Solutions for HW # 20

Problem 1:

According to the footnote in Jackson, the delta function in the coordinate system (x_1, x_2, x_3) is related to the coordinate system (ξ_1, ξ_2, ξ_3) by the following relation:

$$\delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3 - x_3') = \frac{1}{|J(x_i, \xi_i)|}\delta(\xi_1 - \xi_1')\delta(\xi_2 - \xi_2')\delta(\xi_3 - \xi_3'), \tag{1}$$

where

$$J(x_i, \xi_i) = \begin{pmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{pmatrix}, \tag{2}$$

is the Jacobian, (x_1', x_2', x_3') and (ξ_1', ξ_2', ξ_3') are coordinates of the singular point at the two systems respectively. The cylindrical coordinate system (x, y, z) is related to the Cartesian coordinate system (r, θ, z) by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \quad . \tag{3}$$

Inserting eq. (3) into eq. (2) yields

$$J = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

and hence eq. (1) becomes

$$\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) = \frac{1}{r}\delta(r - r_0)\delta(\theta - \theta_0)\delta(z - z_0) \ . \tag{5}$$

Without losing generality, we can pick the Cartesian coordinates system such that, for an electron moving along z axis, its trajectory is given by

$$\begin{aligned} x_0 &= a \\ y_0 &= 0 \quad , \\ z_0 &= vt \end{aligned} \tag{6}$$

and consequently, its charge density is given by inserting eq. (6) into eq. (5):

$$\rho(\vec{x}) = Q\delta(\vec{x}) = Q\frac{1}{r}\delta(r-a)\delta(\theta)\delta(z-vt). \tag{7}$$

As the cylindrical coordinate is not uniquely defined with respect to the azimuthal angle θ, we can write eq. (7) into the following form

$$\rho(r,\theta,z) = Q\frac{1}{r}\delta(r-a)\delta(z-vt)\sum_{n=-\infty}^{\infty}\delta(\theta-2n\pi). \tag{8}$$

Eq. (8) suggests that $\rho(r,\theta,z)$ is a periodic even function in θ with a period of 2π and hence it can be expressed as a summation of its Fourier components:

$$\rho(r,\theta,z) = Q\frac{1}{r}\delta(r-a)\delta(z-vt)\sum_{k=-\infty}^{\infty}a_k\exp(ik\theta) \tag{9}$$

where

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}\delta(\theta-2\pi n)\exp(-ik\theta)\,d\theta = \frac{1}{2\pi} \quad . \tag{10}$$

Inserting eq. (10) into eq. (9) yields

$$\begin{aligned} \rho(r,\theta,z) &= Q\frac{1}{\pi r}\delta(r-a)\delta(z-vt)\left\{\frac{1}{2}+\sum_{m=1}^{\infty}\cos(m\theta)\right\} \\ &= Q\frac{1}{\pi r}\delta(r-a)\delta(z-vt)\frac{1}{1+\delta_{m,0}}\sum_{m=0}^{\infty}\cos(m\theta) \ , \\ &= \sum_{m=0}^{\infty}\frac{Q_m\cos(m\theta)}{\pi a^{m+1}\left(1+\delta_{m,0}\right)}\delta(r-a)\delta(z-vt) \end{aligned} \tag{11}$$

where for the last step, we used the fact that $\delta(r-a)$ implies that r always take value at a.

Problem 2:

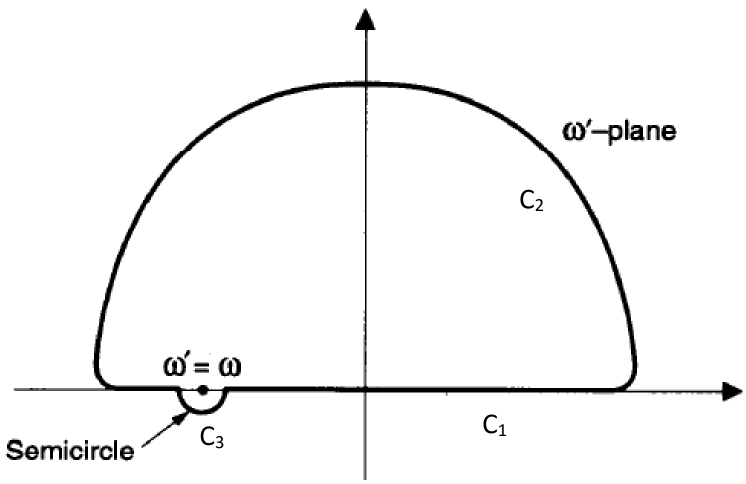

Figure 1: Integration contour in complex ω' plane.

From Cauchy residue theorem, the contour integral can be calculated as

$$\int_C \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' = 2\pi i Z_{//}(\omega) \ . \tag{12}$$

The LHS of (12) can be split into the following form

$$\begin{aligned}\int_C \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' &= \int_{C_1} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' + \int_{C_2} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' + \int_{C_3} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' \\ &= P.V. \int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' + 0 + \int_{e^{i\pi}}^{e^{i2\pi}} \frac{Z_{//}(\omega)}{e^{i\theta}} de^{i\theta} \qquad , \\ &= P.V. \int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' + 0 + i\pi Z_{//}(\omega)\end{aligned} \tag{13}$$

where the integral along C_2 vanishes since we assume $Z_{//}(\omega')$ is well behaved at large $|\omega'|$. From eq. (12) and (13), it follows

$$Z_{//}(\omega) = -\frac{i}{\pi} P.V. \int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' \ . \tag{14}$$

Splitting eq. (14) into the real and imaginary part leads to

$$\mathrm{Re}\left[Z_{//}(\omega)\right] = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\mathrm{Im}\left[Z_{//}(\omega')\right]}{\omega' - \omega} d\omega' \ , \tag{15}$$

and

$$\mathrm{Im}\left[Z_{//}(\omega)\right] = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\mathrm{Re}\left[Z_{//}(\omega')\right]}{\omega' - \omega} d\omega' \ . \tag{16}$$