_1}\right)\cdot\frac{\mathbf{D}^2 - \lambda_1^2\mathbf{I}}{\lambda_2^2 - \lambda_1^2}\cdot\frac{\mathbf{D}^2 - \lambda_3^2\mathbf{I}}{\lambda_2^2 - \lambda_3^2} +$$

$$\left(\mathbf{I}\frac{e^{\lambda_3 s} + e^{-\lambda_3 s}}{2} + \mathbf{D}\frac{e^{\lambda_3 s} - e^{-\lambda_3 s}}{2\lambda_1}\right)\cdot\frac{\mathbf{D}^2 - \lambda_1^2\mathbf{I}}{\lambda_3^2 - \lambda_1^2}\cdot\frac{\mathbf{D}^2 - \lambda_2^2\mathbf{I}}{\lambda_3^2 - \lambda_2^2}$$

2. $det\mathbf{H} = a_o \neq 0$ - there can be only 6, 4 or 2 distinct eigen values. In case of odd number of distinct eigen values, even number of them are non-zero pairs, with odd eigen value equal to its partner with opposite sign – i.e. it is zero requiring zero determinant

Let's first consider case when $\lambda_1=\lambda_2 \neq \lambda_3$:

$$\lambda_{1,2} = \pm\lambda_o;\ m=4;\ l_{1,2} \leq 2;\ \phi_3(\lambda_3) = \frac{e^{\lambda s}}{(\lambda-\lambda_o)^2(\lambda+\lambda_o)^2}$$

$$exp(\mathbf{D}s) = \sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i-1}\frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{D}-\lambda_i\mathbf{I})^j\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right] = e^{\lambda_3 s}\frac{(\mathbf{D}-\lambda_o I)^2\cdot(\mathbf{D}+\lambda_o I)^2}{(\lambda_3-\lambda_o)^2(\lambda_3+\lambda_o)^2} +$$

$$\left(\phi_1(\lambda_o)\cdot\mathbf{I}+\phi_1^{(1)}(\lambda_o)(\mathbf{D}-\lambda_o\mathbf{I})\right)\cdot(\mathbf{D}+\lambda_o I)^2\cdot\left(\mathbf{D}^2-\lambda_3{}^2\mathbf{I}\right)+\left(\phi_2(-\lambda_o)\cdot\mathbf{I}+\phi_2^{(1)}(-\lambda_o)(\mathbf{D}+\lambda_o\mathbf{I})\right)\cdot(\mathbf{D}-\lambda_o\mathbf{I})^2\cdot\left(\mathbf{D}^2-\lambda_3{}^2 I\right)+;$$

$$\phi_{1,2}(\lambda) = \frac{e^{\lambda s}}{(\lambda^2-\lambda_3{}^2)(\lambda\pm\lambda_o)^2};\ \frac{d\phi_{1,2}(\lambda)}{d\lambda} = s\frac{e^{\lambda s}}{(\lambda^2-\lambda_3{}^2)(\lambda\pm\lambda_o)^2} - \frac{2e^{\lambda s}}{(\lambda^2-\lambda_3{}^2)(\lambda\pm\lambda_o)^3} - \frac{2\lambda\cdot e^{\lambda s}}{(\lambda^2-\lambda_3{}^2)(\lambda\pm\lambda_o)^3};$$

$$\phi_{1,2}(\pm\lambda_o) = \frac{e^{\pm\lambda_o s}}{(\lambda^2-\lambda_3{}^2)(2\lambda_o)^2};\ \frac{d\phi_{1,2}(\pm\lambda_o)}{d\lambda} = \frac{e^{\pm\lambda_o s}}{(2\lambda_o)^2(\lambda^2-\lambda_3{}^2)}\left(s\mp\frac{1}{\lambda_o}\mp\frac{2\lambda_o\cdot e^{\pm\lambda_o s}}{(\lambda_o{}^2-\lambda_3{}^2)}\right)$$

