Homework 6. Due September 28

This is first STAR problem. The rules are as follow:

(a) if you find a standard solution (for example from a good Analytical mechanics book or on a Website) – you got a nominal points. I rely on your comment at the end of the derivation.

(b) If you derived your own original proof – you have 5-fold increase in the score.

Problem 1. 15 points. Prove that

(a) A canonical transformation is equivalent to a symplectic map (locality symplectic linear transformation);

(b) a symplectic map (locality symplectic linear transformation) is a Canonical transformation,

Suggestion: you may ask for additional hint during next class.

Solution:

(a) Prove that Canonical transformation is equivalent to a symplectic map.

In other words, we have to show that it is locally symplectic. First, let's write all four forms of generation functions of the same canonical transformation:

$$F(q,\tilde{q},t) \quad \Rightarrow dF = P_i dq_i - \tilde{P}_i d\tilde{q}_i + (H' - H)dt;\; P_i = \frac{\partial F}{\partial q_i}; \tilde{P}_i = -\frac{\partial F}{\partial \tilde{q}_i}; H' = H + \frac{\partial F}{\partial t}.$$

$$\Phi(q,\tilde{P},t) = F + \tilde{q}_i\tilde{P}_i \;\Rightarrow d\Phi = P_i dq_i + \tilde{q}_i d\tilde{P}_i + (H' - H)dt; P_i = \frac{\partial \Phi}{\partial q_i}; \tilde{q}_i = \frac{\partial \Phi}{\partial \tilde{P}_i}; H' = H + \frac{\partial \Phi}{\partial t};$$

$$\Omega(P,\tilde{q},t) = F - P_i q_i \;\Rightarrow d\Omega = -q_i dP_i - \tilde{P}_i d\tilde{q}_i + (H' - H)dt; q_i = -\frac{\partial \Omega}{\partial P_i}; \tilde{P}_i = -\frac{\partial \Omega}{\partial \tilde{q}_i}; H' = H + \frac{\partial \Omega}{\partial t};$$

$$\Lambda(P,\tilde{P},t) = \Phi - P_i q_i \Rightarrow d\Lambda = \tilde{q}_i d\tilde{P}_i - q_i dP_i + (H' - H)dt; q_i = -\frac{\partial \Lambda}{\partial P_i}; \tilde{q}_i = \frac{\partial \Lambda}{\partial \tilde{P}_i}; H' = H + \frac{\partial \Lambda}{\partial t};$$

with

$$\tilde{X} = \tilde{X}(X,s); X = \{q,P\}; \tilde{X} = \{\tilde{q},\tilde{P}\}$$

Then for local transformation we can write

$$\delta\tilde{X} = M(s)\delta X \Leftrightarrow \delta X = M^{-1}(s)\delta\tilde{X}$$

We need to show that

$$M^T S M = S \Leftrightarrow M^{-1} = -S M^T S$$

or in details

$$M^{-1}_{ik} = S_{ij} M_{lj} S_{lk}; \quad S_{ij} = \left\{ \begin{array}{c} 1;\; i = 2m-1, j = 2m \\ -1;\; j = 2m-1, i = 2m \\ 0, \mathit{otherwise} \end{array} \right\}$$

specifically

$$M^{-1}_{2m-1,2k-1}=M_{2k,2m};\ M^{-1}_{2m-1,2n}=-M_{2k-1,2m}$$
$$M^{-1}_{2m,2k-1}=-M_{2k,2m-1};\ M^{-1}_{2m,2k}=M_{2k-1,2m-1}$$
$$k,m=\{1,\dots,n\}$$

Important reminder:

$$x_{2k-1}\equiv q_k;\ \ x_{2k}\equiv P_k;\ \tilde{x}_{2k-1}\equiv\tilde{q}_k;\ \tilde{x}_{2k}\equiv\tilde{P}_k.$$

