# Overview of single particle motion in a magnetic field

Maxwell equations

$$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon_0} \qquad \bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$$

$$\bar{\nabla}\cdot\bar{B} = 0 \qquad \bar{\nabla}\times\bar{B} = \mu_0\bar{J} + \epsilon_0\mu_0\frac{\partial E}{\partial t}$$

$\rho$ & $J$ : charge density & current density attributable to all charged particles

Lorentz force: $\bar{F} = q(\bar{E} + \bar{v}\times\bar{B})$ does not depend on D or H. So if you bring in D & H, you'd have to convert back to E & B to get the force, velocity, etc

Two specific applications of the Maxwell's equations and Lorentz force are very important for accelerators:

1. Motion of a charged particle in a static magnetic field
2. Electromagnetic fields in a pillbox cavity

Let's start with motion of charged particle in static magnetic field:

Let $\bar{B} = \hat{z}B$, $\bar{E} = 0$, consider the non-relativistic case

Intuitively, since acceleration $\propto \bar{v}\times\bar{B}$, and $\bar{v}\times\bar{B} \perp v$, the motion is a centripital motion:

$$\bar{v}\cdot\left[\frac{d\bar{p}}{dt} = q(\underbrace{\bar{E}}_{0} + \bar{v}\times\bar{B})\right]$$

$$\bar{v}\cdot\frac{d\bar{p}}{dt} = q\underbrace{\bar{v}\cdot(\bar{v}\times\bar{B})}_{=0} = 0 \Rightarrow \text{Power} = \frac{dW}{dt} = \bar{v}\cdot\bar{F} = 0$$

i.e. no work is done, $|V|$ remains constant

Centripital acceleration eqn:

$$\frac{mv^2}{r} = qvB$$

$$\therefore \frac{v}{r} = \frac{qB}{m} \longleftarrow \omega = \frac{v}{r} \text{ is called cyclotron frequency}$$

$$\omega = \frac{qB}{m} \longrightarrow \text{Relativistic correction: } \omega = \frac{qB}{\gamma m}$$

$$r_L = \frac{v}{\omega} = \frac{mv}{qB} \longleftarrow \text{referred to as the Larmor radius}$$

relativistic correction: $r_L = \frac{P}{qB}$

In accelerator physics, the ratio of the momentum to charge is often called magnetic rigidity.

The Larmor radius is also identified with symbol $\rho$

$$\therefore B\rho = \frac{P}{e}$$

$\uparrow$ often considered a single symbol.

In mks units, $(B\rho) = \frac{10}{2.9979} P(\text{GeV}/c)$ in T.m

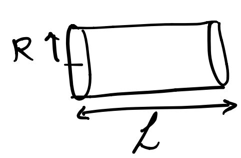
In summary, a particle with charge q in a constant magnetic field will perform circular motion at the cyclotron frequency and with the radius of motion determined by Larmor radius.

The second problem of interest is the fields inside a cylindrical resonator cavity, which is the basic accelerating structure. Edwards and Syphers pg 24-25 states the solution, but here, I want to show how the solution can actually be derived from first principles:

Start with Maxwell equations inside cavity, i.e. no charge / current sources

$$\bar{\nabla} \times \bar{E} = -\frac{\partial B}{\partial t} \qquad \nabla \cdot \bar{E} = 0$$

$$\bar{\nabla} \times \bar{B} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \qquad \nabla \cdot \bar{B} = 0$$




$$\Rightarrow \bar{\nabla} \times \bar{\nabla} \times \bar{E} = -\frac{\partial}{\partial t} \bar{\nabla} \times \bar{B}$$

$$\therefore \quad \bar{\nabla}(\cancel{\bar{\nabla}\cdot\bar{E}}) - \nabla^2 \bar{E} = -\frac{\partial}{\partial t}\mu_0\epsilon_0\frac{\partial E}{\partial t}$$

$$\therefore \quad \left(\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)\bar{E} = 0 \;\ldots (1)$$ (same eqn may be derived for B)

We assume that $\bar{E}$ is a time harmonic function of freq. $\omega$:

$$\bar{E}(\bar{r},t) = \mathrm{Re}\left[\bar{E}(\bar{r})\, e^{j\omega t}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\bar{E}(\bar{r},t)\right) = j\omega\, \bar{E}(\bar{r},t)$$

$$\therefore (1) \Rightarrow \boxed{\left(\bar{\nabla}^2 + \mu_0\epsilon_0\omega^2\right)\bar{E} = 0}$$

This equation is referred to as the Helmholtz eqn.

## Solution in cylindrical coordinates

let $\bar{E}(\bar{r}) = \hat{\rho}\, E_\rho(\bar{r}) + \hat{\varphi}\, E_\phi(\bar{r}) + \hat{z}\, E_z(\bar{r})$

Note $\nabla^2(\hat{\rho}E_\rho) \neq \hat{\rho}\,\nabla^2 E_\rho$ & $\nabla^2(\hat{\varphi}E_\varphi) \neq \hat{\varphi}\,\nabla^2 E_\varphi$

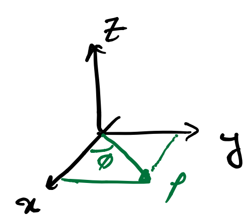
will need to use $\nabla^2\bar{F} = \underbrace{\nabla(\nabla\cdot\bar{F})}_{\text{curl-less function}} - \underbrace{\bar{\nabla}\times(\bar{\nabla}\times\bar{F})}_{\text{divergence-less function}}$ identity