Now got to two distinct eigen values:

$$\lambda_{1,2} = \pm\lambda_o;\ m=2;\ l_{1,2}\leq 3;$$

$$exp(\mathbf{D}s) = \sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i-1}\frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{D}-\lambda_i\mathbf{I})^j\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right] =$$

$$\left(\phi_1(\lambda_o)\cdot\mathbf{I}+\phi_1^{(1)}(\lambda_o)(\mathbf{D}-\lambda_o\mathbf{I})+\frac{\phi_1^{(2)}(\lambda_o)(\mathbf{D}-\lambda_o\mathbf{I})^2}{2}\right)\cdot(\mathbf{D}+\lambda_o I)^3+\left(\phi_2(-\lambda_o)\cdot\mathbf{I}+\phi_2^{(1)}(-\lambda_o)(\mathbf{D}+\lambda_o\mathbf{I})+\frac{\phi_1^{(2)}(\lambda_o)(\mathbf{D}+\lambda_o\mathbf{I})^2}{2}\right)\cdot(\mathbf{D}-\lambda_o\mathbf{I})^3;$$

$$\phi_{1,2}(\lambda) = \frac{e^{\lambda s}}{(\lambda\pm\lambda_o)^3};\ \frac{d\phi_{1,2}(\lambda)}{d\lambda} = s\frac{e^{\lambda s}}{(\lambda\pm\lambda_o)^3} - \frac{3e^{\lambda s}}{(\lambda\pm\lambda_o)^4};\ \frac{d^2\phi_{1,2}(\lambda)}{d\lambda^2} = s^2\frac{e^{\lambda s}}{(\lambda\pm\lambda_o)^3} - \frac{6s\cdot e^{\lambda s}}{(\lambda\pm\lambda_o)^4} + \frac{12e^{\lambda s}}{(\lambda\pm\lambda_o)^5}$$

$$\phi_{1,2}(\pm\lambda_o) = \pm\frac{e^{\pm\lambda_o s}}{(2\lambda_o)^3};\ \phi_1^{(1)}(\pm\lambda_o) = \pm s\frac{e^{\pm\lambda_o s}}{(2\lambda_o)^3} - \frac{3e^{\pm\lambda_o s}}{(2\lambda_o)^4};\ \phi_1^{(2)}(\pm\lambda_o) = \pm s^2\frac{e^{\pm\lambda_o s}}{(\lambda\pm\lambda_o)^3} - \frac{6s\cdot e^{\pm\lambda_o s}}{(\lambda\pm\lambda_o)^4} \pm \frac{12e^{\pm\lambda_o s}}{(\lambda\pm\lambda_o)^5}$$

3. $det\mathbf{H} = a_o = 0$ - means that at least one pair of eigen values is zero we need to consider cases of 5, 3 and 1 distinct values. Start from easy case of one, when all eigen values are zero

We can do the same process as we did for

$$\lambda_1 = 0;\ m = 1;\ l_1 \le 6;\ \ exp(\mathbf{D}\cdot s) = \sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i-1}\frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{D}-\lambda_i\mathbf{I})^j\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right] =$$

$$\phi_1^{(0)}(0)\cdot\mathbf{I} + \frac{\phi_1^{(1)}(0)}{1!}\cdot(\mathbf{D}-0\cdot\mathbf{I}) + \frac{\phi_1^{(2)}(0)}{2!}\cdot(\mathbf{D}-0\cdot\mathbf{I})^2 + \frac{\phi_1^{(3)}(0)}{3!}\cdot(\mathbf{D}-0\cdot\mathbf{I})^3 + \frac{\phi_1^{(4)}(0)}{4!}\cdot(\mathbf{D}-0\cdot\mathbf{I})^5 + \frac{\phi_1^{(5)}(0)}{5!}\cdot(\mathbf{D}-0\cdot\mathbf{I})^5;$$

$$\phi_1(\lambda) = e^{\lambda\cdot s} \rightarrow \phi_1^{(k)}(\lambda) = s^k\cdot e^{\lambda\cdot s} \Rightarrow exp(\mathbf{D}s) = \mathbf{I} + \mathbf{D}\cdot\mathbf{s} + \frac{(\mathbf{D}\cdot s)^2}{2!} + \frac{(\mathbf{D}\cdot s)^3}{3!} + \frac{(\mathbf{D}\cdot s)^4}{4!} + \frac{(\mathbf{D}\cdot s)^5}{3!}$$

or remember that $\mathbf{D}^{l_o} = \mathbf{0};\ exp\left[\mathbf{D}\cdot s\right] = \mathbf{I} + \sum_{k=0}^{l_o}\mathbf{D}^k\cdot\frac{s^k}{k!}.$

