PCA solenoids nonlinearity

VL

January 18, 2023

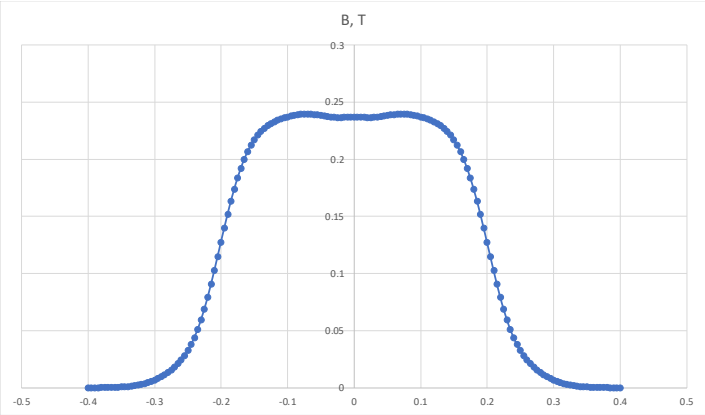
Background

- There was a lot of questions about emittance growth in electron beam propagating through PCA solenoids
- Specifically – there were statements that electron beam with space charge turned off – i.e. single particle ballistic propagation – experiences significant emittance growth because of nonlinear filed in PCA solenoids
- I developed an exact analytical Hamiltonian theory, which clearly separate linear part of the solenoids action (focusing and rotation) from non-linear effects till an arbitrary high order. It is fully coupled non-paraxial motion without any approximation or simplifications.
- I estimated all effects analytically and showed that emittance growth associated with all terms is negligible for CeC beam parameter. It would require ~ 10,000 larger beam emittance (i.e. 100 x beam size or angular spread) are needed to see measurable emittance growth ~ 2-fold
- I also wrote Mathematica tracking program with analytical expression for vector potential and tracking in the rotating frame with separation of the linear and non-linear (up to any desirable order) terms. Again - no measurable emittance growth for CeC beam parameters -

Hamiltonian

$$k(z) = -\frac{eB_s(s)}{2p_o c}$$

$$B_s = B_s(z)$$



$$p_{x,y,z} = \frac{P_z}{p_o}; a_{x,y,j} = \frac{e}{p_o c} A_{x,y,j}; a_x{}^2 + a_y{}^2 \equiv a_j^2$$

$$h = -p_z = -\sqrt{1 - (p_x - a_x)^2 + (p_y - a_y)^2} + ky \cdot p_x - k \cdot x \cdot p_y;$$

$$x' = \frac{\partial h}{\partial p_x} = \frac{p_x - a_x}{p_z} + ky; \quad y' = \frac{\partial h}{\partial p_y} = \frac{p_y - a_y}{p_z} - kx;$$

$$p_x{}' = -\frac{\partial h}{\partial x} = \frac{p_x - a_x}{p_z}\frac{\partial a_x}{\partial x} + \frac{p_y - a_y}{p_z}\frac{\partial a_y}{\partial x} + kp_y;$$

$$p_y{}' = -\frac{\partial h}{\partial y} = \frac{p_y - a_y}{p_z}\frac{\partial a_y}{\partial y} + \frac{p_x - a_x}{p_z}\frac{\partial a_x}{\partial y} - kp_x;$$

$$p_x{}' = \frac{p_x}{p_z}\frac{\partial a_x}{\partial x} + \frac{p_y}{p_z}\left(\frac{\partial a_y}{\partial x} + kp_z\right) - \frac{e^2}{2p_z{}^2c^2}\frac{\partial A_j^2}{\partial x};$$

$$p_y{}' = \frac{p_y}{p_z}\frac{sA_y}{\partial y} + \frac{p_y}{p_z}\left(\frac{\partial a_x}{\partial y} - kp_z\right) - \frac{e^2}{2p_z{}^2c^2}\frac{\partial A_j^2}{\partial y};$$

$$A_\varphi = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}B_s{}^{(2n)} \cdot \left(\frac{r}{2}\right)^{2n+1} = B_s(z) \cdot \frac{r}{2} + \delta A_\varphi; \quad B_s{}^{(n)} \equiv \frac{\partial^n B_s(z)}{\partial z^n};$$

$$\hat{e}_\varphi = \hat{y}\cos\varphi - \hat{x}\sin\varphi; \quad \vec{A} = \hat{e}_\varphi A_\varphi = \hat{y}A_\varphi\frac{x}{r} - \hat{x}A_\varphi\frac{y}{r};$$

$$A_x = -\frac{y}{2}\left(B_s(z) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!(n+1)!}B_s{}^{(2n)} \cdot \left(\frac{x^2+y^2}{4}\right)^{2n}\right);$$

$$A_y = \frac{x}{2}\left(B_s(z) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!(n+1)!}B_s{}^{(2n)} \cdot \left(\frac{x^2+y^2}{4}\right)^{2n}\right);$$

$$\vec{M} = \left[\vec{r} \times \vec{P}\right] \rightarrow M_z = x \cdot P_y - y \cdot P_x;$$

Estimations by an order of magnitude

$$X=\begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix};\frac{d}{dz}X=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -k^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k^2 & 0 \end{bmatrix}X+\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$$f_1=k''\frac{yr^2}{8}+\frac{(p_x-ky)^3}{2}+\frac{(p_x-ky)(p_y+kx)^2}{2}$$

$$f_2=k''\left(k\frac{xr^2}{2}+p_x\frac{xy}{4}+p_y\frac{3x^2+y^2}{8}\right)-\frac{k^2x}{2}\left((p_x-ky)^2+(p_y+kx)^2\right)$$

