PHY554: Solution of Homework 2

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HW 1 (4 pts). Magnet kicker (dipole)

Using the transfer matrix M to show that, when a particle is kicked at s1 by angle θ, the displacement at a downstream location s2 is

$$\Delta x_2 = \theta\sqrt{\beta_1\beta_2} sin\mu,$$

Where β_1 and β_2 are values of betatron functions at s1 and s2 respectively, and μ is the betatron phase advance between s1 and s2. The quantity $\sqrt{\beta_1\beta_2} sin\mu$ is usually called the kicking arm. In the scenario of designing a magnet kicker (which kicks the beam for injection/extraction or other orbit change), to obtain the maximum kick (or minimum kicker strength), what are the requirements for choosing the kicker location?

HW 1 Answer:
In general:

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

Where the transfer matrix :

$$M(s_2, s_1) = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

Therefore, we can express $x(s_2)$ as:

$$x(s_2) = \sqrt{\frac{\beta_2}{\beta_1}} (\cos\mu + \alpha_1 \sin\mu) \cdot x(s_1) + \sqrt{\beta_1\beta_2} \sin\mu \cdot x'(s_1)$$

Since now we added a kick θ at s_1:

$$x(s_2) + \Delta x_2 = \sqrt{\frac{\beta_2}{\beta_1}} (\cos\mu + \alpha_1 \sin\mu) \cdot x(s_1) + \sqrt{\beta_1\beta_2} \sin\mu \cdot (x'(s_1) + \theta)$$

Therefore

$$\Delta x_2 = \theta\sqrt{\beta_1\beta_2}\sin\mu$$

Δx_2 is proportion to $\sqrt{\beta_1}$, and β_1 is the β function at the kicker location.
To obtain the maximum kicker strength, the kicker should be located in the position where β function reaches maximum. To obtain the minimum kicker strength, the kicker should be located in the position where β function reaches minimum.

HW 2 (6 pts). FODO cells

An accelerator is made of 20 FODO cells with circumference of 200 m. The betatron tunes (phase advance per revolution divided by 2π) Qx/Qy are 5/4.8 respectively. What are the maximum/minimum betatron functions (x and y) and where are they located at?

Given the RMS beam emittance ε is 2 mm-mrad, what is the minimum vacuum chamber size to house such beam without losing particles.

HW 2 Answer:
Maximum betatron functions are located at center of QFs, so a FODO cell is arranged as:

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-\frac{L_1^2}{2f^2} & 2L_1\left(1+\frac{L_1}{2f}\right) \\ -\frac{L_1}{2f^2}\left(1-\frac{L_1}{2f}\right) & 1-\frac{L_1^2}{2f^2} \end{pmatrix} \end{aligned}$$

Since M can also be written as:

$$M = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

we can solve that:

$$\begin{aligned} \cos\Phi &= \frac{1}{2}Tr\left(M\right) \\ &= 1 - \frac{L_1^2}{2f^2} \end{aligned}$$

Also

$$\cos\Phi = 1 - 2\sin^2\frac{\Phi}{2}$$
$$\sin\frac{\Phi}{2} = \frac{L_1}{2f}$$

Therefore

$$\alpha = 0$$
$$\beta = \frac{2L_1\left(1+\sin\frac{\Phi}{2}\right)}{\sin\Phi}$$

In our question, number of FODO cells is $n_{FODO} = 20$, circumference is $L = 200m$, so $L1 = L/n_{FODO}/2 = 5m$. The betatron tunes are $Qx = 5; Qy = 4.8$, so the phase advance for each FODO cell should be

$$\begin{aligned}\Phi_x &= \frac{2\pi Q_x}{n_{FODO}} \\ &= \frac{\pi}{2} \\ \Phi_y &= \frac{2\pi Q_y}{n_{FODO}} \\ &= \frac{12\pi}{25}\end{aligned}$$

Therefore

$$\begin{aligned}\beta_{x,max} &= \frac{2L_1\left(1+\sin\frac{\Phi_x}{2}\right)}{\sin\Phi_x} \\ &= 17.07m \\ \beta_{y,max} &= \frac{2L_1\left(1+\sin\frac{\Phi_y}{2}\right)}{\sin\Phi_y} \\ &= 16.88m\end{aligned}$$