The case of three distinct eigen values means that we have four zeros and one non-zero pair:

$$\lambda_1 = 0;\ \{\lambda_2, -\lambda_2\} \neq 0;\ m = 3;\ l_1 \leq 4;\ exp(\mathbf{D}\cdot s) = \sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i - 1}\frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{D}-\lambda_i\mathbf{I})^j\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right] =$$

$$\left(\phi_1^{(0)}(0)\cdot\mathbf{I} + \frac{\phi_1^{(1)}(0)}{1!}\cdot(\mathbf{D}-0\cdot\mathbf{I}) + \frac{\phi_1^{(2)}(0)}{2!}\cdot(\mathbf{D}-0\cdot\mathbf{I})^2 + \frac{\phi_1^{(3)}(0)}{3!}\cdot(\mathbf{D}-0\cdot\mathbf{I})^3\right)\cdot\left(\mathbf{D}^2 - \lambda_2^2 I\right) +$$

$$\phi_2^{(0)}(\lambda_2)\cdot(\mathbf{D}-0\cdot I)^4 + \phi_2^{(0)}(-\lambda_2)(\mathbf{D}-0\cdot I)^4;\ \phi_2^{(0)}(\pm\lambda_2) = \frac{e^{\pm\lambda_2\cdot s}}{(\pm\lambda_2 - \lambda_1)^4} = \frac{e^{\pm\lambda_2\cdot s}}{\lambda_2^4};$$

$$\phi_1(\lambda) = \frac{e^{\lambda\cdot s}}{\lambda^2 - \lambda_2^2};\ \phi_1^{(1)}(\lambda) = s\cdot\frac{e^{\lambda\cdot s}}{\lambda^2-\lambda_2^2} - \frac{2\lambda\cdot e^{\lambda\cdot s}}{(\lambda^2-\lambda_2^2)^2};\ \phi_1^{(2)}(\lambda) = s^2\cdot\frac{e^{\lambda\cdot s}}{\lambda^2-\lambda_2^2} - 4s\frac{\lambda\cdot e^{\lambda\cdot s}}{(\lambda^2-\lambda_2^2)^2} - 2\frac{e^{\lambda\cdot s}}{(\lambda^2-\lambda_2^2)^2} + 8\frac{\lambda^2\cdot e^{\lambda\cdot s}}{(\lambda^2-\lambda_2^2)^2};$$

$$\phi_1^{(2)}(\lambda) = s^3\cdot\frac{e^{\lambda\cdot s}}{\lambda^2-\lambda_2^2} + 3s\cdot e^{\lambda\cdot s}\left(\frac{8\lambda^2}{(\lambda^2-\lambda_2^2)^3} - \frac{2}{(\lambda^2-\lambda_2^2)^2}\right) - 6\cdot e^{\lambda\cdot s}\frac{s^2\lambda}{(\lambda^2-\lambda_2^2)^2} + e^{\lambda\cdot s}\left(\frac{24\lambda}{(\lambda^2-\lambda_2^2)^2} - \frac{48\lambda^3}{(\lambda^2-\lambda_2^2)^2}\right);$$

$$\phi_1(0) = -\frac{1}{\lambda_2^2};\ \phi_1^{(0)}(0) = -\frac{s}{\lambda_2^2};\ \phi_1^{(2)}(0) = -\frac{2+\lambda_2^2 s^2}{\lambda_2^4};\ \phi_1^{(3)}(0) = -\frac{s\left(2+\lambda_2^2 s^2\right)}{\lambda_2^4};$$