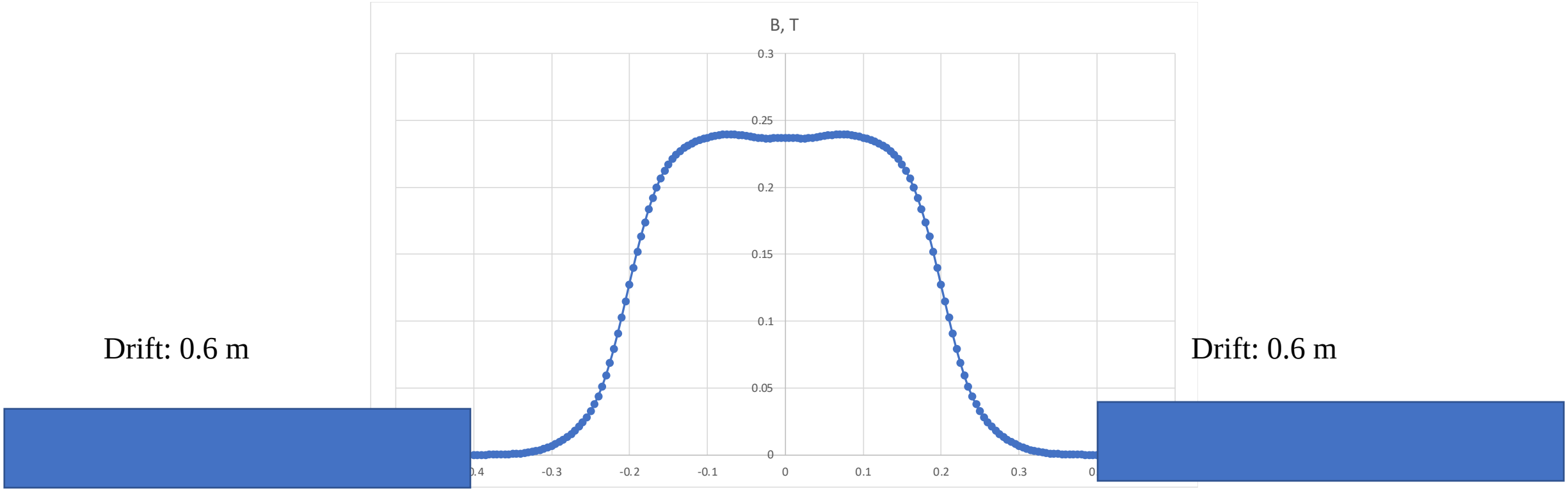
$$f_3=-k''\frac{xr^2}{8}+\frac{(p_y+kx)^3}{2}+\frac{(p_y+kx)(p_x-ky)^2}{2}$$

$$f_2=k''\left(k\frac{yr^2}{2}-p_y\frac{xy}{4}-p_x\frac{x^2+3y^2}{8}\right)-\frac{k^2y}{2}\left((p_x-ky)^2+(p_y+kx)^2\right)$$

σx	9.00E-04	m
σx'	7.31E-04	rad
Estimations		
k^4-κκ''	**47.43**	1/m^3
<x^4>	**3**	
r^2	**1**	
δε g	1.24E-10	m rad
δε n	3.55E-09	m rad
δε/ε	**2.84E-03**	
k^2	**2.18**	1/m
r'^2	**2**	
δε g	1.88E-12	m rad
δε n	5.37E-11	m rad
δε/ε	**4.30E-05**	

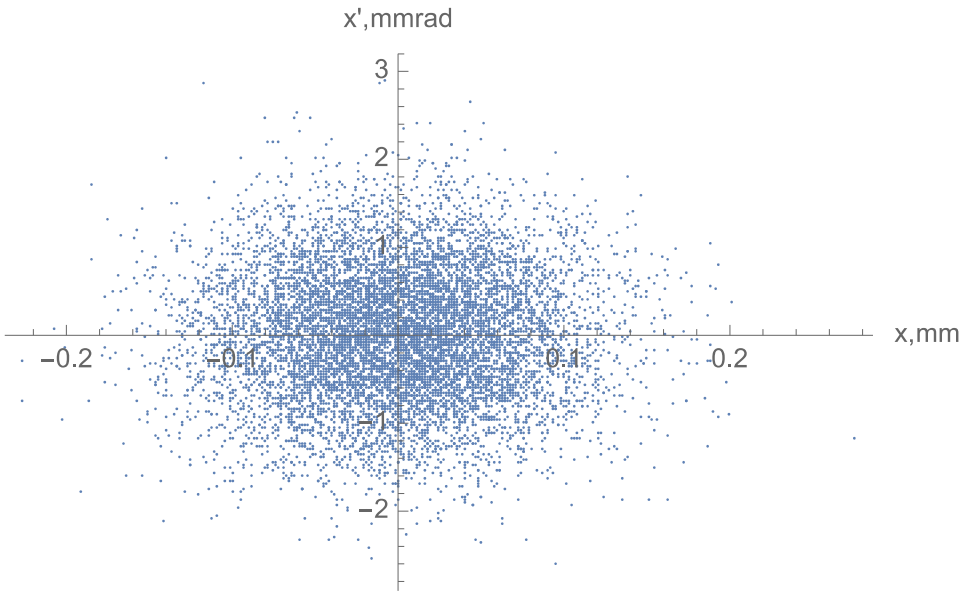
Angular Momentum		
k^3	**5.00**	
M	6.58E-07	
δε g	2.66E-12	m rad
δε n	7.59E-11	m rad
δε/ε	**6.07E-05**	

