Hadron Beam Cooling in Particle Accelerators

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Lectures at 2023 USPAS Winter Session, Houston, Texas

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Our plan for this course

- **Purpose:** The purpose of this course is to introduce methods of hadron beam cooling to reduce the phase-space area of beams in charged particle circular accelerators. Beam cooling enables higher beam brightness and enhanced performance in many accelerator applications. The course is designed for graduate students pursuing accelerator physics as a career, or scientists or engineers having an interest in this topic in accelerator science.
- **Prerequisites:** Classical mechanics, electrodynamics, and applied mathematical methods for scientists and engineers, all at entrance graduate level, are required. Familiarity with accelerator science at the level of the USPAS course *Accelerator Physics* (graduate level) or *Fundamentals of Accelerator Physics and Technology with Simulations and Measurements Lab* (undergraduate level), or equivalent experience, are also required.
- **Instructional Method:** This course includes a series of lectures and exercise sessions. Homework problems will be assigned daily which will be graded and solutions will be reviewed in the exercise sessions the following day. There will be an in-class, open-note final exam at the conclusion of the course.
- **Credit Requirements:** Grades will be evaluated based on the following performances: 50% for home-works and class participation, and 50% for final exam.
- **Course Content;** The course will start with a description of Hamiltonian and non-Hamiltonian processes in particle accelerators. Examples of beam invariants, cooling decrements and diffusion processes will be discussed. Four cooling methods - classical electron cooling, stochastic and optical stochastic cooling, and coherent electron cooling - and their applications will be presented in detail.

Learning goals

- Upon completion of this course, your are expected to understand the basic principles that underline the physics of proton and ion beam cooling in particle accelerators: phase-space density and how increases in the phase-space density are considered "beam cooling".
- Several practical examples will be presented: stochastic and optical stochastic cooling, electron cooling and coherent electron cooling.
- Applying knowledge from these examples, you will develop an insight into the mechanisms of both beam diffusion and beam cooling.
- Insight will be gained on future applications to modern accelerators and colliders, including potential cooling mechanisms to improve performance in the planned Electron-Ion Collider (EIC) project in high energy nuclear physics.

Materials

- *Main reading material will be notes to this course:* http://case.physics.stonybrook.edu/index.php/USPAS_spring_2023

Additional reading suggestions:

- *Accelerator Physics* - fourth edition by S.Y. Lee (World Scientific 2019)
- *Handbook of Accelerator Physics and Engineering* - second edition by Alexander W. Chao and Maury Tigner (World Scientific 2013)
- Fundamentals of Accelerator Physics, http://case.physics.stonybrook.edu/index.php/PHY554_Fall_2021
- Fall: PHY 564: Advanced Accelerator Physics, http://case.physics.stonybrook.edu/index.php/PHY564_fall_2022

The Rules

- Home works will be posed that the course website together with the rest of course materials http://case.physics.stonybrook.edu/index.php/USPAS_spring_2023
- You may collaborate with your classmates on the homework's if you are contributing to the solution. You must personally write up the solution of all problems. It would be appropriate to acknowledge your collaborators – it will not affect your grades.
- We will greatly appreciate your home-works being readable. Few explanatory words between equations will save us a lot of time while checking and grading your home-works. Nevertheless, your writing style will not affect your grades.
- You may (and are encouraged to) use the library and all available resources to help solve the problems. Use of Mathematica, other software tools and spreadsheets are encouraged. Cite your source, if you found the solution somewhere.
- You have return HWs before class starts next day after assignment is given. After that solutions will be posted at the course webstite.
- We will hold Q&A session at 4:30 pm related to HWs
- In addition, we will have Recitation/Discussion session starring at 7:30 pm

Why is beam cooling needed?

- Beam cooling is referred to processes of reducing 6D phase space occupied by beam's particles, i.e. increasing beam brightness
- Beam brightness can deteriorate because instabilities, intra-beam scattering (IBS), mismatch in transport systems, noise in power supplies and RF, scattering on residual gas, quantum fluctuation of radiation….
- **Cooling has multiple applications including**:

  Increase in collider luminosity (i.e. productivity)

$$L = f_c \frac{N_1 N_2}{4\pi\sigma_x\sigma_y} h(\sigma_z/\beta^*_{x,y});\ \sigma_{x,y};\ \sigma_{x,y} \equiv \sqrt{\sigma_{1x,y}^2 + \sigma_{2x,y}^2}; \sigma_{x,y} = \sqrt{\beta^*_{x,y}\varepsilon^*_{x,y}}$$

  Reduction of beam seizes, bunch duration, beam's angular and energy spreads

Hamiltonian systems

We do not have time to follow-up every step of **the Hamiltonian or Canonical Method** – I recommend you either to refresh it either from flipping through one of your favorite Classical mechanics and E&M books or look through Lectures 1&2 of our **Advanced Accelerator Physics** course at CASE website [1].

Here we will have a very brief rehash of the key features specific for Hamiltonian systems, described by it Hamiltonian as a function of *n* coordinates and *n* canonical momenta and independent variable (in this case, *t)*:

$$H = H(q,P,t);\ q = \{q_1,...,q_n\}; P = \{P_1,...,P_n\}. \qquad \text{(M.1.1)}$$

It is important to note that in general the canonical momenta are not equal to corresponding mechanical momenta and using specific number *n=1.2.3…* does not make equations simpler. Evolution of the system in time is described by **Hamilton's equations of motion:**

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial P^i};\ \frac{dP^i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \text{(M1.2)}$$

with symplectic structure of alternating ± signs for the partial derivatives. This symplectic structure is very important and results in large number of invariants of motion and analytical solvability of many problems: Poincare invariants, Liouville theorem , action and phase variables, expelicit expression for inverse matrices, etc, etc, etc… In other words, it is hard to overestimate importance of Hamiltonian method. There is important general theorem for Hamiltonian system by **Emmy Noether** : *Any one-parameter group of dimorphisms operating in a phase space ($(q,\dot{q},t)$ for Lagrangian ((q,P,t) for Hamiltonian) and preserving the Lagrangian/Hamiltonian function equivalent to existence of the (first order) integral of motion.*

[1] http://case.physics.stonybrook.edu/images/4/4c/PHY564_Lectures_1%262_compressed.pdf .

Hamiltonian systems… continued

One of the well-known consequences of Neother's theorem is energy conservation for time independent system, which is result of symplectic

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial H}{\partial P^i}\frac{dP^i}{dt}+\frac{\partial H}{\partial q_i}\frac{dq_i}{dt}\right)=\frac{\partial H}{\partial t}+\sum_{i=1}^{n}\left(-\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial q_i}+\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial P^i}\right)=\frac{\partial H}{\partial t}$$

i.e. energy is preserved when $\frac{\partial H}{\partial t}=0$.

Hamiltonian method gives us very important tool – the general change of variables: $\{P_i,q_i\}\to\{\tilde{P}_i,\tilde{q}_i\}$, called **Canonical transformations**. Generating functions on any combination of old coordinates or old momenta with new coordinates of new momenta are possible, totaling 4= 2 x 2 combinations:

$$F(q,\tilde{q},t)\Rightarrow dF=P_i dq_i-\tilde{P}_i d\tilde{q}_i+(H'-H)dt;\ P_i=\frac{\partial F}{\partial q_i};\ \tilde{P}_i=-\frac{\partial F}{\partial \tilde{q}_i};\ H'=H+\frac{\partial F}{\partial t}.$$

$$\Phi(q,\tilde{P},t)\Rightarrow d\Phi=P_i dq_i+\tilde{q}_i d\tilde{P}_i+(H'-H)dt;\ P_i=\frac{\partial \Phi}{\partial q_i};\ \tilde{q}_i=\frac{\partial \Phi}{\partial \tilde{P}_i};\ H'=H+\frac{\partial \Phi}{\partial t};$$

$$\Omega(P,\tilde{q},t)\Rightarrow d\Omega=-q_i dP_i-\tilde{P}_i d\tilde{q}_i+(H'-H)dt;\ q_i=-\frac{\partial \Omega}{\partial P_i};\ \tilde{P}_i=-\frac{\partial \Omega}{\partial \tilde{q}_i};\ H'=H+\frac{\partial \Omega}{\partial t};\qquad \text{(M.1.3)}$$

$$\Lambda(P,\tilde{P},t)\Rightarrow d\Lambda=\tilde{q}_i d\tilde{P}_i-q_i dP_i+(H'-H)dt;\ q_i=-\frac{\partial \Lambda}{\partial P_i};\ \tilde{q}_i=\frac{\partial \Lambda}{\partial \tilde{P}_{ii}};\ H'=H+\frac{\partial \Lambda}{\partial t};$$

which is an important tool in analysis.

