Homework 15. Due November 2

Problem 1. 20 points. A weak transverse coupling.
***** STAR part - 50 points**

Consider a fully uncoupled x and y betatron motion in a storage ring with circumference C

$$\tilde{h}_o = \frac{\pi_1^2 + \pi_3^2}{2} + f(s)\frac{x^2}{2} + g(s)\frac{y^2}{2}$$

described by eigen vectors:

$$\mu_{x,y} = 2\pi Q_{x,y};\ \ Y_x(s) = \begin{bmatrix} \mathrm{w}_x \\ \mathrm{w}'_x + \dfrac{i}{\mathrm{w}_x} \\ 0 \\ 0 \end{bmatrix}; Y_y(s) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{w}_y \\ \mathrm{w}'_y + \dfrac{i}{\mathrm{w}_y} \end{bmatrix}$$

The eigen vectors and tunes are considered to be known. Introduce a week coupling by SQ-quadrupole and solenoidal fields (for torsion equal zero):

$$\delta\tilde{h} = \delta f \frac{x^2}{2} + \delta n \cdot xy + \delta g \frac{y^2}{2} + \delta L(x\pi_3 - y\pi_1)$$

with

$$\delta n(s) = \frac{e}{2p_o c}\left[\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\right]; \delta L(s) = \frac{e\delta B_s}{2p_o c}; \delta f(s) = \delta g(s) = \delta L^2(s);$$

(a) Write explicitly expressions for new betatron tunes using our developed perturbation method. Show that there is linear term on $\delta n, \delta L$ only in case of coupling resonance when $\mu_x = \pm\mu_y + 2\pi m$.

(b) For the case $\mu_x \neq \mu_y$ write expressions for new Eigen vectors perturbation method developed in class. Normalize them symplecticly.

STAR Part:

(*) Using new eigen vectors and substituting them into the Hamiltonian, find tune change expression on next order of $\delta n, \delta L$ for the case $\mu_x \neq \mu_y$. It is fine if it is just an integral.

(**)What should we do with eigen vectors at coupling resonance, when $\mu_x = \pm\mu_y + 2\pi m$.
Hint – look what is done with two energy degenerated levels in quantum mechanics.

Solution - use perturbation theory from lecture 18, **Sample III:**
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Here is a short re-collection:

$$\frac{dX}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right) \cdot X = \left(\mathbf{SH}(s) + \varepsilon \mathbf{SH}_1(s)\right) \cdot X$$

$$\frac{d\tilde{Y}_k(s)}{ds} = \mathbf{D}(s)\tilde{Y}_k(s); k = 1,..,n.$$

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k}\left(a_{kj}\tilde{Y}_j + b_{kj}\tilde{Y}^*_j\right) + O(\varepsilon^2);\ k = 1,...,n$$

$$\tilde{Y}_{1k}^{\,*} = \tilde{Y}_k^{\,*} e^{-i\delta\phi_k} + \varepsilon c^*_k \tilde{Y}_k + \varepsilon \sum_{j\neq k}\left(a^*_{kj}\tilde{Y}^*_j + b^*_{kj}\tilde{Y}_j\right) + O(\varepsilon^2);$$

$$\frac{d\tilde{Y}_{1k}}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right) \cdot \tilde{Y}_{1k} + o(\varepsilon^2);$$

leads to

$$\delta\phi'_k \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c'_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k}\left(a'_{kj}\tilde{Y}_j + b'_{kj}\tilde{Y}^*_j\right) = \varepsilon \mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k}$$

and symplectic orthogonality of the eigen vectors

$$\tilde{Y}_k^{\,*} S \tilde{Y}_j = -\tilde{Y}_k S \tilde{Y}^*_j = 2i\delta_{ik};\ \tilde{Y}_k S \tilde{Y}_j = \tilde{Y}_k^{\,*} S \tilde{Y}_j^{\,*} = 0$$

multiplying by $\tilde{Y}_m^{\,*} S$ or $\tilde{Y}_m S$ from the left yields:

$$-2\delta\phi'_k = \varepsilon \tilde{Y}^*_k \mathbf{SD}_1(s)\tilde{Y}_k \rightarrow \delta\phi' = \frac{\varepsilon}{2} Y_k^{\,*T}\mathbf{H}_1(s)Y_k;\quad \mathbf{SD}_1 = -\mathbf{H}_1;$$

$$-2ic' = \tilde{Y}^T_k S\mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \rightarrow c' = \frac{1}{2i} Y_k^{\,T}\mathbf{H}_1(s)Y_k e^{i(2\psi_k + \delta\phi_k)} \cong \frac{1}{2i} Y^T{}_k \mathbf{H}_1 Y_k e^{2i\psi_k}$$