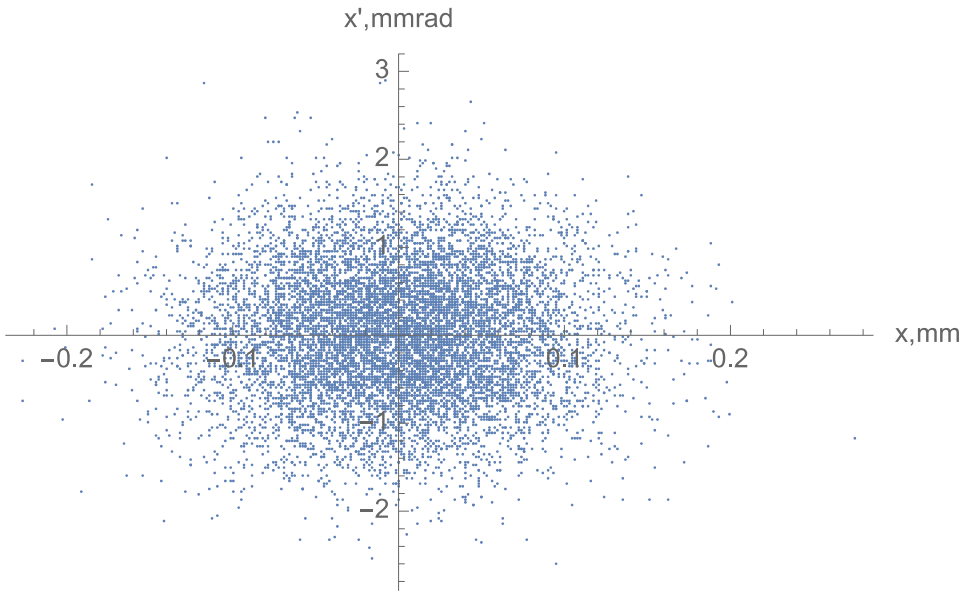
Simulations



Solenoid:±0.4 m

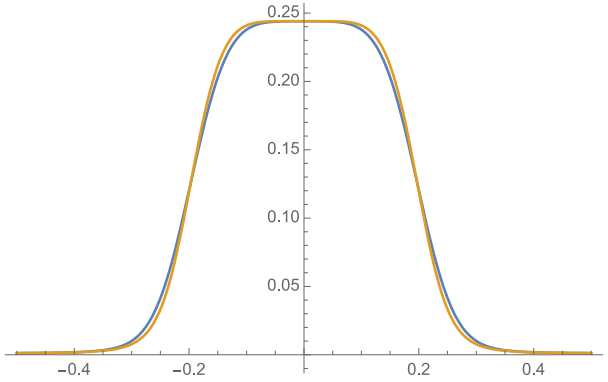
Our initial conditions: in the waist 1 m from the solenoid's center: RMS beam sizes are 60μm; RMS angular spreads are 0.74 mrad

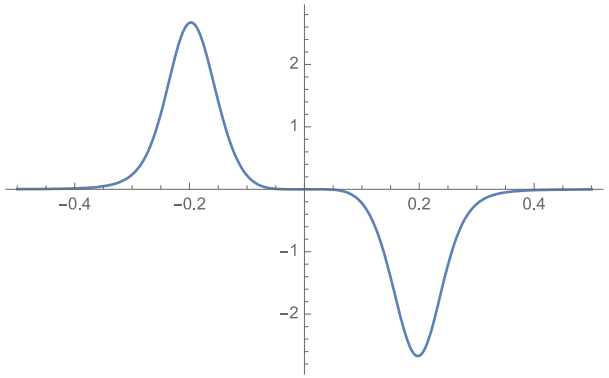


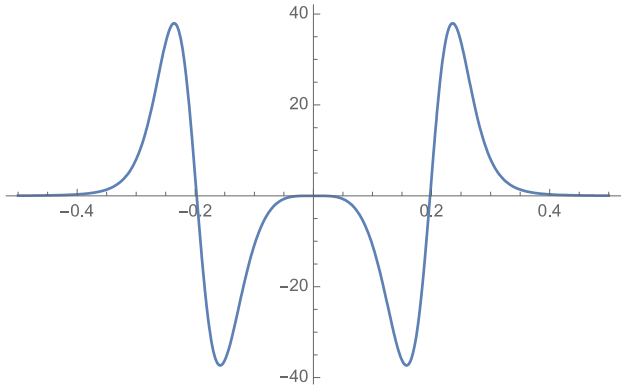


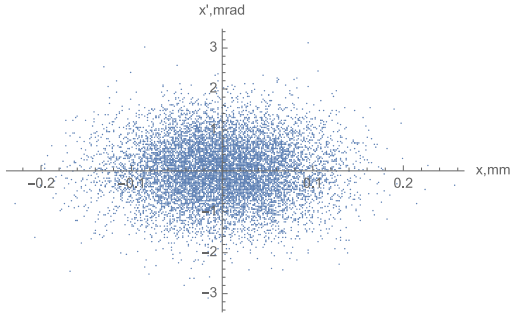
γ	εx, norm	β*,m	αx*	εy, norm	βx*	αx*
	mm mrad			mm mrad		
28.5	**1.25**	0.08	0	1.25	0.08	0