Hamiltonian systems… continued

Canonical transformation preserve phase space volume by the system:

$$d\Gamma = \prod_{i=1}^{n} dq_i dP_i = d\tilde{\Gamma} = \prod_{i=1}^{n} d\tilde{q}_i d\tilde{P}_i \,; \qquad \text{(M.1.5)}$$

which relatively easy to prove for $\Phi(q,\tilde{P},t)\,; P_i = \dfrac{\partial \Phi}{\partial q_i}; \tilde{q}_i = \dfrac{\partial \Phi}{\partial \tilde{P}_i}$

$$\int d\tilde{\Gamma} = \int \det[J] d\Gamma;\; J = \frac{\partial(\tilde{q},\tilde{P})}{\partial(q,P)};\; \det\left[\frac{\partial(\tilde{q},\tilde{P})}{\partial(q,P)}\right] = \det\left[\frac{\partial(\tilde{q},\tilde{P})}{\partial(q,\tilde{P})}\right] \Big/ \det\left[\frac{\partial(q,P)}{\partial(q,\tilde{P})}\right];$$

that non-trivial part Jacobian in the nominator $\dfrac{\partial \tilde{q}_i}{\partial q_j} = \dfrac{\partial^2 \Phi}{\partial \tilde{P}_i \partial q_j}$ and has identical terms in the denominator Jacobian in $\dfrac{\partial P_i}{\partial \tilde{P}_j} = \dfrac{\partial^2 \Phi}{\partial \tilde{P}_j \partial q_i}$, i.e. Jacobians are transpose of each other. Hence, there determinates are equal. The same method can be used for all four form of the Canonical transformation – in next lecture we will find even more elegant prove using symplectic matricies.

Hamiltonian systems… continued

Hamiltonian method is result of the most general physics principle: the least action principle [1] with action integral taken along the system trajectory from point A to point B in the phases space *(q, P)*:

$$S = \int_A^B \left(\sum_{i=1}^{n} P^i dq_i - H(q,P,t)dt \right); \qquad \text{(M.1.5)}$$

The most non-trivial finding from the Hamiltonian method is that the motion of a system, i.e., the evolution of coordinates and momenta also entails a Canonical transformation:

$$q_i(t+\tau) = \tilde{q}_i(q_i(t), P_i(t), t);\ P_i(t+\tau) = \tilde{P}_i(q_i(t), P_i(t), t)\,,$$

with generation function being the action integral along a real trajectory:

$$F(\tilde{q},q,t) = S(t+\tau) - S(t) = \int_t^{t+\tau} \left(P_i dq_i - Hdt\right);\ dF = \tilde{P}_i d\tilde{q}_i - P_i dq_i + (H_{t+\tau} - H_t)dt;$$

$$P_i(t+\tau) = \frac{\partial F}{\partial \tilde{q}_i}; P_j = \frac{\partial F}{\partial q_i};$$

Hence, this is a prove that phase space occupied by Hamiltonian system is one of invariants of motion:

$$\Gamma = \int \prod_{i=1}^{n} dq_i dP_i = inv\,. \qquad \text{(M.1.6)}$$

We will learn about others invariants later today.

Relativistic particles in EM field

In accelerator physics we are mostly dealing with relativistic charged particles propagating in electro-magnetic (EM) field created by accelerator systems. Particles are described by position in space

$$q \equiv \left\{\vec{r}\right\} = \left\{x, y, z\right\}$$

and velocity, mechanical momentum and energy:

$$\vec{\mathrm{v}} = \frac{d\vec{r}}{dt}; \vec{\beta} = \frac{\vec{\mathrm{v}}}{c}; \ \gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}}; \ \vec{p} = \gamma m \vec{\mathrm{v}}; \ \mathrm{E} = \gamma mc^2 \equiv \sqrt{m^2c^4 + \vec{p}^2c^2}\,.$$

The EM field is fully described by its 4-potential and its components of electric and magnetic field:

$$A^i = (\varphi, \vec{A}); \ \vec{E} = -grad\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}; \ \vec{B} = curl\vec{A}; \qquad \text{(M.1.7)}$$

with 4-poternial flexibility of selecting gauge:

$$\vec{A} \to \vec{A} + \vec{\nabla} f\left(\vec{r}, t\right); \varphi \to \varphi - \frac{\partial f\left(\vec{r}, t\right)}{\partial t},$$

which does not change electric and magnetic fields.

Relativistic particles in EM field

But choice of the EM gauge changes expressions for Canonical momenta and Hamiltonian:

$$H=\sqrt{m^2c^4+\left(\vec{P}-\frac{e}{c}\vec{A}\left(\vec{r},t\right)\right)}+\mathrm{e}\varphi\left(\vec{r},t\right),\ \vec{P}=\vec{p}+\frac{e}{c}\vec{A};$$
$$\frac{d\vec{r}}{dt}=\frac{\partial H}{\partial\vec{P}};\ \frac{d\vec{P}}{dt}=-\frac{\partial H}{\partial\vec{r}}. \qquad \text{(M1.8)}$$

Actually, selecting proper gauge is important when you trying to solve equations of motion. While, in principle, it is possible to use *t* as independent variable, it is customary in accelerator physics to use s, th length along trajectory of a reference (or ideal, or equilibrium…) particle $\vec{r}=\vec{r}_o\left(t\right)$:

$$s(t)=\int_{t_i}^{t}\left|d\vec{r}_o(t)\right|=\int_{t_i}^{t}\left|\vec{\mathrm{v}}_o(t)\right|dt;$$
$$\vec{\mathrm{v}}_o(t)=\frac{d\vec{r}_o(t)}{dt};\gamma\left(t\right)=1/\sqrt{1-\vec{\mathrm{v}}_o^{\,2}(t)/c^2};\ \vec{p}_o(t)=\gamma\left(t\right)m\vec{\mathrm{v}}_o(t);\ E_o(t)=\gamma\left(t\right)mc^2 \qquad \text{(M1.9)}$$

as an independent variable. The reason for this preference is the fact that accelerators are bolted to the floor with majority of its elements (magnets, drift spaces, diagnostics, etc.) be either time-independent or slow-varying in time but having fixed location in the space. T

Note: we use here *e* as a generic symbol for particle charge (we will use positive **+e** for positrons and protons, it is negative **-e** for electrons, and **Ze** for ions, when we talk about specific cases), as well as *m* is a generic symbol for particle's mass.

Accelerator Hamiltonian

This is especially important for periodic system, such periods in the accelerator lattice or for storage ring, where particles see identical lattice repeating at each turn with element appearing in the same places again and again. It is not true about if time is used as independent variable – arrival time of particles can both differ and also oscillate from turn to turn. Hence, we are for an additional treat with Hamiltonian

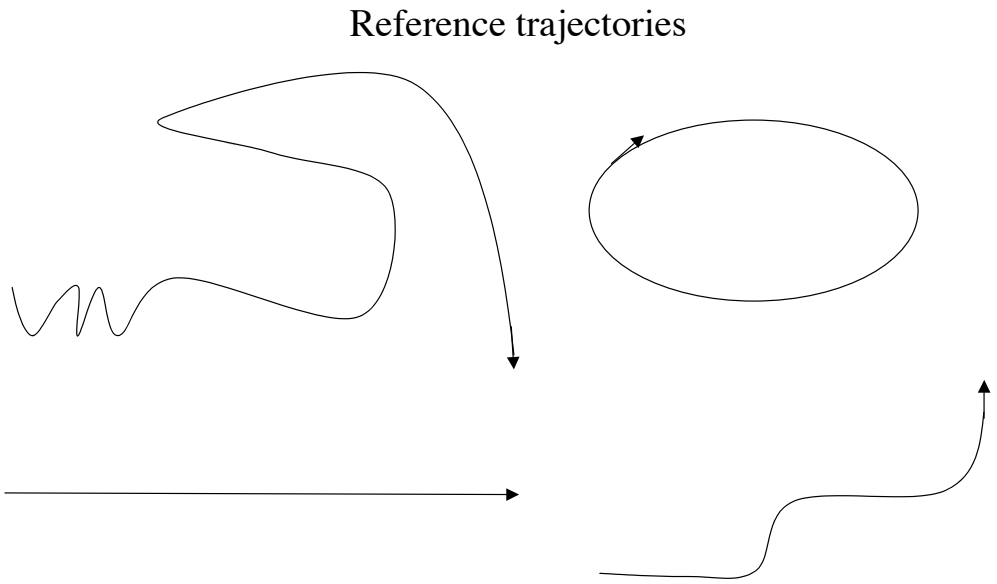


Fig. M.1.1. Various possible reference trajectories, from a simple straight pass to a circular one, though all other possibilities.

It is important for independent variable to be a monotonous function (as is time), which requires that the reference particle never stops moving (except possibly at the beginning and the end of the reference trajectory). In this case equation (M1.9) $s = s(t)$ has unique solution for the arrival time of the referense particle to the azimuth s:

$$t = t_o(s) \qquad \text{(M1.10)}$$

Accelerator Coordinates

The reference trajectory is determined by initial 4-momentum of the reference particle and the EM field along its trajectory. We should consider that trajectory is given (and from $\vec{r}_o(t)$ we also know the particle's 4-momentum in each point of trajectory) and so satisfy the equation of motion. Usually, EM fields are designed for the existence of such a trajectory (within constrains of Maxwell equation). Herein, the words reference trajectory and orbit are used interchangeably.

