Homework 12. Due October 21

Problem 1. 10 points. Matrix Gymnastics.

For a one-dimensional motion consider parameterization via the eigen vectors and their propagation along *s*:

$$Y(\mathrm{s})=\begin{bmatrix} \mathrm{w(s)} \\ \mathrm{w}'(\mathrm{s})+\dfrac{i}{\mathrm{w(s)}} \end{bmatrix};Y^*(\mathrm{s})=\begin{bmatrix} \mathrm{w(s)} \\ \mathrm{w}'(\mathrm{s})-\dfrac{i}{\mathrm{w(s)}} \end{bmatrix};$$

$$\beta(\mathrm{s})\equiv \mathrm{w}^2(\mathrm{s});\ \ \alpha(\mathrm{s})=-\frac{\beta'(\mathrm{s})}{2}\equiv -\mathrm{w(s)w}'(\mathrm{s});\ \ \frac{d\psi}{ds}=\frac{1}{\mathrm{w}^2(\mathrm{s})}\equiv\frac{1}{\beta(\mathrm{s})};$$

$$Y(\mathrm{s}_2)\mathrm{e}^{i\Delta\psi}=\mathrm{M}\left(\mathrm{s}_1|\mathrm{s}_2\right)Y(\mathrm{s}_1)\Leftrightarrow Y^*(\mathrm{s}_2)\mathrm{e}^{-i\Delta\psi}=\mathrm{M}\left(\mathrm{s}_1|\mathrm{s}_2\right)Y^*(\mathrm{s}_1);\Delta\psi=\psi(\mathrm{s}_2)-\psi(\mathrm{s}_1).$$

or in other form as:

$$\left[Y(\mathrm{s}_2)\mathrm{e}^{i\Delta\psi},Y^*(\mathrm{s}_2)\mathrm{e}^{-i\Delta\psi}\right]=\mathrm{M}\left(\mathrm{s}_1|\mathrm{s}_2\right)\left[Y(\mathrm{s}_1),Y^*(\mathrm{s}_1)\right];$$

$$\tilde{W}(\mathrm{s})=\left[Y(\mathrm{s})\mathrm{e}^{i\psi(\mathrm{s})},Y^*(\mathrm{s})\mathrm{e}^{-i\psi(\mathrm{s})}\right];\tilde{W}(\mathrm{s}_2)=\mathrm{M}\left(\mathrm{s}_1|\mathrm{s}_2\right)\tilde{W}(\mathrm{s}_1);$$

Show that then matrix can be expressed through the values of envelopes **w**, and its derivatives **w'** and the "betatron" phase advance $\Delta\psi=\psi(\mathrm{s}_2)-\psi(\mathrm{s}_1)$ as:

$$\mathrm{M}\left(\mathrm{s}_1|\mathrm{s}_2\right)=\begin{bmatrix} \dfrac{\mathrm{w}_2}{\mathrm{w}_1}\cos\Delta\psi-\mathrm{w}_1'\mathrm{w}_2\sin\Delta\psi & \mathrm{w}_1\mathrm{w}_2\sin\Delta\psi \\ -\sin\Delta\psi\left(\dfrac{1}{\mathrm{w}_1\mathrm{w}_2}+\mathrm{w}_1'\mathrm{w}_2'\right)-\cos\Delta\psi\left(\dfrac{\mathrm{w}_1'}{\mathrm{w}_2}-\dfrac{\mathrm{w}_2'}{\mathrm{w}_1}\right) & \dfrac{\mathrm{w}_1}{\mathrm{w}_2}\cos\Delta\psi+\mathrm{w}_2'\mathrm{w}_1\sin\Delta\psi \end{bmatrix}$$

or in traditional terms:

$$\mathrm{M}\left(\mathrm{s}_1|\mathrm{s}_2\right)=\begin{bmatrix} \sqrt{\dfrac{\beta_2}{\beta_1}}\left(\cos\Delta\psi+\alpha_1\sin\Delta\psi\right) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\dfrac{\sin\Delta\psi\left(1+\alpha_1\alpha_2\right)+\cos\Delta\psi\left(\alpha_2-\alpha_1\right)}{\sqrt{\beta_1\beta_2}} & \sqrt{\dfrac{\beta_1}{\beta_2}}\left(\cos\Delta\psi-\alpha_2\sin\Delta\psi\right) \end{bmatrix}$$

Hint: if you are using $\tilde{W}(\mathrm{s})$, *than use that* $\tilde{W}^T SW=-2iS$. *If you hate complex variables, than construct* $\tilde{U}(\mathrm{s})=\left[\tilde{R}(\mathrm{s}),\tilde{Q}(\mathrm{s})\right];\tilde{R}(\mathrm{s})=\mathrm{Re}\,\tilde{Y}(\mathrm{s});\tilde{Q}(\mathrm{s})=\mathrm{Im}\,\tilde{Y}(\mathrm{s});$ *show that it is symplectic, and define how it transformed with s.*