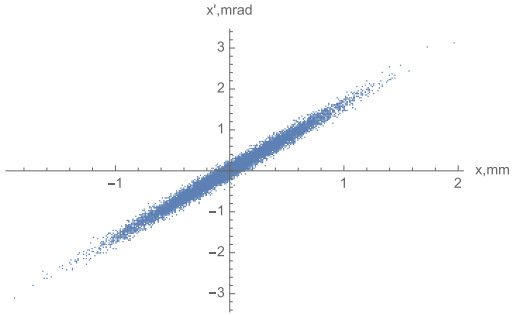
Standard setting: β^*=8 cm, ε_n=1.25 μm
Center of Solenoid: RMS Size 0.679265 mm

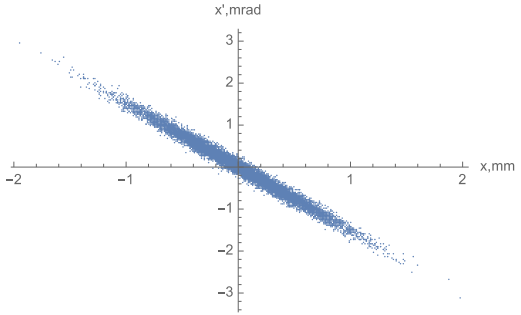


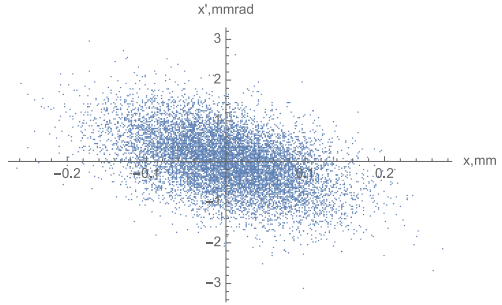


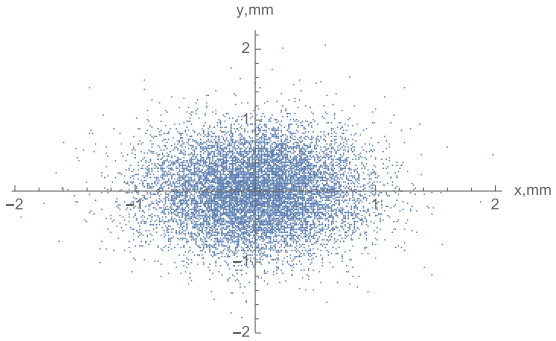


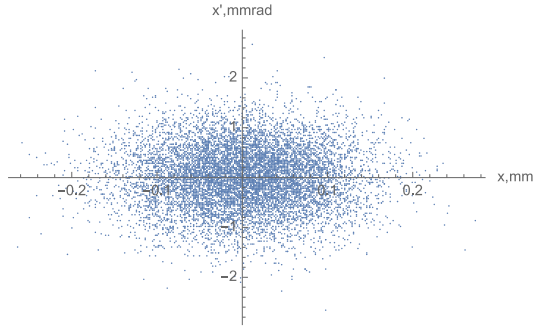








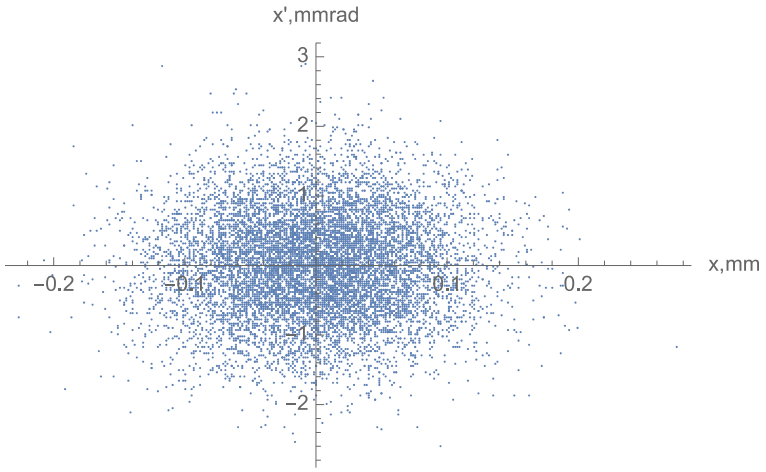




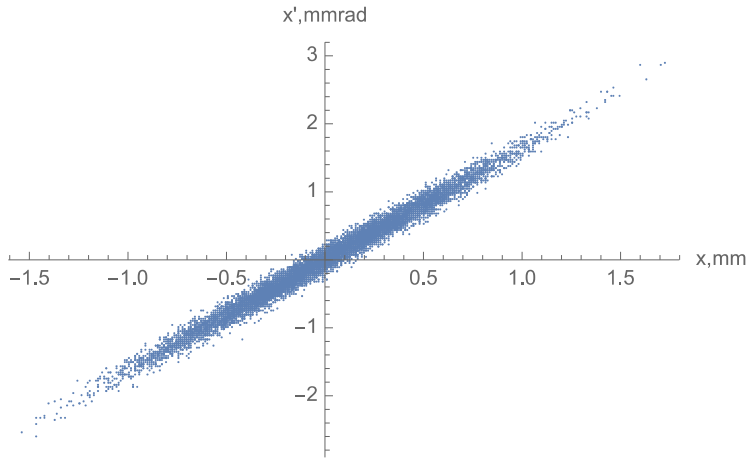
x: -3.68313*10^-6, y: +4.45739*10^-6

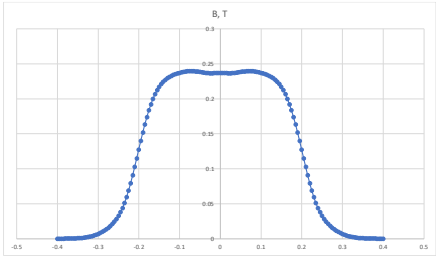
Nominal setting: change of emittances is <10-6