Using uniqueness of (M1.10) we can write the reference partilce trajectory at the function of *s*:

$$\vec{r} = \vec{r}_o(s);\ t = t_o(s);\ \vec{p} = \vec{p}_o(s), E = E_o(s). \qquad \text{(M.1.11))}$$

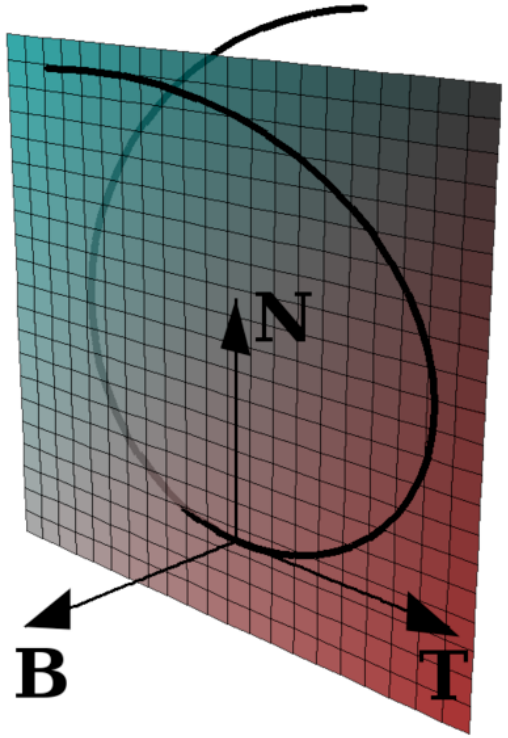
with the charge to the designer of accelerator to make it real trajectory:

$$\frac{d\vec{p}_o(s)}{ds} = \frac{dt_o(s)}{ds}\left(e\vec{E}\left(\vec{r}_o(s),t_o(s)\right) + \frac{e}{c}\left[\vec{\mathrm{v}}_o(s)\times\vec{B}\left(\vec{r}_o(s),t_o(s)\right)\right]\right)$$

Starting from this point, we use following conventions: Derivatives of any function with respect to the time will be shown by appropriate number of dots, while appropriate number of symbol ′ will be used to indicate derivatives with respect to ***s***:

$$f' = \frac{df}{ds};\ f'' = \frac{d^2 f}{ds^2} \ldots\ldots\ \dot{f} = \frac{df}{dt};\ \ddot{f} = \frac{d^2 f}{dt^2}. \qquad \text{(M.1.11)}$$

There is infinite variety of possible reference trajectories. The most popular ones are flat, i.e. they lie in a plane. A typical example is the circular orbit of a storage ring with a horizontal trajectory. Many of reference orbits are piece-wise combinations of trajectories lying in various planes. Still, there are 3D reference orbits by design. As the matter of fact, all real reference orbits are 3D because of the field errors in magnets, and errors in aligning these magnets.



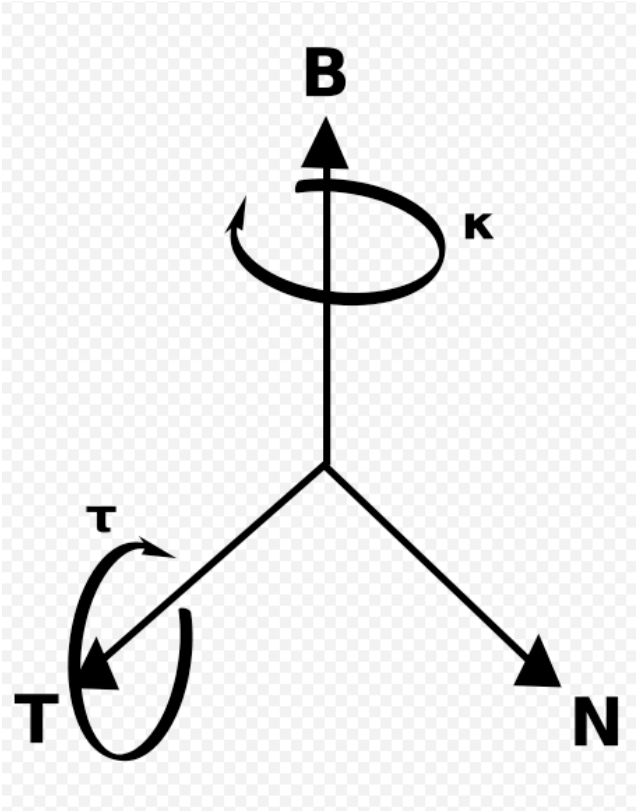
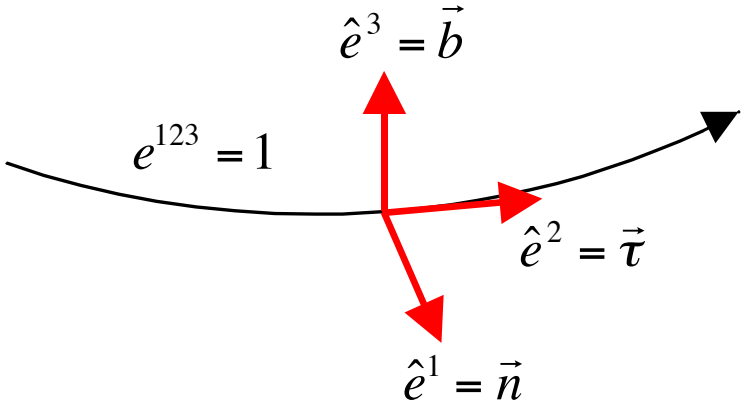


Fig. M1.2. Illustration of Frenet-Serret formulas and system from http://en.wikipedia.org/wiki/Frenet-Serret

Hence, there is no good reason not to start this discussion from general 3D reference trajectory. Fortunately two French mathematicians, Jean Frédéric Frenet and Joseph Alfred Serret, in the mid-nineteenth century developed such a coordinate system, which is described by the Frenet-Serret formulas in classical differential geometry *(O.Struik, Dirk J., Lectures on Classical Differential Geometry, Addison-Wesley, Reading, Mass, 1961)*. The Frenet-Serret coordinate system often is called the natural coordinate system. One important feature is that it has non-diagonal metrics. Hence, we have a bit of differential geometry to spice the mix.

Figures illustrate the Frenet-Serret coordinate system and define 3 orthogonal unit vectors: Normal $\hat{e}^1 = \vec{n}(s)$, tangent $\hat{e}^2 = \vec{\tau}(s)$, and normal and bi-normal $\hat{e}^3 = \vec{b}(s) = \left[\vec{n} \times \vec{\tau}\right]$:

$$\left(\vec{n} \cdot \vec{\tau}\right) = \left(\vec{b} \cdot \vec{n}\right) = \left(\vec{b} \cdot \vec{\tau}\right) = 0.$$



$$\vec{\tau} = \frac{d\vec{r}_o(s)}{ds} = \vec{r}_o{}'$$

$$\vec{n} = -\frac{\vec{r}_o{}''}{\left|\vec{r}_o{}''\right|}$$

$$\vec{b} = \left[\vec{n} \times \vec{\tau}\right]$$

Fig. M.1.3. Unit vectors in the Frenet-Serret coordinate system and their definitions

The reference trajectory must be smooth, with finite second derivatives, etc….etc… The position of any particle located in close proximity to the reference trajectory can uniquely expressed as

$$\vec{r} = \vec{r}_o(s) + x \cdot \vec{n}(s) + y \cdot \vec{b}(s). \qquad \text{(M.1.12)}$$

Proximity to the reference orbit is important for the uniqueness of the extension (M.1.12): As shown on the figure above, equation (101-2) may have multiple solutions if the requirement of proximity is not applied, i.e, the expansion may have multiple branches and mathematically become too involved.

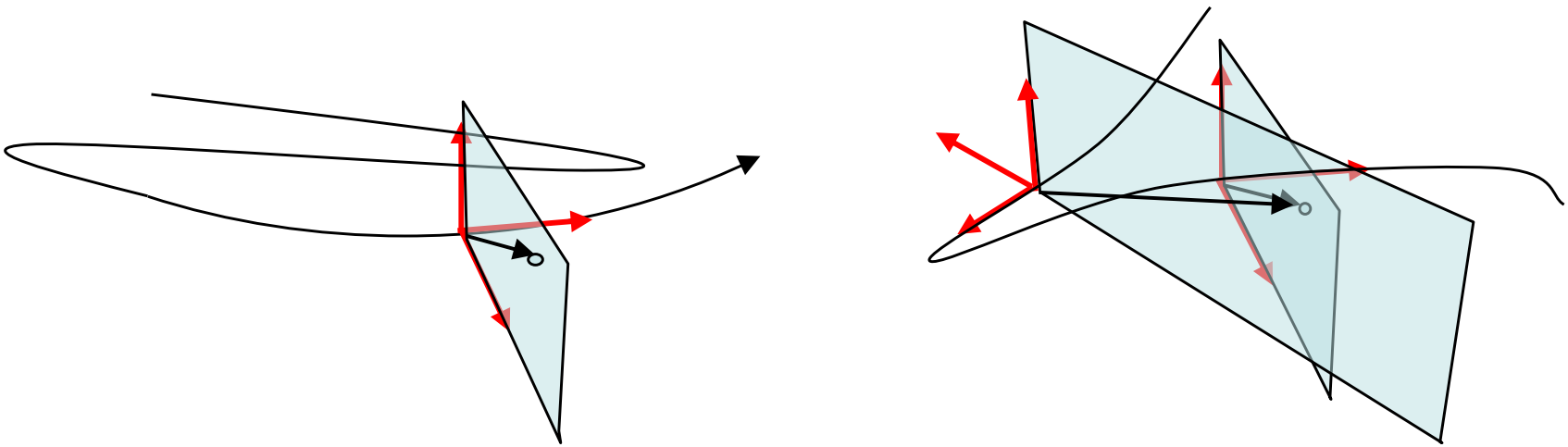


Fig. M.1.4. Expansion of particle's position in Frenet-Serret frame.