Similarly, minimum betatron functions are located at center of DFs, so a FODO cell is arranged as:

$$\begin{aligned}M &= \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-\frac{L_1^2}{2f^2} & 2L_1\left(1-\frac{L_1}{2f}\right) \\ -\frac{L_1}{2f^2}\left(1+\frac{L_1}{2f}\right) & 1-\frac{L_1^2}{2f^2} \end{pmatrix}\end{aligned}$$

Therefore

$$\begin{aligned}\beta_{x,min} &= \frac{2L_1\left(1-\sin\frac{\Phi_x}{2}\right)}{\sin\Phi_x} \\ &= 2.93m \\ \beta_{y,min} &= \frac{2L_1\left(1-\sin\frac{\Phi_y}{2}\right)}{\sin\Phi_y} \\ &= 3.16m\end{aligned}$$

Now we have the maximum of betatron functions and RMS beam emittance, we can calculate the RMS beam size

$$\begin{aligned}\sigma_x &= \sqrt{\varepsilon\beta_{x,max}}\\ &= \sqrt{2e-6\times 17.07}\\ &= 5.84mm\\ \sigma_y &= \sqrt{\varepsilon\beta_{y,max}}\\ &= \sqrt{2e-6\times 16.88}\\ &= 5.81mm\end{aligned}$$

As we know, in a 1D normal distribution, integral of density function from -8σ to 8σ will be more than 99%. So if we take $8\times max(\sigma_x;\sigma_y)$ as the chamber radius, the vacuum chamber will be large enough to house such beam. So the vacuum chamber size should be at least 46.75mm in radius or 93.49mm in diameter.

HW 3 (10 point): localized orbit correction

The closed orbit can be locally corrected by using steering dipoles. A commonly used algorithm is based on the "three-bumps'' method, where three steering dipoles are used to adjust local-orbit distortion.

Let θ_1 , θ_2 and θ_3 be the three bump angles. For the orbit distortion to be localized between first and third dipoles, show that these angles must be related by

$$\theta_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\frac{\sin\psi_{31}}{\sin\psi_{32}}, \qquad \theta_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\frac{\sin\psi_{21}}{\sin\psi_{32}},$$

where β_1, β_2 and β_3 are the beta functions at local bumps and ψ_{ij} is the phase advance between ith and jth steering dipoles.

Show under what condition, the "three-bumps" method can become "two-bumps" method, i.e., only two steering dipoles are used for local orbit distortion.

HW 3 Answer:
Each bump is considered as a delta angle kick along the in the ring. It's effect is described by the Greens function $G(s, s_0 = s_i)$ and the kick angle θ_i:

$$G(s,s_0)=\frac{\sqrt{\beta\left(s\right)\beta\left(s_0\right)}}{2\sin\pi\nu}\cos\left(\psi\left(s\right)-\psi\left(s_0\right)-\pi\nu\right)$$

Therefore we have:

$$x_{co}\left(s\right)\propto\sum_{i=1}^{3}\beta^{1/2}\left(s_i\right)\theta_i\cos\left(\psi\left(s\right)-\psi\left(s_i\right)-\pi\nu\right)$$

We demand that $x_{co}\left(s_1^-\right)=x\left(s_3^+\right)=0$.

$$\sum_{i=1}^{3}\beta_i^{1/2}\theta_i\cos\left(\psi\left(s_1+C\right)-\psi\left(s_i\right)+\pi\nu\right)=0$$

$$\sum_{i=1}^{3}\beta_i^{1/2}\theta_i\cos\left(\psi\left(s_3\right)-\psi\left(s_i\right)-\pi\nu\right)=0$$

$$\sqrt{\beta_1}\theta_1\cos\left(\pi\nu\right)+\sqrt{\beta_2}\theta_2\cos\left(-\psi_{21}+\pi\nu\right)+\sqrt{\beta_3}\theta_3\cos\left(-\psi_{31}+\pi\nu\right)=0$$

$$\sqrt{\beta_1}\theta_1\cos\left(\psi_{31}-\pi\nu\right)+\sqrt{\beta_2}\theta_2\cos\left(\psi_{32}-\pi\nu\right)+\sqrt{\beta_3}\theta_3\cos\left(\pi\nu\right)=0$$