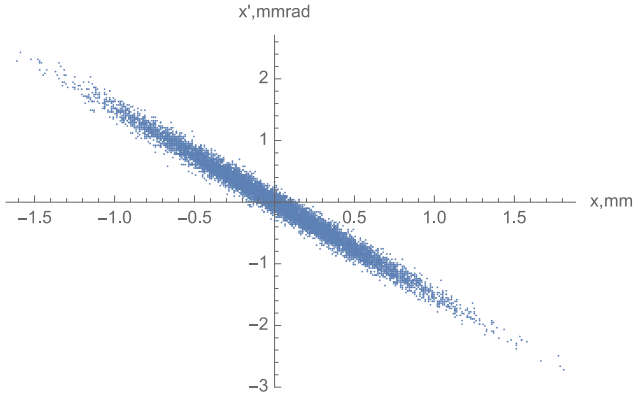
γ	εx, norm	β*,m	αx*	εy, norm	βx*	αx*	drifts, m	z,m	steps	I, A	npm
	mm mrad			mm mrad							
28.5	1.25	0.08	0	1.25	0.08	0	0.6	-0.4 to 0.4	100	121	10,000



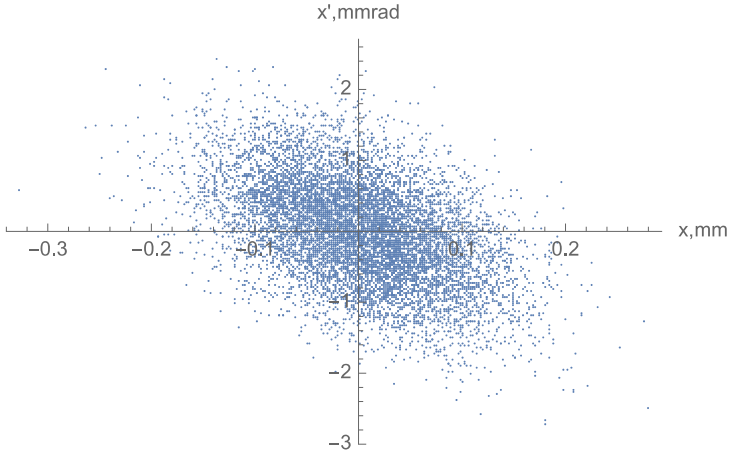
0.6 m





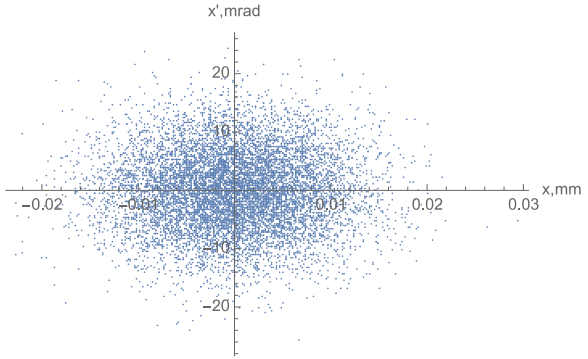


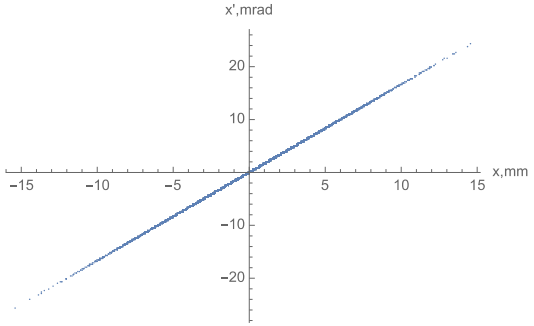
0.6 m

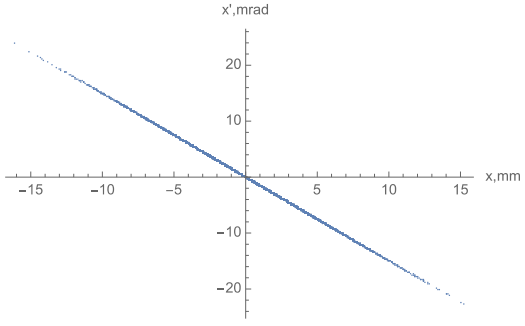


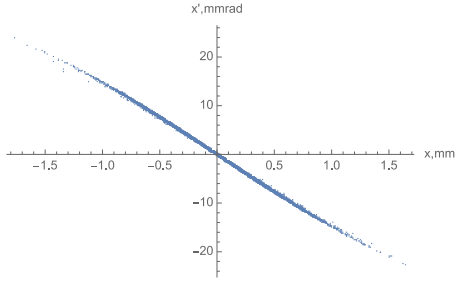
Abnormal setting: **β^*=0.1 cm**, ε_n=1.25 μm; **RMS Size 6.6 μm; RMS spread 6.7 mrad**