Hence, the position of any particle is uniquely describes as

$$\vec{r} = \vec{r}_o(s) + x \cdot \vec{n}(s) + y \cdot \vec{b}(s).$$

i.e., it is fully described by 3 contra-variant coordinates:

$$q^1 = x;\ q^2 = s,\ q^3 = y. \tag{M.1.13}$$

The vectors $\left\{\vec{n},\vec{\tau},\vec{b}\right\}$ satisfy Frenet-Serret formulae:

$$\frac{d\vec{\tau}}{ds} = -K(s)\cdot\vec{n};\ \frac{d\vec{n}}{ds} = K(s)\cdot\vec{\tau} - \kappa(s)\cdot\vec{b};\ \frac{d\vec{b}}{ds} = \kappa(s)\cdot\vec{n};. \tag{M.1.14}$$

where

$$K(s) = 1/\rho(s) \tag{M.1.15}$$

is the curvature of the trajectory, and $\kappa(s)$ is its torsion.

If the torsion is equal to zero, the trajectory remains in one plane, as designed for majority (99.99%) of accelerators. Curvature of trajectory is more common – each dipole magnet makes trajectory to curve. Hence for plane trajectories:

$$\kappa = 0 \Rightarrow \vec{b} = const;\ \frac{d\vec{\tau}}{ds} = -K(s)\cdot\vec{n};\ \frac{d\vec{n}}{ds} = K(s)\cdot\vec{\tau}\ . \tag{M.1.16}$$

I will spare you exercise in differential geometry with general curvilinear coordinates – if you are interested, read Lecture 4 in our Advanced Accelerator Physics course:

[2] http://case.physics.stonybrook.edu/index.php/PHY564_fall_2022

Accelerator Hamiltonian

The Hamiltonian of a charged particle in EM field in Cartesian coordinate system is

$$H(\vec{r},\vec{P},t) = c\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2} + e\varphi,$$

where the canonical momentum is $\vec{P} = \vec{p} + \frac{e}{c}\vec{A}$. Let us explore how we can make the transformation to our "curved and twisted" coordinate system. The easiest way is to apply Canonical transformation with generation function:

$$F(\vec{P},q^i) = -\vec{P}\cdot\left(\vec{r}_o(s) + x\cdot\vec{n}(s) + y\cdot\vec{b}(s)\right). \qquad \text{(M.1.17)}$$

to our new coordinates (M.1.13): $q^1 = x;\ q^2 = s,\ q^3 = y$ with new Canonical momenta obtained by simple differentiation

$$P_1 = P_x; P_2 = (1+Kx)P_s + \kappa\left(P_x y - P_y x\right);\ P_3 = P_y; \qquad \text{(M.1.18)}$$

give us accelerator Hamiltonian with time as independent variable:

$$H = c\sqrt{(1+Kx)^{-2}\left(\left(P_2 - \frac{e}{c}A_2\right) + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right)\right)^2 + \left(P_1 - \frac{e}{c}A_1\right)^2 + \left(P_3 - \frac{e}{c}A_3\right)^2 + m^2c^2} + e\varphi$$

(M.1.19)

Absence of torsion, again, simplifies it significantly:

$$P_1 = P_x; P_2 = (1+Kx)P_s.; P_3 = P_y;$$

$$H = c\sqrt{(1+Kx)^{-2}\left(P_2 - \frac{e}{c}A_2\right)^2 + \left(P_1 - \frac{e}{c}A_1\right)^2 + \left(P_3 - \frac{e}{c}A_3\right)^2 + m^2c^2}.$$

Accelerator Hamiltonian

This is still the Hamiltonian with ***t*** as independent variable and three sets of canonical pairs $\{q^1,P_1\},\{q^2,P_2\},\{q^3,P_3\}$. To change the independent variable to ***s*** let's observe structure of the particle action integral:

$$S=\int_A^B P_1\,dq^1+P_2\,ds+P_3\,dq^3-H\cdot dt;\ q^2=s\,,$$

where canonical pairs $\{q^1,P_1\},\{q^2,P_2\}\{q^3,P_3\},\{-t,H\}$ play identical role. When we use time *t* as independent variable, *H* serves as the Hamiltonian of the system. If we switch to $q^2=s$ as independent variable, then $h^*=-P_2$ will serve as corresponding Hamiltonian, with $q_t=-t$ becoming a new dependent variable *(t* is arrival time to azimuth *s*) and $P_t=H$ serving as corresponding (conjugate) Canonical momentum – again, see details in [2]. What we need just to solve (M1.19) with respect to P_2 to get most general Accelerator Hamiltonian:

$$h^*=-(1+Kx)\sqrt{\frac{(H-e\varphi)^2}{c^2}-m^2c^2-\left(P_1-\frac{e}{c}A_1\right)^2-\left(P_3-\frac{e}{c}A_3\right)^2}$$
$$-\frac{e}{c}A_2+\kappa x\left(P_3-\frac{e}{c}A_3\right)-\kappa y\left(P_1-\frac{e}{c}A_1\right) \qquad \text{(M.1.20)}$$

and equation of monition with *s* as independent variable:

$$x'=\frac{dx}{ds}=\frac{\partial h^*}{\partial P_1};\ \ \frac{dP_1}{ds}=-\frac{\partial h^*}{\partial x};\qquad y'=\frac{dy}{ds}=\frac{\partial h^*}{\partial P_3};\ \ \frac{dP_3}{ds}=-\frac{\partial h^*}{\partial y};$$
$$\{q_t=-t,\ P_t=H\},\ q_t'=\frac{\partial h^*}{\partial P_t}\rightarrow\frac{dt}{ds}=-\frac{\partial h^*}{\partial H};\ \frac{dP_t}{ds}=\frac{\partial h^*}{\partial q_t}\rightarrow\frac{dH}{ds}=\frac{\partial h^*}{\partial t}. \qquad \text{(M.1.21)}$$

Choosing a specific gauge

to express 4-potential as explicit function of electric and magnetic fields.

Again, I'll spare you of tedious details how to choose a gauge for the 4-potential. One good choice (my preference) is to make the vector potential equal to zero at the reference trajectory – no need to carry something useless for reference particle… Two other auxiliary conditions will allow us to express the components of the 4-vector potential in a form of the Taylor series:

$$a)\ \vec{A}(s,0,0,t)=0;\ b)\ \partial_x^n A_1\big|_{s,0,0,t}=\partial_y^n A_3\big|_{s,0,0,t}=0;\ c)\ \frac{\partial A_1}{\partial y}\bigg|_{s,0,0,t}+\frac{\partial A_3}{\partial x}\bigg|_{s,0,0,t}=0 \qquad \text{(M.1.17)}$$

that can be achieved by gauge transformation

$$\vec{A}=\tilde{\vec{A}}-\vec{\nabla}f;\ \varphi=\varphi+\frac{1}{c}\frac{\partial f}{\partial t};\ f=f_a=f_b+f_c$$

$$f_a=\int_0^s \tilde{A}_2(s_1,0,0,t)ds_1+\tilde{A}_1(s_1,0,0,t)\cdot x+\tilde{A}_3(s_1,0,0,t)\cdot y$$

$$f_b=\sum_{n=1}\left(\partial_x^n\tilde{A}_1\big|_{s,0,0,t}\frac{x^{n+1}}{(n+1)!}+\partial_y^n\tilde{A}_3\big|_{s,0,0,t}\frac{y^{n+1}}{(n+1)!}\right) \qquad \text{(M.1.18)}$$

$$f_c=\frac{1}{2}\sum_{n,k=0}\partial_x^n\partial_y^k\left(\partial_y\tilde{A}_1+\partial_x\tilde{A}_3\right)\frac{x^{n+1}}{(n+1)!}\frac{y^{k+1}}{(n+1)!}$$

Conditions (116) have following important consequences:

$$a)\ \partial_s^k\partial_t^l\vec{A}(s,0,0,t)\equiv 0;\ b)A_1(s,x,0,t)\equiv 0;$$

$$A_3(s,0,y,t)\equiv 0;\ c)\partial_s^k\partial_t^l\partial_x^m\partial_y^n\left(\frac{\partial A_1}{\partial y}+\frac{\partial A_3}{\partial x}\right)\equiv 0 \qquad \text{(M.1.19)}$$