Let consider all four cases: $M_{2k-1,2m-1}$ vs $M^{-1}_{2m,2k}$. It is easy to prove that they equal

$$P_m=\frac{\partial\Phi}{\partial q_m};\tilde{q}_k=\frac{\partial\Phi}{\partial\tilde{P}_k};$$
$$M^{-1}_{2m,2k}=\frac{\partial P_m}{\partial\tilde{P}_k}=\frac{\partial}{\partial\tilde{P}_k}\frac{\partial\Phi}{\partial q_m}=\frac{\partial^2\Phi}{\partial\tilde{P}_k\,\partial q_m}=\frac{\partial^2\Phi}{\partial q_m\,\partial\tilde{P}_k}$$
$$M_{2k-1,2m-1}=\frac{\partial\tilde{q}_k}{\partial q_m}=\frac{\partial}{\partial q_m}\frac{\partial\Phi}{\partial\tilde{P}_k}=\frac{\partial^2\Phi}{\partial q_m\,\partial\tilde{P}_k}=M^{-1}_{2m,2k}$$

Let consider all four cases: $M_{2m,2k}$ vs $M^{-1}_{2k-1,2m-1}$. It is easy to prove that they equal

$$q_k=-\frac{\partial\Omega}{\partial P_k};\tilde{P}_m=-\frac{\partial\Omega}{\partial\tilde{q}_m};$$
$$M_{2m,2k}=\frac{\partial\tilde{P}_m}{\partial P_k}=-\frac{\partial}{\partial P_k}\left(\frac{\partial\Omega}{\partial\tilde{q}_m}\right)=-\frac{\partial^2\Omega}{\partial P_k\,\partial\tilde{q}_m}=-\frac{\partial^2\Phi}{\partial\tilde{q}_m\,\partial P_k}$$
$$M^{-1}_{2k-1,2m-1}=\frac{\partial q_k}{\partial\tilde{q}_m}=-\frac{\partial}{\partial\tilde{q}_m}\left(\frac{\partial\Omega}{\partial P_k}\right)=-\frac{\partial^2\Phi}{\partial\tilde{q}_m\,\partial P_k}=M_{2m,2k}$$

Let consider all four cases: $M_{2k-1,2m}$ vs $M^{-1}_{2m-1,2k}$. It is easy to prove that they equal

$$\tilde{q}_k=\frac{\partial\Lambda}{\partial\tilde{P}_k};q_m=-\frac{\partial\Lambda}{\partial P_m};$$
$$M_{2k-1,2m}=\frac{\partial\tilde{q}_k}{\partial P_m}=\frac{\partial}{\partial P_m}\left(\frac{\partial\Lambda}{\partial\tilde{P}_k}\right)=\frac{\partial^2\Lambda}{\partial P_m\,\partial\tilde{P}_k}$$
$$M^{-1}_{2m-1,2k}=\frac{\partial q_m}{\partial\tilde{P}_k}=-\frac{\partial}{\partial\tilde{P}_k}\left(\frac{\partial\Lambda}{\partial P_m}\right)=-\frac{\partial^2\Lambda}{\partial P_m\,\partial\tilde{P}_k}=-M_{2k-1,2m}$$

And using the same drill for $M_{2k,2m-1}$ vs $M^{-1}_{2m,2k-1}$

$$P_m=\frac{\partial F}{\partial q_m};\tilde{P}_k=-\frac{\partial F}{\partial\tilde{q}_k}$$
$$M_{2k,2m-1}=\frac{\partial\tilde{P}_k}{\partial q_m}=-\frac{\partial}{\partial q_m}\left(\frac{\partial F}{\partial\tilde{q}_k}\right)=-\frac{\partial^2 F}{\partial q_m\,\partial\tilde{q}_k}$$
$$M^{-1}_{2m,2k-1}=\frac{\partial P_m}{\partial\tilde{q}_k}=\frac{\partial}{\partial\tilde{q}_k}\left(\frac{\partial F}{\partial q_m}\right)=\frac{\partial^2 F}{\partial\tilde{q}_k\,\partial q_m}=-M_{2k,2m-1}$$

Note, that we used all four forms of otherwise identical Canonical transformation to prove that every Canonical transformation is a symplectic map.