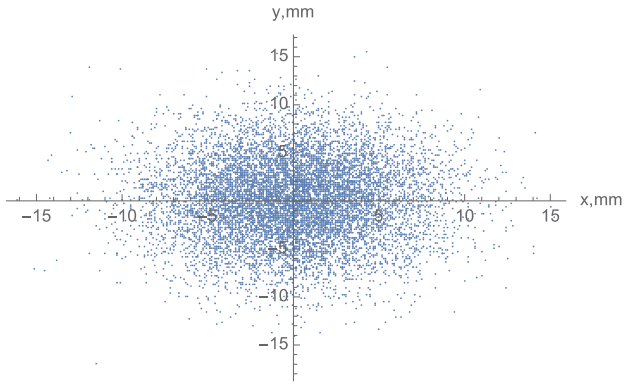
Center of Solenoid: RMS Size 6.7 mm

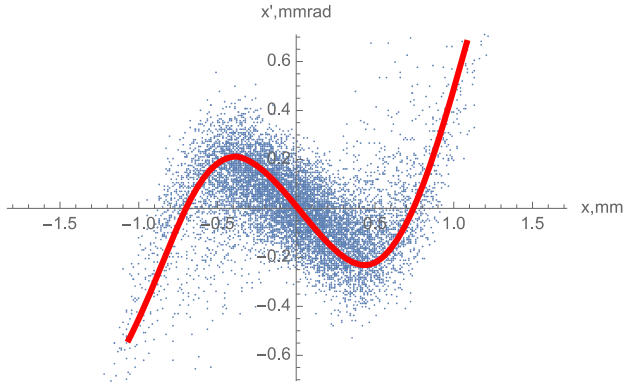








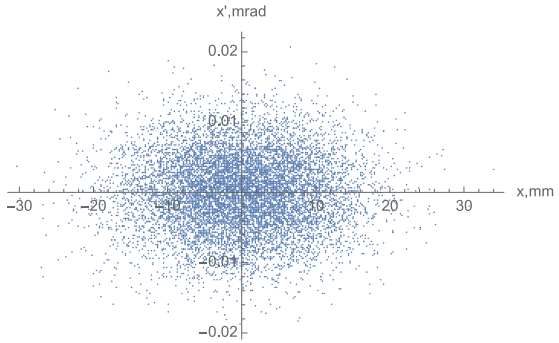


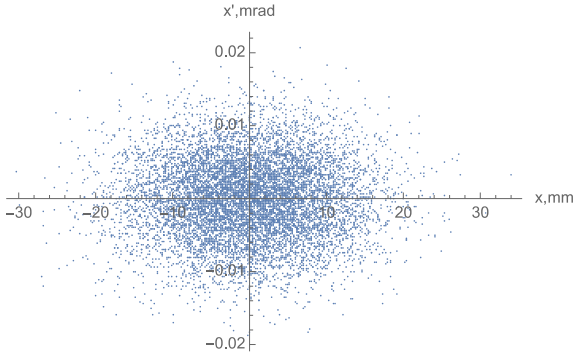


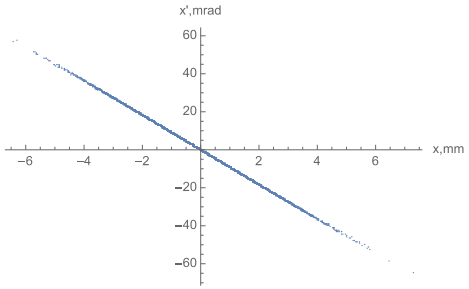
Emittance increased 2-fold

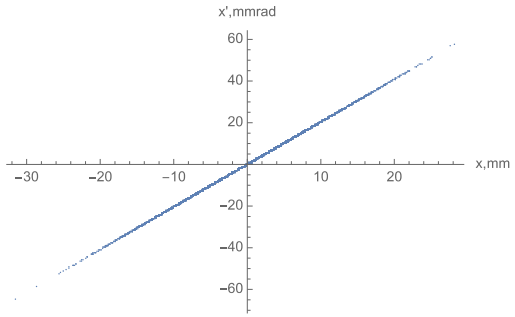
Abnormal setting: **βx^*=1,500 m, βy^*=1 mm** ε_n=1.25 μm; **RMS x 8.1 mm; RMS x" 0.005 mrad**