Expression for 4-potentials

After a one-page-long exercise, using the first pair of Maxwell equations and conditions identified at previous slide, one can express the 4-potential in this gauge though the components of the magnetic- and electric- fields, in other words, make an unique vector potential:

$$A_1 = \frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\big|_{ro} \frac{x^k}{k!}\frac{y^{n+1}}{(n+1)!};\; A_3 = -\frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\big|_{ro} \frac{x^{k+1}}{(k+1)!}\frac{y^n}{n!}$$

$$A_2 = \sum_{n=1}^{\infty}\left\{\partial_x^{n-1}\left((1+Kx)B_y + \kappa x B_s\right)_{ro}\frac{x^n}{n!} - \partial_y^{n-1}\left((1+Kx)B_x - \kappa y B_s\right)_{ro}\frac{y^n}{n!}\right\} +$$

$$+\frac{1}{2}\sum_{n.k=1}^{\infty}\left\{\partial_x^{n-1}\partial_y^k\left((1+Kx)B_y + \kappa x B_s\right)_{ro}\frac{x^n}{n!}\frac{y^k}{k!} - \partial_x^n\partial_y^{k-1}\left((1+Kx)B_x - \kappa y B_s\right)_{ro}\frac{x^n}{n!}\frac{y^k}{k!}\right\}; \qquad \text{(M.1.20)}$$

$$\varphi = \varphi_o(s,t) - \sum_{n=1}^{\infty}\partial_x^{n-1}E_x\big|_{ro}\frac{x^n}{n!} - \sum_{n=1}^{\infty}\partial_y^{n-1}E_y\big|_{ro}\frac{y^n}{n!} - \frac{1}{2}\sum_{n.k=1}^{\infty}\left(\partial_x^{n-1}\partial_y^k E_x\big|_{ro} + \partial_x^n\partial_y^{k-1}E_y\big|_{ro}\right)\frac{x^n}{n!}\frac{y^k}{k!};$$

where $f\big|_{ro}$; $(f)_{ro}$ denotes that the value of the function $\boldsymbol{f}$ is taken at the reference orbit $r_o(s)$: i.e., at $x=0;\;\; y=0$, but in an arbitrary moment of time $\boldsymbol{t}$. Removing torsion for plane trajectories removes only few terms (indicated in red) and does not significantly simplify expressions.

We should note that $\varphi_o(s,t)$ is determined with the accuracy of an arbitrary constant, which can be eliminated by requesting $\varphi_o\left(s_o, t_o(s_o)\right) = 0$ at some point along the reference trajectory. The coefficients in (118) can be expanded further using a trivial time series

$$f(t) = f(t_o(s)) + \sum_{n=1}^{\infty}\left.\frac{d^n f}{dt^n}\right|_{t=t_o(s)}\frac{(t-t_o(s))^n}{n!}.$$

An equilibrium particle

One important feature of the field expansion that no conditions in the EM field are assumed; thus, it can be in free-space field (typical for single-particle dynamics) or a field with sources (for example, charges and currents of beam are examples). Hence, the expansion is applicable to any arbitrary accelerator problem. We reserve the notions $f\big|_{ref}$; $(f)_{ref}$ for values taken at the reference trajectory $\vec{r} = \vec{r}_o(s)$ at the reference time $t = t_o(s)$.

It is noteworthy that the value of our new Hamiltonian for the reference particle is the full particle's momentum with the minus sign:

$$h^*\big|_{ref} = -p_o(s)$$

A particle that follows the reference trajectory is called an equilibrium (or reference) one:

$$\vec{r} = \vec{r}_o(s); \quad t = t_o(s); \quad H = H_o(s) = E_o(s) + \varphi_o(s, t_o(s)),$$

with $x \equiv 0; \quad y \equiv 0; \; p_x \equiv 0; p_y \equiv 0$. This is where condition $\vec{A}\big|_{ref} = 0$ is useful, i.e., for

$$x\big|_{ref} = 0; \; y\big|_{ref} = 0; \; P_1\big|_{ref} = p_x\big|_{ref} + \frac{e}{c} A_1\big|_{ref} \equiv 0; \quad P_3\big|_{ref} = p_y\big|_{ref} + \frac{e}{c} A_3\big|_{ref} \equiv 0 \,. \tag{M.1.21}$$

or in the differential form

$$\begin{aligned} &\left.\frac{dx}{ds}\right|_{ref} = \left.\frac{\partial h^*}{d\,P_1}\right|_{ref} = 0; \quad \left.\frac{dy}{ds}\right|_{ref} = \left.\frac{\partial h^*}{dP_3}\right|_{ref} = 0; \\ &\left.\frac{dP_1}{ds}\right|_{ref} = -\left.\frac{\partial h^*}{dx}\right|_{ref} = 0; \; \left.\frac{dP_3}{ds}\right|_{ref} = -\left.\frac{\partial h^*}{dy}\right|_{ref} = 0; \end{aligned} \tag{M.1.22}$$

An equilibrium particle: HW1

It will be your homework to prove that condition for reference (equilibrium) particle result in following relations:

$$K(s) \equiv \frac{1}{\rho(s)} = -\frac{e}{p_o c}\left(B_y\Big|_{ref} + \frac{E_o}{p_o c} E_x\Big|_{ref} \right); \qquad \text{(M.1.23)}$$

$$B_x\Big|_{ref} = \frac{E_o}{p_o c} E_y\Big|_{ref}; \qquad \text{(M.1.24)}$$

$$\frac{dt_o(s)}{ds} = \frac{1}{\mathrm{v}_o(s)} \qquad \text{(M.1.25)}$$

$$\frac{dE_o(s)}{ds} = -e\frac{\partial \varphi}{\partial s}\Big|_{ref} \equiv eE_2(s, t_o(s)). \qquad \text{(M.1.26)}$$

Useful hint – use Hamiltonian equations and evaluate terms at the reference orbit (x=0, y=0, $P_{1,2}$ =0) and for refence particle momentum – it will eliminate a lot. Only after that bring it to reference time…

Going infinitesimal

By selecting the reference trajectory as basis for our coordinate system, we set the transverse coordinates and momenta at zero at the reference orbit - two canonical pairs $\{q^1,P_1\},\{q^3,P_3\}$ have solid origin. The third pair $\{-t,H\}$ is odd; it is not zero for the reference particle and has different dimensions from $\{q^1,P_1\},\{q^3,P_3\}$. A more natural Canonical pair $\{q_\tau=-ct,p_\tau=H/c\}$ can be produced by generating function $\Phi\left(q=-t,\tilde{P}=p_\tau\right)=-ct\cdot p_\tau$ with $q_\tau=-ct$ having dimension of distance and $p_\tau=H/c$ - the dimension of momentum.

To select variables that are zero at the reference orbit

$$\{\tau=-c(t-t_o(s)),\ \delta=\left(H-E_o(s)-e\varphi_o(s,t)\right)/c\},\qquad \text{(M.1.27)}$$

we can used generation function

$$\Phi(q,\tilde{P},s)=\tilde{P}_1x\ +\tilde{P}_3y-\left(E_o(s)+c\delta\right)\left(t-t_o(s)\right)-e\int^t\varphi_o(s,t_1)dt_1,$$

to produce desired:

$$P_1=\frac{\partial\Phi}{\partial x}=\tilde{P}_1;\ P_3=\frac{\partial\Phi}{\partial y}=\tilde{P}_3;H=\frac{\partial\Phi}{\partial(-t)}=E_o+c\delta+e\varphi_o(s,t);$$

$$\tilde{q}_1=\frac{\partial\Phi}{\partial\tilde{P}_1}=x;\ \tilde{q}_3=\frac{\partial\Phi}{\partial\tilde{P}_3}=y;\tilde{q}_\delta=\frac{\partial\Phi}{\partial\delta}=-c\left(t-t_o(s)\right)=\tau\qquad \text{(M.1.28)}$$

$$\tilde{h}=h^*+\frac{\partial\Phi}{\partial s}=h+\frac{E_o(s)+c\delta}{\mathrm{v}_o(s)}+E_o'(s)\tau/c-e\int^t\varphi_o'(s,t_1)\,dt_1$$

Full Hamiltonian

The change to the Hamiltonian comprised only of meaningful terms as well as just a trivial function of ***s***, ***g(s):***

$$\frac{\partial \Phi}{\partial s} = \frac{c}{\mathrm{v}_o(s)}\delta - e\varphi_{//}(s,\tau) + g(s); \quad g(s) = E_o(s)/\mathrm{v}_o(s) - e\int^{t_o(s)} \varphi_o'(s,t_1)dt_1$$

$$\varphi_{//}(s,\tau) =_{def} \frac{\partial}{\partial s}\int_0^{-\tau/c}\left(\varphi_o(s,t_o(s)+\zeta) - \varphi_o(s,t_o(s))\right)d\zeta \equiv -\int_0^{-\tau/c}\left(E_2(s,t_o(s)+\zeta) - E_2\big|_{ref}\right)d\zeta$$

where we used $E_o'(s) = -\partial\varphi/\partial s\big|_{ref}$. Additive ***g(s)*** simply can be dropped from the Hamiltonian - it does not change equations of motion. Now the only remaining task is to express the new Hamiltonian function with an updated canonical pair (130) and (115):

$$\tilde{h} = -(1+Kx)\sqrt{p_o^2 + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_\perp\right) + \left(\delta - \frac{e}{c}\varphi_\perp\right)^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2} + \\ -\frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right) + \frac{c}{\mathrm{v}_o}\delta - \frac{e}{c}\varphi_{//}(s,\tau) \qquad \text{(M.1.29)}$$

where we used following expansion and definition:

$$\frac{\left(E_o + c\delta + e\varphi_o(s,t) - e\varphi\right)^2}{c^2} - m^2c^2 = p_o^{\,2} + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_\perp\right) + \left(\delta - \frac{e}{c}\varphi_\perp\right)^2; \qquad \text{(M.1.30)}$$

$$\varphi_{\perp def} = \varphi(s,x,y,t) - \varphi_o(s,t) \equiv \varphi(s,x,y,t) - \varphi(s,0,0,t)$$

Scaling variables

Frequently, it is useful to scale one of canonical variables. Typical scaling in accelerator physics involves dividing the canonical momenta P_1, P_3, δ by the momentum of the reference particle:

$$\pi_1 = \frac{P_1}{p_o};\ \pi_3 = \frac{P_3}{p_o};\ \ \pi_o = \frac{\delta}{p_o}.$$

These variables are dimensionless and are close to $x', y', \delta E / p_o c$ for small deviations. Such scaling only is allowed in Hamiltonian mechanics when the scaling parameter is constant, i.e., is not function of s.