$$exp(\mathbf{D}\cdot s) = \mathbf{D}^4\frac{e^{\lambda_2\cdot s} + e^{-\lambda_2\cdot s}}{\lambda_2^4} - \left(\frac{1}{\lambda_2^2}\mathbf{I} + \frac{s}{\lambda_2^2}\mathbf{D} + \frac{2+\lambda_2^2 s^2}{\lambda_2^4}\frac{\mathbf{D}^2}{2!} + \frac{s\left(2+\lambda_2^2 s^2\right)}{\lambda_2^4}\frac{\mathbf{D}^3}{3!}\right)\left(\mathbf{D}^2 - \lambda_2^2\mathbf{I}\right).$$

The case of five distinct number means that we have two zeros and two distinct non-zero pairs:

$$\lambda_1 = 0;\ \{\lambda_{2,3}, -\lambda_{2,3}\} \neq 0;\ m = 5;\ l_1 \leq 2;\ \mathbf{D} - \lambda_1 \cdot \mathbf{I} \equiv \mathbf{D};$$

$$exp(\mathbf{D}\cdot s) = \sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i-1}\frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{D}-\lambda_i\mathbf{I})^j\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right] = \left(\phi_1^{(0)}(0)\cdot\mathbf{I} + \frac{\phi_1^{(1)}(0)}{1!}\cdot(\mathbf{D}-0\cdot\mathbf{I})\right)\cdot(\mathbf{D}^2-\lambda_2^2 I)(\mathbf{D}^2-\lambda_3^2 I)$$

$$+\left(\phi_2^{(0)}(\lambda_2)+\phi_2^{(0)}(-\lambda_2)\right)\cdot\mathbf{D}^2\cdot(\mathbf{D}^2-\lambda_3^2 I) + +\left(\phi_2^{(0)}(\lambda_3)+\phi_2^{(0)}(-\lambda_3)\right)\cdot\mathbf{D}^2\cdot(\mathbf{D}^2-\lambda_2^2\mathbf{I}) +$$

$$\phi_2^{(0)}(\pm\lambda_2) = \frac{e^{\pm\lambda_2\cdot s}}{(\pm\lambda_2-\lambda_1)^2(\lambda_2^2-\lambda_3^2)} = \frac{e^{\pm\lambda_2\cdot s}}{\lambda_2^2(\lambda_2^2-\lambda_3^2)};\phi_2^{(0)}(\pm\lambda_3) = = \frac{e^{\pm\lambda_3\cdot s}}{\lambda_3^2(\lambda_3^2-\lambda_2^2)}$$

$$\phi_1(\lambda) = \frac{e^{\lambda\cdot s}}{(\lambda^2-\lambda_2^2)(\lambda^2-\lambda_3^2)};\ \phi_1^{(1)}(\lambda) = s\cdot\frac{e^{\lambda\cdot s}}{(\lambda^2-\lambda_2^2)(\lambda^2-\lambda_3^2)} - \frac{2\lambda\cdot e^{\lambda\cdot s}}{(\lambda^2-\lambda_2^2)^2(\lambda^2-\lambda_3^2)}\ \frac{2\lambda\cdot e^{\lambda\cdot s}}{(\lambda^2-\lambda_2^2)(\lambda^2-\lambda_3^2)^2};$$

$$\phi_1(0) = \frac{1}{\lambda_2^2\lambda_3^2};\phi_1^{(1)}(0) = \frac{s}{\lambda_2^2\lambda_3^2};\ exp(\mathbf{D}\cdot s) = \frac{1}{\lambda_2^2\lambda_3^2}(\mathbf{I}+s\mathbf{D})(\mathbf{D}^2-\lambda_2^2\mathbf{I})(\mathbf{D}^2-\lambda_3^2\mathbf{I}) +$$

$$\frac{\mathbf{D}^2}{\lambda_2^2-\lambda_3^2}\cdot\left(\frac{e^{\lambda_2\cdot s}+e^{-\lambda_2\cdot s}}{\lambda_2^2}(\mathbf{D}^2-\lambda_3^2\mathbf{I}) - \frac{e^{\lambda_3\cdot s}+e^{-\lambda_3\cdot s}}{\lambda_3^2}(\mathbf{D}^2-\lambda_2^2\mathbf{I})\right)$$

What we learned?