(b). Let's now show that a symplectic map (e.g. generating symplectic matrices) is equivalent to a Canonical transformation.
Let start from a generic transformation in the phase space:

$$X = F(\tilde{X}, s) \tag{1}$$

with

$$M_{ij} = \frac{\partial X_i}{\partial \tilde{X}_j} \equiv \frac{\partial F_i(\tilde{X}, s)}{\partial \tilde{X}_j}; M = \left[\frac{\partial F}{\partial \tilde{X}}\right]; \;\; M^T S M = S$$

Since our initial system ($\tilde{X}$) is a Hamiltonian system, than

$$\frac{d\tilde{X}}{ds} = S\frac{\partial \tilde{H}}{\partial \tilde{X}}$$

and we need to show that

$$\frac{dX}{ds} = S\frac{\partial H}{\partial X}$$

where H is a new Hamiltonian in phase space X.

$$\frac{dX}{ds} = \frac{dF(\tilde{X}, s)}{ds} = \frac{\partial F(\tilde{X}, s)}{\partial s} + \frac{\partial F(\tilde{X}, s)}{\partial \tilde{X}}\frac{d\tilde{X}}{ds}$$

There is a number of steps to prove this. First, lets expand eq. (1) near real trajectory $\frac{d\tilde{X}}{ds} = S\frac{\partial \tilde{H}}{\partial \tilde{X}}$

$$\frac{d\tilde{X}_o(s)}{ds} = S\left.\frac{\partial \tilde{H}}{\partial \tilde{X}}\right|_{\tilde{X}=\tilde{X}_o(s)}; \;\; \tilde{X} = \tilde{X}_o(s) + \delta\tilde{X};$$

and the same for

$$X = X_o(s) + \delta X; \;\; \delta X(s) = M(s)\delta\tilde{X}(s)$$

$$X_o(s) = F\left(\tilde{X}_o(s), s\right);$$

e.g. we determined the local symplectic transformation in the entire phase space as function of $\left(s, \tilde{X}\right)$. It allows us to rewrite the transformation (1) function as

$$\frac{d}{ds}\delta X = \frac{dM}{ds}\delta\tilde{X} + M\frac{d}{ds}\delta\tilde{X}$$

$$\frac{d}{ds}\delta\tilde{X} = S\frac{\partial \tilde{H}}{\partial \delta\tilde{X}} = SM^T\frac{\partial \tilde{H}}{\partial \delta X}; \;\; \frac{\partial}{\partial \delta\tilde{X}} = \left(\frac{\partial \delta X}{\partial \delta\tilde{X}}\right)^T\frac{\partial}{\partial \delta X}$$

Thus, we have

$$
\begin{aligned}
&MSM^T = S \Rightarrow \frac{d}{ds}\delta X = \frac{dM}{ds}M^{-1}\delta X + \frac{\partial \tilde{H}}{\partial \delta X} \\
&H = \tilde{H}\left(\tilde{X}(X,s),s\right) + \frac{1}{2}\delta X^T\left(\frac{dM}{ds}M^{-1}\right)\delta X^T
\end{aligned}
\tag{2}
$$

Thus, infinitesimal motion around any trajectory correspond to a Hamiltonian system. The only thing left to show that any smooth $X_o(s)$ is possible. It will correspond to the a simple linear s-dependent Hamiltonian

$$
\begin{aligned}
&H_o = -X^T S \frac{dX_o(s)}{ds}; \\
&\frac{dX}{ds} = S\frac{\partial H}{\partial X} = -S^2\frac{dX_o(s)}{ds} = \frac{dX_o(s)}{ds}
\end{aligned}
\tag{3}
$$

Expansion around the $X_o(s)$ adds quadratic terms (2) to the Hamiltonian.

Combination of (2) and (3) proves that any symplectic map corresponds to local canonical transformation from one Hamiltonian system to another. Since map is symplectic everywhere, we can expand this to the entire phase space.