Scaling by a constant is easy; divide the Hamiltonian by the constant and rename the variables. Hence, transforming (134) with constant, called p_o, will make Hamiltonian (132) into

$$\tilde{h} = -(1+Kx)\sqrt{1+\frac{2E_o}{p_o c}\left(\delta - \frac{e}{p_o c}\varphi_\perp\right) + \left(\delta - \frac{e}{p_o c}\varphi_\perp\right)^2 - \left(\pi_1 - \frac{e}{p_o c}A_1\right)^2 - \left(\pi_3 - \frac{e}{p_o c}A_3\right)^2} + \tag{M.1.31}$$
$$-\frac{e}{p_o c}A_2 + \kappa x\left(\pi_3 - \frac{e}{p_o c}A_3\right) - \kappa y\left(\pi_1 - \frac{e}{p_o c}A_1\right) + \frac{c}{v_o}\delta - \frac{e}{p_o c}\varphi_{//}(s,\tau)$$

Usage of this Hamiltonian is very popular for storage rings or transport channels, wherein the energy of the particles remains constant in time. It should not be employed for particles undergoing acceleration or deceleration.

Expanding the Hamiltonian

Expanding the Hamiltonian is the main tool in accelerator physics to separate effects of various orders and use of perturbation-theory approaches assuming that

$$\left|\frac{P_1}{p_o}\right| << 1;\ \left|\frac{P_3}{p_o}\right| << 1;\ \left|\frac{\delta}{p_o}\right| << 1;$$

and that the EM fields are sufficiently smooth around the reference trajectory to allow expansion in terms of x; y; τ. We will consider* all six variables to be of the same order (of infinitesimally, $\varepsilon<<1$). The order of expansion is the maximum total power in a product of $x, y, \tau, P_1, P_2, \delta$. Unless there is a good reason not to do so, we truncate the series using this rule.

The general expansion of Hamiltonian can be accomplished via the already derived expansion for 4-potential and the well-known expansion of the square root:

$$\sqrt{1+g} = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{g^n}{2^n} \frac{(2n-3)!!)}{n!} = 1 + \frac{g}{2} + O\left(g^2\right).$$

First order terms in expansion of the Hamiltonian:

$$\tilde{h} = C_x x + C_y y + C_\tau \tau + C_{P_1} \frac{P_1}{p_o} + C_{P_3} \frac{P_3}{p_o} + + C_\delta \frac{\delta}{p_o} + O(\varepsilon^2)$$

must be zero to satisfy conditions for the reference particle: $\{x = 0, y = 0, \tau = 0, P_1 = 0; P_2 = 0, \delta = 0\}_{ref}$, i.e.

$$x'_{ref} = \frac{C_{P_1}}{p_o} = 0; y'_{ref} = \frac{C_{P_3}}{p_o} = 0; \tau'_{ref} = \frac{C_\delta}{p_o} = 0; P_1'_{ref} = -C_x = 0; P_3'_{ref} = -C_y = 0; \delta'_{ref} = -C_\tau = 0.$$

*Sometimes, one can keep explicit the time dependence of fields and expand only the rest of the variables. One such case is an approximate, and useful, description of synchrotron oscillations.

Expanding the Hamiltonian

Second order (oscillator) expansion. We continue with ideal condition and expand the Hamiltonian to the most important - second order

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L(xP_3 - yP_1) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta + g_y y\delta + F_x x\tau + F_y y\tau \quad ; \qquad \text{(M.1.32)}$$

$$\frac{F}{p_o} = \left[-K\cdot\frac{e}{p_o c}B_y - \frac{e}{p_o c}\frac{\partial B_y}{\partial x} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_x}{\partial x} - 2K\frac{eE_x}{p_o \mathrm{v}_o} + \left(\frac{meE_x}{p_o^2}\right)^2;$$

$$\frac{G}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial y} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{meE_z}{p_o^2}\right)^2;$$

$$\frac{2N}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial x} - \frac{e}{p_o c}\frac{\partial B_y}{\partial y}\right] - K\cdot\frac{e}{p_o c}B_x - \frac{e}{p_o \mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o \mathrm{v}_o} + \left(\frac{meE_z}{p_o^2}\right)\left(\frac{meE_x}{p_o^2}\right)$$

$$L = \kappa + \frac{e}{2p_o c}B_s; \qquad \frac{U}{p_o} = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \quad g_x = \frac{(mc)^2\cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; \; g_y = \frac{(mc)^2\cdot eE_y}{p_o^3}; \qquad \text{(M1.33)}$$

$$F_x = \frac{e}{c}\frac{\partial B_y}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_x}{\partial ct}; F_y = -\frac{e}{c}\frac{\partial B_x}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_y}{\partial ct}.$$

Expanding the Hamiltonian

If momentum p_o is constant, we can use (134) and rewrite Hamiltonian of the linearized motion (140) as

$$\tilde{h}_n = \frac{\pi_1^{\,2} + \pi_3^{\,2}}{2} + f\frac{x^2}{2} + n\cdot xy + g\frac{y^2}{2} + L\left(x\pi_3 - y\pi_1\right) + \frac{\pi_o^{\,2}}{2}\cdot\frac{m^2c^2}{p_o^{\,2}} + u\frac{\tau^2}{2} + g_x x\pi_o + g_y y\pi_o + f_x x\tau + f_y y\tau \quad ; \qquad \text{(M.1.34)}$$

$$f = \frac{F}{p_o};\ n = \frac{N}{p_o};\ g = \frac{G}{p_o};\ u = \frac{U}{p_o};\ f_x = \frac{F_x}{p_o};\ f_y = \frac{F_y}{p_o};$$

Note that

$$x' = \frac{\partial h_n}{\partial \pi_1} = \pi_1 - Ly; \quad y' = \frac{\partial h_n}{\partial \pi_3} = \pi_3 + Lx; \ ;$$

i.e. as soon as L=0, we can use traditional x' and y' as reduced momenta.

For a flat reference orbit - $\kappa = 0$, in the absence of transverse coupling (L=0, N=0) and transverse electric fields, the accelerator Hamiltonian has the form which is used in most of the text books and papers:

$$\tilde{h} = \frac{P_1^{\,2} + P_3^{\,2}}{2p_o} + F\frac{x^2}{2} + G\frac{y^2}{2} + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^{\,2}} + U\frac{\tau^2}{2} + g_x x\delta$$

or ; (M.1.34)

$$\tilde{h}_n = \frac{x'^2 + y'^2}{2p_o} + f\frac{x^2}{2} + g\frac{y^2}{2} + \frac{\pi_o^{\,2}}{2}\cdot\frac{m^2c^2}{p_o^{\,2}} + u\frac{\tau^2}{2} + g_x x\delta$$

$$f = \left[-K\cdot\frac{e}{p_o c}B_y - \frac{e}{p_o c}\frac{\partial B_y}{\partial x}\right];\ g = \frac{e}{p_o c}\frac{\partial B_x}{\partial y};\ u = \frac{e}{pc^2}\frac{\partial E_s}{\partial t};\ g_x = -K\frac{c}{v_o}.$$

Linear equations of motion.

Hamiltonian system is described by the set of coordinates, Canonical momenta, and Hamiltonian. In matrix form, the Hamiltonian equations can written compactly:

$$H = H(Q,P,s);\ X = \begin{bmatrix} Q_1 \\ P_1 \\ \dots \\ Q_n \\ P_n \end{bmatrix} \equiv \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_{2n-1} \\ x_{2n} \end{bmatrix};\ \frac{dX}{ds} = S\frac{\partial H}{\partial X};\ S = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{bmatrix};\ \text{M1.35)}$$

where S is a generator (norm) of a symplectic group of matrixes (two different but closely related types of mathematical groups). The space of coordinates and momenta is called phase space of the system with dimension ***2n***. There is no reason to select specific n, unless n=1 and motion is one-dimensional.