- It is important to pay attentions when using Sylvester formulae, especially when we are considering degenerates cases
- Because of (λ,-λ) pairing for linear Hamiltonian system, we have analytical expressions (long for general 3D case) for eigen values
- For each case, depending on number of distinct eigen values, we also can write analytical expression for $exp[\mathbf{D}s]$, i.e. the transport matrix of any element in accelerator

Let's for a distraction take square root of 2x2 matrix:

$$\mathbf{M}=\begin{bmatrix} a & b \\ c & d \end{bmatrix};\ det(\mathbf{M}-\boldsymbol{\lambda}\cdot\mathbf{I})=\lambda^2-\lambda\cdot tr\mathbf{M}+det\mathbf{M}=0;\ \lambda_{1,2}=\frac{tr\mathbf{M}}{2}\pm\sqrt{\left(\frac{tr\mathbf{M}}{2}\right)^2-det\mathbf{M}}$$

There is degenerated case only when square root is zero, otherwise we have simple

$$\lambda_1=\frac{tr\mathbf{M}}{2}+d;\ \lambda_2=\frac{tr\boldsymbol{M}}{2}-d;\ \lambda_1-\lambda_2=2d;\ \ d=+\sqrt{\left(\frac{tr\boldsymbol{M}}{2}\right)^2-det\boldsymbol{M}};\ \lambda_1\lambda_2=det\mathbf{M}$$

$$\sqrt{\mathbf{M}}=\pm\sqrt{\lambda_1}\cdot\frac{\mathbf{M}-\lambda_2\cdot\mathbf{I}}{\lambda_1-\lambda_2}\begin{matrix}\pm\\ \mp\end{matrix}\sqrt{\lambda_2}\cdot\frac{\mathbf{M}-\lambda_1\cdot\mathbf{I}}{\lambda_2-\lambda_1}=\pm\sqrt{\lambda_1}\cdot\frac{\mathbf{M}-\lambda_2\cdot\mathbf{I}}{2d}\begin{matrix}\mp\\ \pm\end{matrix}\sqrt{\lambda_2}\cdot\frac{\mathbf{M}-\lambda_1\cdot\mathbf{I}}{2d}$$

$$\sqrt{\boldsymbol{M}}_{+-}=\frac{\sqrt{\lambda_1}\cdot(\boldsymbol{M}-\lambda_2\cdot\boldsymbol{I})+\sqrt{\lambda_2}\cdot(\boldsymbol{M}-\lambda_1\cdot\boldsymbol{I})}{\boldsymbol{2d}}=\frac{\boldsymbol{M}\ -\boldsymbol{I}\cdot\sqrt{\lambda_1\lambda_2}}{2d}\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}\right)$$

$$\sqrt{\boldsymbol{M}}_{++}=\frac{\sqrt{\lambda_1}\cdot(\boldsymbol{M}-\lambda_2\cdot\boldsymbol{I})-\sqrt{\lambda_2}\cdot(\boldsymbol{M}-\lambda_1\cdot\boldsymbol{I})}{2d}=\frac{\boldsymbol{M}\ +\boldsymbol{I}\cdot\sqrt{\lambda_1\lambda_2}}{2d}\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)$$

$$\sqrt{\boldsymbol{M}}_{-+}=-\sqrt{\boldsymbol{M}}_{+-};\ \sqrt{\boldsymbol{M}}_{--}=-\sqrt{\boldsymbol{M}}_{++}$$

i.e. we have 4 solutions, but two pairs are simple result of $(-1)^2=1^2=1$. But two solutions are distinct:

$$\lambda_1=\frac{tr\mathbf{M}}{2}+d;\ \lambda_2=\frac{tr\mathbf{M}}{2}-d;\ \lambda_1-\lambda_2=2d;\ \ d=+\sqrt{\left(\frac{tr\mathbf{M}}{2}\right)^2-det\mathbf{M}};\ \lambda_1\lambda_1=det\mathbf{M}$$

$$\sqrt{\mathbf{M}}_1=\pm\frac{\mathbf{M}\ -\mathbf{I}\cdot\sqrt{det\mathbf{M}}}{2d}\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}\right);\ \left(\sqrt{\mathbf{M}}_1\right)^2=\frac{(\boldsymbol{M}\ -\boldsymbol{I}\cdot\sqrt{det\mathbf{M}})^2}{\boldsymbol{4d^2}}\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}\right)^2$$

$$\sqrt{\mathbf{M}}_2=\pm\frac{\mathbf{M}\ -\mathbf{I}\cdot\sqrt{det\mathbf{M}}}{2d}\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right);\ (\sqrt{\boldsymbol{M}}_1)^2=\frac{(\boldsymbol{M}\ +\boldsymbol{I}\cdot\sqrt{det\mathbf{M}})^2}{4d^2}\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)^2$$

$$(\mathbf{M}\ \pm\mathbf{I}\cdot\sqrt{det\mathbf{M}})^2=\mathbf{M}^2\pm2\sqrt{\mathbf{detM}}\cdot\mathbf{M}+\mathbf{I}\cdot det\mathbf{M};\ 4d^2=(\mathbf{trM})^2-4det\mathbf{M};$$

$$\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}\right)^2=\lambda_1+\lambda_2+2\sqrt{\lambda_1\lambda_2}=tr\mathbf{M}+2\sqrt{det\mathbf{M}}$$

$$\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)^2=\lambda_1+\lambda_2-2\sqrt{\lambda_1\lambda_2}=tr\mathbf{M}-2\sqrt{det\mathbf{M}}$$

with additional work to do

$$(\sqrt{M}_{1,2})^2 = \frac{M^2 \pm 2\sqrt{detM}\cdot M + I\cdot detM}{trM \pm 2\sqrt{detM}};$$

$$\mathbf{M}^2 + \mathbf{I}\cdot det\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{bmatrix} = \mathbf{M}\cdot tr\mathbf{M};$$

$$\mathbf{M}^2 \pm 2\sqrt{det\mathbf{M}}\cdot\mathbf{M} + \mathbf{I}\cdot det\mathbf{M} = \mathbf{M}\cdot(trM \pm 2\sqrt{detM})\#$$

to prove that all four solutions are indeed square roots of 2x2 matrix with distinct eigen values.

Now let's look at degenerated case with $d=0$:

$$\lambda_o = \lambda_1 = \lambda_2 = \frac{tr\mathbf{M}}{2}\ d = +\sqrt{\left(\frac{trM}{2}\right)^2 - detM} = 0;\ \lambda_1\lambda_2 = det\mathbf{M} \equiv \left(\frac{tr\mathbf{M}}{2}\right)^2;$$

$$m=1,\ l_1 \le 2;\ f(\mathbf{M}) = \sum_{\mathbf{i}=1}^{\mathbf{m}}\left[\left(\sum_{\mathrm{j}=0}^{\mathrm{l_i}-1}\frac{\phi_i^{(j)}(\lambda_i)}{j!}(\mathbf{M}-\lambda_\mathbf{i}\mathbf{I})^{\mathrm{j}}\right)\prod_{k\neq i}(\mathbf{D}-\lambda_k I)^{l_k}\right] = \phi_1(\lambda_o)\cdot\mathbf{I} + \phi_1^{(1)}(\lambda_o)\cdot(\mathbf{M}-\lambda_o\cdot\mathbf{I});$$

$$\phi_1(\lambda) = f(\lambda)\,/\prod_{j\neq 1}(\lambda-\lambda_j)^{l_j};\ \phi_1(\lambda_o) = \sqrt{\lambda_o};\ \phi_1^{(1)}(\lambda) = \left.\frac{df\sqrt{\lambda}}{d\lambda}\right|_\lambda = \frac{1}{2\sqrt{\lambda_o}};$$