### Side note

$$\hat{\rho} = \frac{\bar{\rho}}{\rho} = \frac{\rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y}}{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{z} = \hat{z}$$

$$\hat{\phi} = \hat{z}\times\hat{\rho} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$





$$\frac{\partial\hat{z}}{\partial\rho} = \frac{\partial\hat{z}}{\partial\phi} = \frac{\partial\hat{z}}{\partial z} = 0$$

$$\begin{cases} \frac{\partial \hat{\rho}}{\partial \rho} = 0 \\ \frac{\partial \hat{\rho}}{\partial \phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi = \hat{\phi} \\ \frac{\partial \rho}{\partial \hat{z}} = 0 \end{cases}$$

$$\frac{\partial \hat{\phi}}{\partial \rho} = 0$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\hat{x}\cos\phi - \sin\phi\,\hat{y} = -\hat{\rho}$$

$$\frac{\partial \hat{\phi}}{\partial z} = 0$$

The three scalar equations are

$$\left.\begin{array}{l} (\nabla^2 + \beta^2) E_\rho - \frac{1}{\rho^2} E_\rho - \frac{2}{\rho^2} \frac{\partial E_\phi}{\partial \phi} = 0 \\ (\nabla^2 + \beta^2) E_\varphi - \frac{1}{\rho^2} E_\varphi + \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \varphi} = 0 \end{array}\right\} \text{ coupled equations.}$$

$(\nabla^2 + \beta^2) E_z = 0 \longrightarrow$ uncoupled equation, useful for finding $TE_z$ & $TM_z$ mode solutions

$\beta = \epsilon_0 \mu_0 \omega^2$

The uncoupled equations in *z* (which is the accelerating field in the pillbox cavity) can be solved analytically using the ***separation of variables method***:

let $E_z(r) = E_z(\rho, \varphi, z) = f(\rho)\, g(\varphi)\, h(z)$

$$\nabla^2 E_z = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial E_z}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \varphi^2} + \frac{\partial^2 E_z}{\partial z^2}$$

$$(\nabla^2 + \beta^2) E_z = 0$$

$$\therefore \quad gh\left[\frac{1}{\rho} \frac{d}{d\rho}(\rho f)\right] + \frac{fh}{\rho^2} \frac{d^2 g}{d\varphi^2} + fg \frac{d^2 h}{dz^2} + \beta^2 fgh = 0$$

divide by $fgh \Rightarrow$

$$\frac{1}{f}\frac{d^2 f(\rho)}{d\rho^2} + \frac{1}{f}\cdot\frac{1}{\rho}\frac{df}{d\rho} + \underbrace{\frac{1}{g}\cdot\underbrace{\frac{1}{\rho^2}}\frac{d^2 g}{d\phi^2}}_{\text{function of } \rho \,\&\, \phi} + \underbrace{\frac{1}{h}\frac{d^2 h}{dz^2}}_{\text{only function of } z} + \beta^2 = 0 \quad \dots (2)$$

The three terms above have to add up to a constant

$$z: \quad \frac{1}{h(z)}\frac{d^2 h(z)}{dz^2} = -\underbrace{\beta_z^2}_{\text{constant}} \Rightarrow \left(\frac{d^2}{dz^2} + \beta_z^2\right) h(z) = 0$$

second order ODE $\Rightarrow$ $\boxed{h(z) = A_1 \cos(\beta_z z) + B_1 \sin(\beta_z z)}$

Multiply (2) by $\rho^2$:

$$\frac{\rho^2}{f}\frac{d^2 f}{d\rho^2} + \frac{1}{f}\cdot\rho\frac{df}{d\rho} + \underbrace{\frac{1}{g}\frac{d^2 g}{d\phi^2}}_{\text{The only function of } \phi} + (\beta^2 - \beta_z^2)\rho^2 = 0 \dots (3)$$

$$\frac{1}{g}\frac{d^2 g(\varphi)}{d\phi^2} = \underbrace{-m^2}_{\text{constant}} \Rightarrow \left(\frac{d^2}{d\phi^2} + m^2\right) g(\phi) = 0$$

$$\boxed{\therefore \; g(\phi) = A_2 \cos(m\phi) + B_2 \sin(m\phi)}$$

note: m can only be integeter because of periodicity of space in $\phi = 2n\pi$.

$$(3) \Rightarrow \left[\rho^2 \frac{d^2}{d\rho^2} + \rho\frac{d}{d\rho} + (\beta_\rho^2 \rho^2 - m^2)\right] f = 0$$ $\Leftarrow$ Bessel differential Equation

where $\beta_\rho = \beta^2 - \beta_z^2$

$$\therefore \boxed{f(\rho) = A_3 J_m(\beta_\rho \rho) + B_3 Y_m(\beta_\rho \rho)}$$

$J_m$ & $Y_m$ are Bessel functions of first & second kind, respectively,

The choice of functional forms depends on boundary value nature of the problem.

Any function $E_z(\bar{r}) = f(\rho)\, g(\phi)\, h(z)$ is a solution to the helmholtz equation, but the boundary conditions of the pillbox resonator place a more restrictive condition on the form of $E_z$. The complete field description and the properties of the resonator cavities will be the subject of next class.