We finished the accelerator Hamiltonian expansion by concluding that the first not-trivial term in the accelerator Hamiltonian expansion is a quadratic term of canonical momenta and coordinates. This Hamiltonian can be written in the matrix form (letting *n* be a dimension of the Hamiltonian system with n canonical pairs $\{q_i, P_i\}$)

$$H = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T \cdot \mathbf{H}(s) \cdot X; \qquad \text{(M1.36)}$$

$$X^T = \begin{bmatrix} q^1 & P_1 & \dots\dots & \dots\dots & q^n & P_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots\dots & \dots\dots & x_{2n-1} & x_{2n} \end{bmatrix},$$

with the self-evident feature that Hamiltonian matrix is symmetric:

$$\mathbf{H^T} = \mathbf{H} \qquad \text{(M1.37)}$$

The equations of motion are just a set of ***2n*** linear ordinary differential equations with ***s***-dependent coefficients:

$$\frac{dX}{ds} = \mathbf{D}(s) \cdot X; \mathbf{D} = \mathbf{S} \cdot \mathbf{H}(s) \tag{M1.38}$$

One important feature of this system is that

$$Trace[\mathbf{D}] = 0, \tag{M1.39}$$

(the trivial proof is based on $Trace[\mathbf{AB}] = Trace[\mathbf{BA}]$; $Trace[\mathbf{A}^T] = Trace[\mathbf{A}]$ and $(\mathbf{SH})^T = -(\mathbf{HS})$). i.e., the Wronskian determinant of the system (http://en.wikipedia.org/wiki/Wronskian) is equal to one. The famous Liouville theorem comes from well-known operator formula $\frac{d\det[\mathbf{W}(s)]}{ds} = Trace[\mathbf{D}]$; we do not need it here because we will have an easier method of proof. **You also have it as a homework problem.**

One important consequence of this observation is that Hamiltonian systems preserve phase space occupied by particles and that dissipative (non-Hamiltonian) forces are needed to make trace of the **D** non-zero.

The solution of any system of first-order linear differential equations can be expressed through its ***2n*** initial conditions $\boldsymbol{X_o}$ at azimuth $\boldsymbol{s_o}$

$$X(s_o) = X_o, \tag{M.1.40}$$

through the transport matrix $\boldsymbol{M(s_o/s)}$:

$$X(s) = \mathbf{M}(s_o|s) \cdot X_o. \tag{M.1.41}$$

There are two simple proofs of this theorem. The first is an elegant one: Let us consider the matrix differential equation

$$\mathbf{M}' \equiv \frac{d\mathbf{M}}{ds} = \mathbf{D}(s) \cdot \mathbf{M}; \tag{M.1.42}$$

with a unit matrix as its initial condition at azimuth $\boldsymbol{s_o}$

$$\mathbf{M}(s_o) = \mathbf{I}. \tag{M.1.43}$$

Such solution exists* and then we readily see that

$$X(s) = \mathbf{M}(s) \cdot X_o. \tag{M.1.44}$$

satisfies $\dfrac{dX}{ds} = \mathbf{D}(s) \cdot X$:

$$\frac{dX}{ds} = \frac{d\mathbf{M}(s)}{ds} \cdot X_o = \mathbf{D}(s) \cdot \mathbf{M}(s) \cdot X_o \equiv \mathbf{D}(s) \cdot X\#.$$

* Mathematically, it is nothing else but

$$M(s) = \lim_{N\to\infty} \prod_{k=1}^{N} \left(\mathbf{I} + \mathbf{D}(s_k)\Delta s\right);\ \Delta s = (s - s_o)/N;\ s_k = s_o + k \cdot \Delta s.$$

A more traditional approach to the same solution is to use the facts that a) there exists a solution of equation $\frac{dX}{ds} = \mathbf{D}(s) \cdot X$ with arbitrary initial conditions (less-trivial statement); and, b) any linear combination of the solutions also is a solution of $\frac{dX}{ds} = \mathbf{D}(s) \cdot X$ (very trivial one). Considering a set of solutions $M_k(s)$, k=1,…2n, with initial conditions at azimuth $\boldsymbol{s_o}$,then

$$M_1(s_o) = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix}; M_2(s_o) = \begin{bmatrix} 0 \\ 1 \\ \dots \\ 0 \\ 0 \end{bmatrix}; \dots\dots\dots M_{2n}(s_o) = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}; \quad \frac{dM_k(s)}{ds} = \mathbf{D}(s) \cdot M_k(s);$$

and their linear combination

$$X(s) = \sum_{k=1}^{2n} x_{ko} \cdot M_k(s),$$

which satisfies the initial condition (167)

$$X(s_o) = \sum_{k=1}^{2n} x_{ko} \cdot M_k(s_o) = \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ \dots \\ x_{2n-1,0} \\ x_{2n,0} \end{bmatrix} = X_o \cdot$$

Now, we recognize that this solution is nothing other than the transport matrix with matrix **M(s)** being a simple combination of 2n columns $M_k(s)$:

$$\mathbf{M}(s) = [M_1(s), M_2(s), \dots, M_{2n}(s)].$$

In differential calculus, the solution is defined as

$$\mathbf{M}\left(s_o\middle|s\right)=\exp\left[\int_{s_o}^{s}\mathbf{D}(s)\,ds\right]=\lim_{N\to\infty}\prod_{k=1}^{N}\left(\mathbf{I}+\mathbf{D}(s_k)\Delta s\right);\qquad\text{(M.1.45)}$$

$$\Delta s=(s-s_o)/N;\ s_k\in\{s_o+(k-1)\cdot\Delta s, s_o+k\cdot\Delta s\}$$

The fact that the transport matrix for a linear Hamiltonian system has unit determinant (i.e., the absence of dissipation!)

$$\det\mathbf{M}=\exp\left[\int_{s_o}^{s}Trace(\mathbf{D}(s))ds\right]=1\cdot\qquad\text{(M.1.46)}$$

is the first indicator of the advantages that follow. Let us consider the invariants of motion characteristic of linear Hamiltonian systems, i.e., invariants of the symplectic phase space*. Starting from the bilinear form of two independent solutions of eq. (165), $X_1(s)$ and $X_2(s)$, *(it is obvious that* $X^T\mathbf{S}X=0$*)* we show that

$$X_2^T(s)\cdot\mathbf{S}\cdot X_1(s)=X_2^T(s_o)\cdot\mathbf{S}\cdot X_1(s_o)=inv\,.\qquad\text{(M.1.47)}$$

The proof is straightforward

$$\frac{d}{ds}\left(X_2^T\cdot\mathbf{S}\cdot X_1\right)=X_2^{T\prime}\cdot\mathbf{S}\cdot X_1+X_2^T\cdot\mathbf{S}\cdot X_1'=X_2^T\cdot\left((\mathbf{SD})^{\mathbf{T}}\mathbf{S}+\mathbf{SSD}\right)\cdot X_1'\equiv 0\,.$$

** Phase space is defined as the 2n-dimentional space of canonical variable* $\{q_i,P^i\}$*, that is, the space where this Hamiltonian system evolves.*

Proving that transport matrices for Hamiltonian system are symplectic is very similar:

$$\mathbf{M}^{\mathbf{T}} \cdot \mathbf{S} \cdot \mathbf{M} = \mathbf{S}. \qquad \text{(M.1.48)}$$

Beginning from the simple fact that the unit matrix is symplectic: $\mathbf{I}^{\mathbf{T}} \cdot \mathbf{S} \cdot \mathbf{I} = \mathbf{S}$, i.e. $\mathbf{M}(s_o|s_o)$ is symplectic, and following with the proof that $\mathbf{M}^{\mathbf{T}}(s_o|s) \cdot \mathbf{S} \cdot \mathbf{M}(s_o|s) = \mathbf{M}^{\mathbf{T}}(s_o|s_o) \cdot \mathbf{S} \cdot \mathbf{M}(s_o|s_o) = \mathbf{S}$:

$$\frac{d}{ds}\left(\mathbf{M}^{\mathbf{T}} \cdot \mathbf{S} \cdot \mathbf{M}\right) = \mathbf{M}^{\mathbf{T}'} \cdot \mathbf{S} \cdot \mathbf{M} + \mathbf{M}^{\mathbf{T}} \cdot \mathbf{S} \cdot \mathbf{M}' = \mathbf{M}^{\mathbf{T}} \cdot \left((\mathbf{SD})^{\mathbf{T}}\mathbf{S} + \mathbf{SSD}\right) \cdot \mathbf{M} \equiv \mathbf{0} \ \#$$

Symplectic square matrices of dimensions *2n x 2n*, which include unit matrix **I**, create a symplectic group, where the product of symplectic matrices also is a symplectic matrix*. The symplectic condition (177) is very powerful and should not be underappreciated. Before going further, we should ask ourselves several questions: How can the inverse matrix of **M** be found? Are there invariants of motion to hold-on to? Can something specific be said about a real accelerator wherein there are small but all-important perturbations beyond the linear equation of motions?