$$\sqrt{M} = \sqrt{\lambda_o}\cdot\mathbf{I} + \frac{\mathbf{M}-\lambda_o\cdot I}{2\sqrt{\lambda_o}} = \frac{\mathbf{M}+\lambda_o\cdot\mathbf{I}}{2\sqrt{\lambda_o}} \Rightarrow \sqrt{M}_{1,2} = \pm\frac{\mathbf{M}+\lambda_o\cdot\mathbf{I}}{2\sqrt{\lambda_o}}.$$

Let's check that indeed this is a solution

$$\lambda_o = \lambda_1 = \lambda_2 = \frac{tr\mathbf{M}}{2}\ d = +\sqrt{\left(\frac{trM}{2}\right)^2 - detM} = 0;\ \lambda_1\lambda_2 = det\mathbf{M} \equiv \left(\frac{tr\mathbf{M}}{2}\right)^2;$$

$$\left(\sqrt{M}_{1,2}\right)^2 = \pm\frac{\mathbf{M}^2 + 2\lambda_o\cdot\mathbf{M} + \lambda_o^2\cdot\mathbf{I}}{4\lambda_o} = \frac{\mathbf{M}}{2} + \frac{\mathbf{M}^2 + \lambda_o^2\cdot\mathbf{I}}{2(a+d)}$$

$$\left(\frac{trM}{2}\right)^2 - detM \Rightarrow 4 = (a+d)^2 \Rightarrow \lambda_o^2 = \frac{(a+d)^2}{4} = ad-bc;$$

$$\mathbf{M}^2 + \lambda_o^2\cdot\mathbf{I} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} a^2+ad & b(a+d) \\ b(a+d) & d^2+ad \end{bmatrix} = 2\lambda_o\cdot\mathbf{M};$$

$$(\sqrt{\mathbf{M}}_{1,2})^2 = \frac{\mathbf{M}}{2} + \frac{\mathbf{M}}{2} = \mathbf{M}\#$$

The only case we cannot consider here when eigen vectors are zero, i.e. both the trace and determinant of matric are zero

$$\mathbf{M}=\begin{bmatrix} a & b \\ c & -a \end{bmatrix};\ a=\pm\sqrt{-bc};\ \lambda_0=\lambda_1=\lambda_2=0;\ \mathbf{M}^2=\begin{bmatrix} \mathrm{a}^2+\mathrm{bc} & 0 \\ 0 & \mathrm{a}^2+\mathrm{bc} \end{bmatrix}\equiv\mathbf{0}.$$

and we have zero in the denominator for our solution. This is the case when Sylvester formula does not provide us with solution. It does not mean that does not exist – it means that method we relied upon (Taylor expansion) is no longer useful because derivatives at the value(s) of eigen number(s) become infinite. But as we can see from the above equation, there is infinite number of solutions for square root of zero matrix.

When you meet such challenge, one of the ways to expand the function at location where derivatives exist, for example expand $\sqrt{\mathbf{I}+(\mathbf{M}-\mathbf{I})}$ and look for solution of matrix equation $(\mathbf{I}+\mathbf{A})^2=\mathbf{M}-\mathbf{I}$ but this is entire new area of discussions, and we do not have time to go there.

Square roots of real matrix are not necessarily real with an obvious example

$$\sqrt{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}=\pm i\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

but what is probably even more interesting for us that it has infinite number of real solutions:

$$\mathbf{A}^2=\begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2=\begin{bmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\ \forall\mathbf{A}\ \textit{with}\ tr\mathbf{A}=0\ \textit{and}\ \ det\mathbf{A}=-\left(a^2+bc\right)=1$$

again, pointing to the fact that not all matrix equations have unique roots and, in some cases, can have infinite number of "solutions" (branches). I suggest that as exercise at home you would check when square root expressions, we derived above are matrices with real numbers.