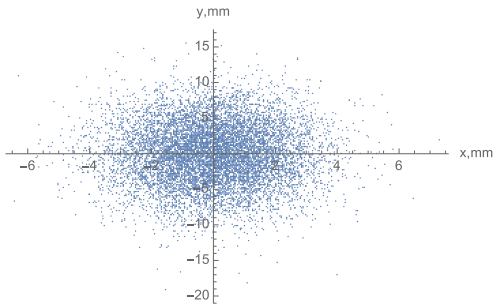
Center of Solenoid: RMS Size 7.3 mm

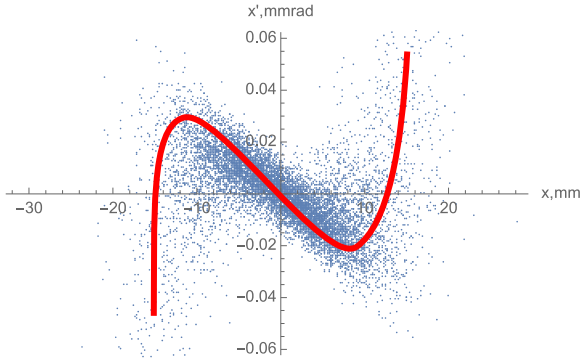










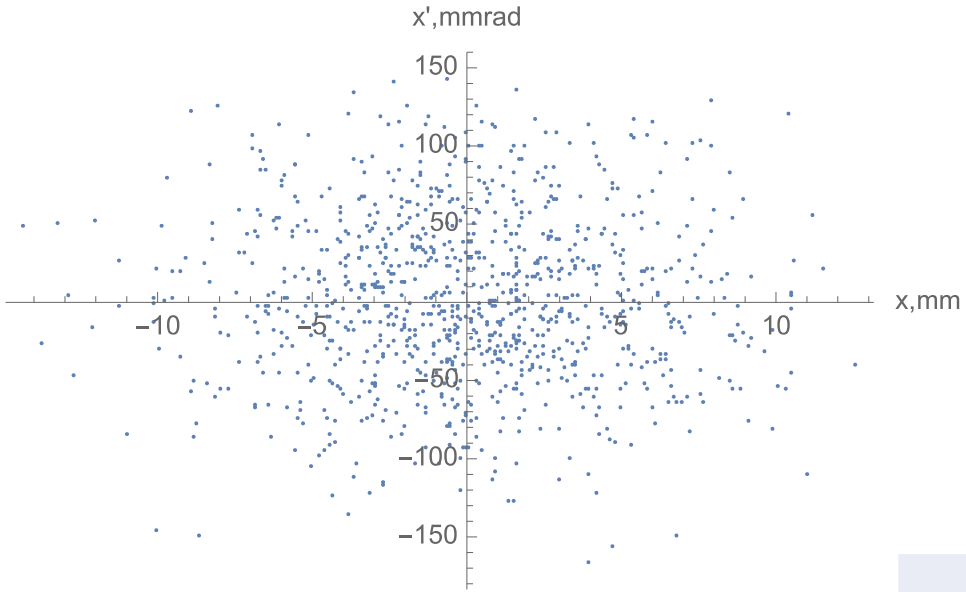


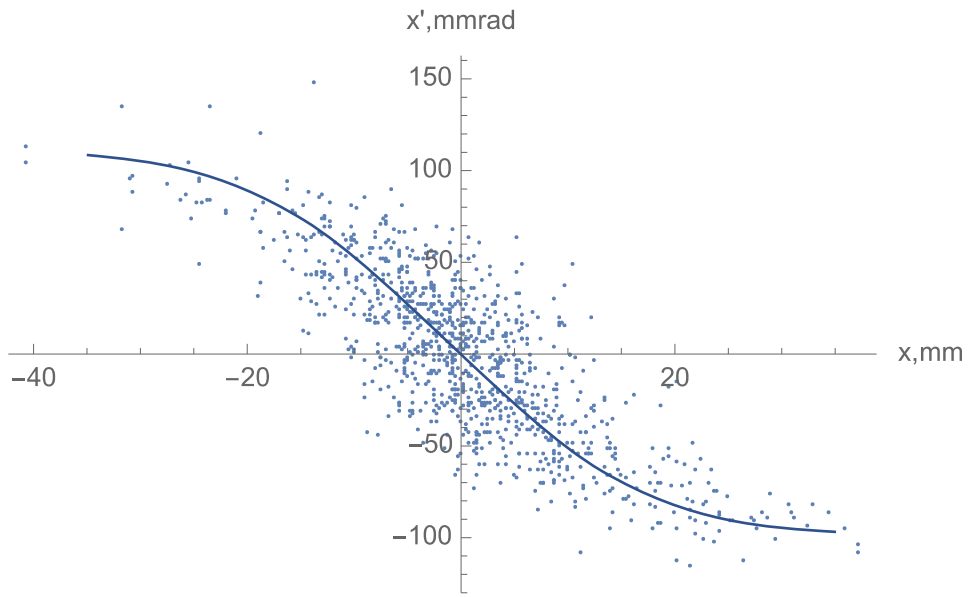
Emittance x increased 2.6-fold

Emittance y increased 4-fold

5,000 fold increase in emittance

γ	εx, norm	β*,m	αx*	εy, norm	βx*	αx*	drifts, m	z,m	steps	I, A	npm
	mm mrad			mm mrad							
28.5	6,250	0.08	0	6,250	0.08	0	0.6	-0.4 to 0.4	100	121	1000