**Group G is defined as a set of elements, with a definition of a product of any two elements of the group;* $P = A \bullet B \in G$; $A, B \in G$. *The product must satisfy the associative law :* $A \bullet (B \bullet C) = (A \bullet B) \bullet C$; *there is an unit element in the group* $I \in G; I \bullet A = A \bullet I = A : \forall A \in G$; *and inverse elements:* $\forall A \in G; \exists B(called\ A^{-1}) \in G : A^{-1}A = AA^{-1} = I.$

As you probably concluded, the Hamiltonian method yield many answers, and is why it is so vital to research.

We can count them: The general transport matrix $\mathbf{M}$ (solution of $\mathbf{M}' = \mathbf{D}(s)\cdot\mathbf{M}$ with arbitrary $\mathbf{D}$) has $(2n)^2$ independent elements. Because the symplectic condition $\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M}-\mathbf{S}=\mathbf{0}$ represents an asymmetric matrix with n-diagonal elements equivalently being zeros, and the conditions above and below the diagonal are identical – then only the $n(2n-1)$ condition remains and only the $n(2n+1)$ elements are independent. For $n=1$ (1D) there is only one condition, for $n=2$ there are 6 conditions, and n=3 (3D) there are 15 conditions. Are these facts of any use in furthering this exploration?

First, symplecticity makes the matrix determinant to be unit*:

$$\det\left[\mathbf{M}^{\mathbf{T}}(s)\cdot\mathbf{S}\cdot\mathbf{M}(s)\right]=\det\mathbf{S}\ \rightarrow\left(\det\mathbf{M}(s)\right)^2=1\rightarrow\det\mathbf{M}=\pm1;\quad\det\mathbf{M}(0)=1\rightarrow\det\mathbf{M}=1\ \#$$

i.e., it preserves the 2n-D phase space volume occupied by the ensemble of particles (system):

$$\int\prod_{i=1}^{n}dq_i dP^i=inv \tag{M.1.49}$$

The other invariants preserved by symplectic transformations are called Poincaré invariants and are the sum of projections onto the appropriate over- manifold in two, four…. (2n-2) dimensions:

$$\sum_{i=1}^{n}\iint dq_i dP^i=inv;\sum_{i\neq j}\iint\iint dq_i dP^i dq_j dP^j=inv.....\tag{M.1.50}$$

**HW: for n=1 case with 2x2 matrices to verify that the symplectic product is reduced to determine*

$$\mathbf{M}_{2x2}=\begin{bmatrix}a & b\\ c & d\end{bmatrix};\mathbf{S}_{2x2}=\sigma;\Rightarrow\mathbf{M}^{\mathbf{T}}\cdot\sigma\cdot\mathbf{M}=\det\mathbf{M}\cdot\sigma \tag{M.1.51}$$

For example, matrix **M** can be represented as n^2 combinations of 2x2 matrices M_{ij}:

$$\mathbf{M}=\begin{bmatrix} M_{11} & \dots & M_{1n} \\ \dots & \dots & \dots \\ M_{n1} & \dots & M_{nn} \end{bmatrix};\ \mathbf{S}=\begin{bmatrix} \sigma & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \sigma \end{bmatrix};\ \sigma=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

$$\mathbf{M}^T\mathbf{S}\mathbf{M}=\begin{bmatrix} \sum_{i=1}^{n} M_{1i}{}^T\sigma M_{1i} & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \sum_{i=1}^{n} M_{ni}{}^T\sigma M_{ni} \end{bmatrix}=\begin{bmatrix} \sigma & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \sigma \end{bmatrix};. \qquad \text{(M.1.52)}$$

$$\sum_{i=1}^{n} M_{1i}{}^T\sigma M_{1i}=\sigma\cdot\sum_{i=1}^{n}\det M_{1i}.$$

Using equation (M.1.51), we easily demonstrate the requirement for the symplectic condition (M.1.48) is that the sum of determinants in each row of these 2x2 matrices is equal to one; the same is true for the columns:

$$\sum_{i=1}^{n}\det\left[M_{ij}\right]=\sum_{j=1}^{n}\det\left[M_{ij}\right]=1 \qquad \text{(M.1.53)}$$

with a specific prediction for decoupled matrices, which are block diagonal:

$$\mathbf{M}=\begin{vmatrix} M_{11} & 0\dots & 0 \\ 0 & \dots & 0 \\ 0 & 0\dots & M_{nn} \end{vmatrix};\ \det\left[M_{ii}\right]=1. \qquad \text{(M.1.54)}$$

Other trivial and useful features are: for the columns

$$\mathbf{M}=\begin{bmatrix} C_1 & C_2 & .... & C_{2n-1} & C_{2n} \end{bmatrix} \Rightarrow$$

$$C_{2k-1}^T S C_{2k} = -C_{2k}^T S C_{2k-1} = 1, k = 1,..,n \qquad \text{(M.1.55)}$$

$$others\ are\ 0$$

or lines of the symplectic matrix:

$$\mathbf{M}=\begin{bmatrix} L_1 \\ L_2 \\ .... \\ L_{2n-1} \\ L_{2n} \end{bmatrix} \Rightarrow -L_{2k} S L_{2k-1}^T = L_{2k-1} S L_{2k}^T = 1, others\ are\ 0 \qquad \text{(M.1.56)}$$

We could go further, but we will stop here by showing the most incredible feature of symplectic matrices, viz., that it is easy to find their inverse (recall there is no general rule for inverting a 2n x 2n matrix!) Thus, multiplying $\mathbf{M^T \cdot S \cdot M = S}$ from left by **–S** we get

$$\mathbf{-S \cdot M^T \cdot S \cdot M = I \ \Rightarrow \ M^{-1} = -S \cdot M^T \cdot S}. \qquad \text{(M.1.57)}$$

As an easy exercise for 2x2 symplectic (i.e. with unit determinant – see note below) matrices, you can show that $\mathbf{M}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (183) gives $\mathbf{M}=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. It is a much less trivial task to invert 6x6 matrix; hence, the power of symplecticity allows us to enact many theoretical manipulations that otherwise would be impossible. Obviously, and easy to prove, transposed symplectic and inverse symplectic matrices also are also symplectic:

$$\mathbf{M^{-1T} \cdot S \cdot M^{-1} = S}; \qquad \mathbf{M \cdot S \cdot M^T = S}. \qquad \text{(M.1.58)}$$

Still the most powerful conclusion for us is that

$$\mathbf{M}^{-1} = -\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S} = -\begin{bmatrix} 0 & -1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & .. & 1 & 0 \end{bmatrix}\begin{bmatrix} m_{11} & m_{21} & \dots & m_{2n-1,1} & m_{2n,1} \\ m_{12} & m_{22} & \dots & m_{2n-1,2} & m_{2n,2} \\ \dots & \dots & \dots & \dots & \dots \\ m_{1,2n-1} & m_{2,2n-1} & \dots & m_{2n-1,2n-1} & m_{2n,2n-1} \\ m_{1,2n} & m_{2,2n} & .. & m_{2n-1,2n} & m_{2n,2n} \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} m_{12} & m_{22} & \dots & m_{2n-1,2} & m_{2n,2} \\ -m_{11} & -m_{21} & \dots & -m_{2n-1,1} & -m_{2n,1} \\ \dots & \dots & \dots & \dots & \dots \\ m_{1,2n} & m_{2,2n} & \dots & m_{2n-1,2n} & m_{2n,2n} \\ -m_{1,2n-1} & -m_{2,2n-1} & .. & -m_{2n-1,2n-1} & -m_{2n,2n-1} \end{bmatrix}\cdot\begin{bmatrix} 0 & -1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & .. & 1 & 0 \end{bmatrix} =$$

$$\mathbf{M}^{-1} = \begin{bmatrix} m_{22} & -m_{21} & \dots & m_{2n,2} & -m_{2n-1,2} \\ -m_{12} & m_{11} & \dots & -m_{2n,1} & m_{2n-1,1} \\ \dots & \dots & \dots & \dots & \dots \\ m_{2,2n} & -m_{1,2n} & \dots & m_{2n,2n} & -m_{2n,2n-1} \\ -m_{2,2n-1} & m_{1,2n-1} & .. & -m_{2n-1,2n} & m_{2n-1,2n-1} \end{bmatrix}$$

e.g. as simple as it can go:

$$\left(\mathbf{M}^{-1}\right)_{2k-1,2j-1} = \left(\mathbf{M}\right)_{2j,2k}; \left(\mathbf{M}^{-1}\right)_{2k,2j} = \left(\mathbf{M}\right)_{2j-1,2k-1};\ j,k = 1,\dots,n$$

$$\left(\mathbf{M}^{-1}\right)_{2k,2j-1} = -\left(\mathbf{M}\right)_{2j,2k-1}; \left(\mathbf{M}^{-1}\right)_{2k-1,2j} = -\left(\mathbf{M}\right)_{2j-1,2k};\ j,k = 1,\dots,n$$

Anybody who tried to write analytical expression for inverse of 6x6 matrix would really appreciate this wonderful simplicity

To conclude

- We walked through a Hamiltonian method of describing particle's motion in accelerators
- We learned that transport matrices are symplectic, which corresponds to invariants of motion: one for 1D, six for 2D and 15 for 3D motion.
- One of the important implication is that we have analytical expression for inverse matrices