We know that

$$\begin{aligned}\det\begin{pmatrix}\cos\left(-\psi_{21}+\pi\nu\right) & \cos\left(-\psi_{31}+\pi\nu\right)\\ \cos\left(\psi_{32}-\pi\nu\right) & \cos\left(\pi\nu\right)\end{pmatrix} &= \cos\left(\pi\nu\right)\cos\left(\psi_{32}-\psi_{31}+\pi\nu\right)\\ &\quad-\cos\left(-\psi_{32}+\pi\nu\right)\cos\left(-\psi_{31}+\pi\nu\right)\\ &=\frac{1}{2}\left[\cos\left(2\pi\nu+\psi_{32}-\psi_{31}\right)-\cos\left(2\pi\nu-\psi_{32}-\psi_{31}\right)\right]\\ &=-\sin\left(2\pi\nu-\psi_{31}\right)\sin\left(\psi_{32}\right)\end{aligned}$$

Then we can get:

$$\begin{aligned}\begin{pmatrix}\sqrt{\beta_2}\theta_2\\ \sqrt{\beta_3}\theta_3\end{pmatrix} &= \frac{\sqrt{\beta_1}\theta_1}{\sin\left(2\pi\nu-\psi_{31}\right)\sin\left(\psi_{32}\right)}\begin{pmatrix}\cos\left(\pi\nu\right) & -\cos\left(-\psi_{31}+\pi\nu\right)\\ -\cos\left(\psi_{32}-\pi\nu\right) & \cos\left(-\psi_{21}+\pi\nu\right)\end{pmatrix}\begin{pmatrix}\cos\left(\pi\nu\right)\\ \cos\left(\psi_{31}-\pi\nu\right)\end{pmatrix}\\ &=\frac{\sqrt{\beta_1}\theta_1}{\sin\left(2\pi\nu-\psi_{31}\right)\sin\left(\psi_{32}\right)}\begin{pmatrix}\cos^2\left(\pi\nu\right)-\cos^2\left(-\psi_{31}+\pi\nu\right)\\ -\cos\left(\pi\nu\right)\cos\left(\psi_{32}-\pi\nu\right)+\cos\left(-\psi_{21}+\pi\nu\right)\cos\left(\psi_{31}-\pi\nu\right)\end{pmatrix}\\ &=\frac{\sqrt{\beta_1}\theta_1}{\sin\left(2\pi\nu-\psi_{31}\right)\sin\left(\psi_{32}\right)}\begin{pmatrix}\cos\left(2\pi\nu\right)/2-\cos\left(-2\psi_{31}+2\pi\nu\right)/2\\ -\cos\left(2\pi\nu-\psi_{32}\right)/2+\cos\left(2\pi\nu-\psi_{21}-\psi_{31}\right)/2\end{pmatrix}\\ &=\frac{\sqrt{\beta_1}\theta_1}{\sin\left(2\pi\nu-\psi_{31}\right)\sin\left(\psi_{32}\right)}\begin{pmatrix}-\sin\psi_{31}\sin\left(2\pi\nu-\psi_{31}\right)\\ \sin\psi_{21}\sin\left(2\pi\nu-\psi_{31}\right)\end{pmatrix}\\ &=\frac{\sqrt{\beta_1}\theta_1}{\sin\left(\psi_{32}\right)}\begin{pmatrix}-\sin\psi_{31}\\ \sin\psi_{21}\end{pmatrix}\end{aligned}$$

When $\psi_{31}=n\pi$, we find $\theta_2=0$, i.e. only two steering dipoles are needed for a local bump. Since $\psi_{32}=\psi_{31}-\psi_{21}=n\pi-\psi_{21}$, we have $\sin\psi_{32}=(-1)^{n-1}\sin\psi_{31}$, and $\theta_3=(-1)^{n-1}\sqrt{\beta_1/\beta_2}\theta_1$.