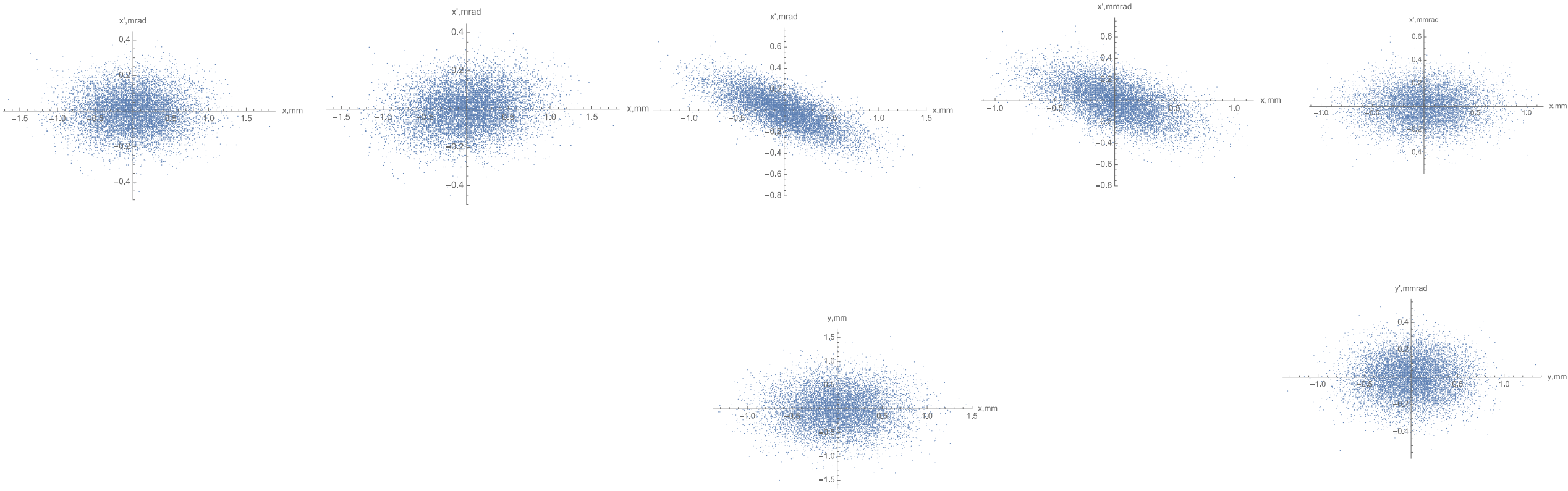




εxf/εx	εyf/εy	Relative change
1.16756	1.17312	0.16

Relaxed lattice I=50 A: **β^*=4 m,** ε_n=1.25 μm; **RMS size 0.42 mm; RMS spread 0.1 mrad**

Center of Solenoid: RMS Size 0.42 mm



No emittance increase at all: $\Delta\varepsilon/\varepsilon \sim 10^{-7}$

Summary

Test/Parameter	γ	εx, norm	β*,m	αx*	εy, norm	βy*	αy*	drifts, m	z,m	steps	I, A	npm	order	εxf/εx	εyf/εy	Maximum change
		mm mrad			mm mrad											εf/εi-1
1	28.5	**1.25**	**0.08**	0	1.25	**0.08**	0	0.6	-0.4 to 0.4	400	**121**	30000	5	0.999995	0.999999	-5.00E-06
2	28.5	**1250**	0.08	0	**1250**	0.08	0	0.6	-0.4 to 0.4	100	**121**	1000	5	1.02136	1.04093	4.00E-02
3	28.5	**6,250**	0.08	0	**6,250**	0.08	0	0.6	-0.4 to 0.4	100	**121**	1000	5	1.16756	1.17312	0.16
4	28.5	**1.25**	0.08	0	1.25	0.08	0	0.6	-0.4 to 0.4	100	**121**	1000	5	0.999995	0.999999	-5.00E-06
5	28.5	**1.25**	**1500**	0	**1.25**	**1500**	0	0.6	-0.4 to 0.4	200	**121**	10000	5	**2.6**	**2.6**	**1.6**
6	28.5	**1.25**	**15**	0	**1.25**	**15**	0	0.6	-0.4 to 0.4	200	**121**	10000	5	1	1	1.00E-05
7	28.5	**1.25**	**0.8**	0	**1.25**	**0.8**	0	0.6	-0.4 to 0.4	200	**121**	10000	5	1	1	1.50E-07
8	28.5	**1.25**	**0.08**	0	**1.25**	**0.08**	0	0.6	-0.4 to 0.4	400	**121**	30000	5	1	0.999996	-5.00E-06
9	28.5	**1.25**	**0.01**	0	**1.25**	**0.01**	0	0.6	-0.4 to 0.4	400	**121**	30000	5	1.0001	1.00012	1.00E-04
10	28.5	**1.25**	**0.08**	0	1.25	**0.08**	0	0.6	-0.4 to 0.4	800	**121**	10000	3	0.999997	1.00001	5.00E-06
11	28.5	**1.25**	**0.001**	0	**1.25**	**0.001**	0	0.6	-0.4 to 0.4	800	**121**	10000	3	**2.03**	**1.99**	**1.0**
12	28.5	**1.25**	**1500**	0	**1.25**	**0.001**	0	0.6	-0.4 to 0.4	800	**121**	10000	3	**4**	**2.5**	**3.0**
13	28.5	**1.25**	**4**	0	**1.25**	**4**	0	0.6	-0.4 to 0.4	800	**50**	10000	3	**1**	**1**	2.00E-07

Conclusions

- For the CeC beam parameters, there is no measurable emittance growth associated with nonlinearity of solenoidal focusing or, for the sake of the completeness, from the finite angular spread ~ 1mrad.
- It is correct for PCA settings of the solenoids
- Emittance growth is proportional to square of the solenoidal fields, it means that there is nothing at all for relaxed lattice
- Huge emittance or 100-fold mismatch is needed to generate mesurable emittance growth.