$$2ia'_{kj} = \tilde{Y}^*_j \mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \rightarrow a'_{kj} = \frac{-1}{2i} Y_j^{\,*T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k - \psi_j + \delta\phi_k\right)} \cong \frac{-1}{2i} Y_j^{\,*T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k - \psi_j\right)};\ j \neq k$$

$$-2ib'_{kj} = \tilde{Y}^*_j \mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \rightarrow b'_{kj} = \frac{1}{2i} Y_j^{\,T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k + \psi_j + \delta\phi_k\right)} \cong \frac{1}{2i} Y_j^{\,T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k + \psi_j\right)};\ j \neq k.$$

$$\delta\phi(s) = \phi_o + \frac{\varepsilon}{2}\int_0^s Y_k^{\,*T}\mathbf{H}_1 Y_k\, d\xi;\ c(s) = c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\,T}\mathbf{H}_1 Y_k e^{i(2\psi_k + \delta\phi_k)};$$

$$a_{kj} = a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\,*T}\mathbf{H}_1 Y_k e^{i\left(\psi_k - \psi_j + \delta\phi_k\right)}; b_{kj} = b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\,T}\mathbf{H}_1 Y_k e^{i\left(\psi_k + \psi_j + \delta\phi_k\right)};$$

$$\tilde{Y}_{1k} e^{-i\left(\psi_k + \delta\phi_k\right)} = Y_k + \varepsilon c_k Y_k^{\,*} e^{-i(2\psi_k + \delta\phi_k)}\left(c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\,T}\mathbf{H}_1 Y_k e^{i(2\psi_k + \delta\phi_k)}\right) +$$

$$\varepsilon \sum_{j\neq k}\left(\begin{array}{l} Y_j e^{-i\left(\psi_k - \psi_j + \delta\phi_k\right)}\left(a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\,*T}\mathbf{H}_1 Y_k e^{i\left(\psi_k - \psi_j + \delta\phi_k\right)}\right) + \\ Y^*{}_j e^{-i\left(\psi_k + \psi_j + \delta\phi_k\right)}\left(b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\,T}\mathbf{H}_1 Y_k e^{i\left(\psi_k + \psi_j + \delta\phi_k\right)}\right) \end{array}\right) + O(\varepsilon^2)$$

Now we want to have periodic eigen vectors, e.g.

$$\tilde{Y}_{1k}(s+C)=\tilde{Y}_{1k}(s)e^{i\mu_{1k}};\mu_{1k}=\mu_k+\frac{\varepsilon}{2}\int_0^C Y_k^{*T}\mathbf{H}_1 Y_k\,d\xi;$$

we need to choose the initial conditions to make a coefficient looking like:

$$d(s)=e^{-i\theta(s)}\left(d_o-\frac{1}{2i}\int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right);\rightarrow\left(d_o+\int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right)=\frac{1}{e^{i\Delta\theta(C)}-1}\int_o^{s+C}d\xi f(\xi)e^{i\theta(\xi)}.$$

giving us:

$$\tilde{Y}_{1k}e^{-i(\psi_k+\delta\phi_k)}=Y_{1k}(s)=Y_k+\varepsilon\frac{Y_k^{*}e^{-i(2\psi_k+\delta\phi_k)}}{2i\left(1-e^{i(2\mu_k+\delta\mu_k)}\right)}\int_s^{s+C}d\xi Y_k^{T}\mathbf{H}_1Y_ke^{i(2\psi_k+\delta\phi_k)}+$$

$$\varepsilon\sum_{j\neq k}\left(\begin{array}{l}-\dfrac{Y_je^{i(\psi_j-\psi_k-\delta\phi_k)}}{2i\left(1-e^{i(\mu_k-\mu_j+\delta\mu_k)}\right)}\displaystyle\int_s^{s+C}d\xi Y_j^{*T}\mathbf{H}_1Y_ke^{i(\psi_k-\psi_j+\delta\phi_k)}+\\ \dfrac{Y_j^{*}e^{-i(\psi_j+\psi_k+\delta\phi_k)}}{2i\left(1-e^{i(\mu_j+\mu_k+\delta\mu_k)}\right)}\displaystyle\int_s^{s+C}d\xi Y_j^{T}\mathbf{H}_1Y_ke^{i(\psi_k+\psi_j+\delta\phi_k)}\end{array}\right)+O(\varepsilon^2)\qquad(18\text{-}15)$$

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We shall start from writing deviation of the Hamiltonian (desirably in the matrix form):

$$\delta K=\begin{bmatrix}\delta f & 0 & \delta n & \delta L\\ 0 & 0 & -\delta L & 0\\ \delta n & -\delta L & \delta g & 0\\ \delta L & 0 & 0 & 0\end{bmatrix};\delta D=\begin{bmatrix}0 & 0 & -\delta L & 0\\ -\delta f & 0 & -\delta n & -\delta L\\ \delta L & 0 & 0 & 0\\ -\delta n & \delta L & -\delta g & 0\end{bmatrix};\qquad(1)$$

In our case we have only 2D case

$$\mu_{x,y}=2\pi Q_{x,y};\;\;Y_x(s)=\begin{bmatrix}\mathrm{w}_x\\ \mathrm{w}'_x+\dfrac{i}{\mathrm{w}_x}\\ 0\\ 0\end{bmatrix};Y_y(s)=\begin{bmatrix}0\\ 0\\ \mathrm{w}_y\\ \mathrm{w}'_y+\dfrac{i}{\mathrm{w}_y}\end{bmatrix}$$

First, let's find tune changes

$$\tilde{Y}_x^{*T}\delta\mathbf{H}(\zeta)\tilde{Y}_x=\mathrm{w}^2{}_x\delta f=\beta_x\cdot\delta L^2$$

$$\tilde{Y}_y^{*T}\delta\mathbf{H}(\zeta)\tilde{Y}_y=\mathrm{w}^2{}_y\delta g=\beta_y\cdot\delta L^2$$

$$\delta Q_{x,y}\cong\frac{1}{4\pi}\int_0^C ds\beta_{x,y}\cdot\delta L^2.$$

to prove that they indeed change only in the second order of δL. If one gets to the next (second order) perturbation – there will be also second order term of δn. Still, no first order chance. A special case of the resonance $\mu_x = \pm\mu_y + 2\pi m$ - we will consider later.

To find perturbed eigen vectors using straight-forward convolutions

$$Y_y^{*T}\delta\mathbf{H}Y_x = \delta n \mathrm{w}_y\mathrm{w}_x + \delta L\left(\mathrm{w}_x\mathrm{w}_y' - \mathrm{w}_y\mathrm{w}_x' - \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x}\right) = \left(Y_x^{*T}\delta\mathbf{H}Y_y\right)^*$$

$$Y_y^{T}\delta\mathbf{H}Y_x = \delta n \mathrm{w}_y\mathrm{w}_x + \delta L\left(\mathrm{w}_x\mathrm{w}_y' - \mathrm{w}_y\mathrm{w}_x' + \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x}\right) = Y_x^{T}\delta\mathbf{H}Y_y$$

$$Y_x^{T}\delta\mathbf{H}Y_x = \left(\mathrm{w}_x\delta L\right)^2; Y_y^{T}\delta\mathbf{H}Y_y = \left(\mathrm{w}_2\delta L\right)^2$$

and dropping the last two, since they are of the second order we have

$$Y_1 \cong Y_x + a_x Y_y + b_x Y_y^*$$

$$Y_2 \cong Y_y + a_y Y_x + b_y Y_x^*$$

$$a_x(s) = \frac{-e^{-i(\psi_x-\psi_y)}}{2i\left(e^{i(\mu_x-\mu_y)}-1\right)}\int_s^{s+C} e^{i(\psi_x-\psi_y)}\left(\delta n\cdot \mathrm{w}_y\mathrm{w}_x + \delta L\left(\mathrm{w}_x\mathrm{w}_y' - \mathrm{w}_y\mathrm{w}_x' - \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x}\right)\right)d\xi$$

$$b_x(s) = \frac{e^{-i(\psi_x+\psi_y)}}{2i\left(e^{i(\mu_x+\mu_y)}-1\right)}\int_o^{C} e^{i(\psi_x+\psi_y)}\left(\delta n\cdot \mathrm{w}_y\mathrm{w}_x + \delta L\left(\mathrm{w}_x\mathrm{w}_y' - \mathrm{w}_y\mathrm{w}_x' + \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x}\right)\right)d\xi$$

$$a_x(s) = \frac{-e^{-i(\psi_y-\psi_x)}}{2i\left(e^{i(\mu_y-\mu_x)}-1\right)}\int_o^{C} e^{i(\psi_y-\psi_x)}\left(\delta n\cdot \mathrm{w}_y\mathrm{w}_x + \delta L\left(-\mathrm{w}_x\mathrm{w}_y' + \mathrm{w}_y\mathrm{w}_x' - \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x}\right)\right)d\xi$$

$$b_x(s) = \frac{e^{-i(\psi_x+\psi_y)}}{2i\left(e^{i(\mu_x+\mu_y)}-1\right)}\int_o^{C} e^{i(\psi_x+\psi_y)}\left(\delta n\cdot \mathrm{w}_y\mathrm{w}_x + \delta L\left(-\mathrm{w}_x\mathrm{w}_y' + \mathrm{w}_y\mathrm{w}_x' - \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} + \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x}\right)\right)d\xi$$

It is easy to see that attained expression is periodic. We can show that new set is symplectic-orthogonal for a general case - it is not more complicated that for specific case. We just need to re-write (8-15) in a compact form.

$$Y_{1k} = Y_k + \varepsilon\left(c_k Y_k^* + \sum_{j\neq k}\left(a_{kj}Y_j + b_{kj}Y^*{}_j\right)\right);$$

$$Y_{1k}^* = Y_k^* + \varepsilon\left(c^*{}_k Y_k + \varepsilon\sum_{j\neq k}\left(a_{kj}{}^*Y^*{}_j + b_{kj}{}^*Y_j\right)\right);$$

With obvious $Y_{1k}{}^T\mathbf{S}Y_{1k} \equiv 0$ let's check first normal pairs $Y^*{}_{1k}, Y_{1k}$:

$$Y_{1k}{}^{*T}\mathbf{S}Y_{1k} = Y_k{}^{*T}\mathbf{S}Y_k + \varepsilon\left(c_k Y_k{}^{*T}\mathbf{S}Y_k{}^* + \sum_{j\neq k}\left(a_{kj}Y_k{}^{*T}\mathbf{S}Y_j + b_{kj}Y_k{}^{*T}\mathbf{S}Y_j{}^*\right)\right); Y_k{}^*$$

$$+\varepsilon\left(c^*{}_k Y_k{}^T\mathbf{S}Y_k + {}^*\sum_{j\neq k}\left(a_{kj}{}^*Y_j{}^{*T}\mathbf{S}Y_k + b_{kj}{}^*Y_j\mathbf{S}Y_k\right)\right) + O\left(\varepsilon^2\right) = 2i$$

where all term is red are equal zero. One can also evaluate second order term to be

$$O\left(\varepsilon^2\right) = 2i\varepsilon^2 \sum_{j\neq k}\left(\left|a_{kj}\right|^2 - \left|b_{kj}\right|^2\right)$$

Let's now look at $Y_{1k}{}^T \mathbf{S} Y_{1m}; m \neq k$

$$Y_{1k} = Y_k + \varepsilon\left(c_k Y_k{}^* + \sum_{j\neq k}\left(a_{kj}Y_j + b_{kj}Y^*{}_j\right)\right);$$

$$Y_{1m} = Y_k + \varepsilon\left(c_m Y_m{}^* + \sum_{j\neq k}\left(a_{mj}Y_j + b_{mj}Y^*{}_j\right)\right);$$

$$Y_{1k}{}^T \mathbf{S} Y_{1m} = Y_k{}^T \mathbf{S} Y_m + \varepsilon\left(c_m Y_k{}^T \mathbf{S} Y_m{}^* + \sum_{j\neq k}\left(a_{mj}Y_k{}^T \mathbf{S} Y_j + b_{mj} Y_k{}^T \mathbf{S} Y^*{}_j\right)\right) +$$

$$\varepsilon\left(c_k Y_k{}^{*T} \mathbf{S} Y_m + \sum_{j\neq k}\left(a_{kj} Y_j{}^T \mathbf{S} Y_m + b_{kj} Y_j{}^{*T} \mathbf{S} Y_m\right)\right) + O\left(\varepsilon^2\right);$$

$$Y_k{}^T \mathbf{S} Y^*{}_j = -2i\delta_{kj}; Y_j{}^{*T} \mathbf{S} Y_m = 2i\delta_{jm}$$

$$Y_{1k}{}^T \mathbf{S} Y_{1m} = 2i\varepsilon\left(b_{km} - b_{mk}\right);$$

$$b_{km} = b_{mk} = \frac{e^{-i(\psi_m + \psi_k)}}{2i\left(1 - e^{i(\mu_m + \mu_k)}\right)} \int_s^{s+C} d\xi Y_j{}^T \mathbf{H}_1 Y_k e^{i(\psi_k + \psi_m)}$$

Again, red term are equal zero and can be dropped. Similar equation work for $Y_{1k}{}^{*T} \mathbf{S} Y_{1m}; m \neq k$

$$Y_{1m}{}^{*T} \mathbf{S} Y_{1k} = 2i\varepsilon\left(a_{mk}{}^* - a_{kn}\right);$$

$$a_{km} = a_{mk}{}^* = \frac{-e^{i(\psi_m - \psi_{kk})}}{2i\left(1 - e^{i(\mu_k - \mu_m)}\right)} \int_s^{s+C} d\xi Y_j{}^{*T} \mathbf{H}_1 Y_k e^{i(\psi_k - \psi_m)}$$

Note, that this is all approximation with error of $O\left(\varepsilon^2\right)$. Naturally, not to have any errors, we should get the transport matrices for the motion and then do everything perfectly – but not analytically!

STAR PART – we will discuss it when we are discussing resonances.