Homework 2. PHY 564

Problem 1. 4 points. Plane bent trajectory
Consider a curved but plane (not torsion, only curvature) reference trajectory laying in y=0 plane. Use Lorenz force and mechanical momentum to derive equations (123), (124), (125) and (126) from Lecture 4.

Solution: Let's write equation of motion for particle in y=0 plane:

$$\vec{r} = \vec{r}_o(t);\ \vec{\mathrm{v}}_o(t) = \frac{d\vec{r}_o(t)}{dt} \equiv \vec{\tau}(t)\cdot \mathrm{v}_o(t);\ \vec{p}_o(t) = \frac{m\vec{\mathrm{v}}_o(t)}{\sqrt{1-\frac{\vec{\mathrm{v}}_o^{\,2}(t)}{c^2}}} \equiv \vec{\tau}(t)\cdot m\,\gamma_o(t)\mathrm{v}_o(t)$$

$$\frac{d\vec{p}_o(t)}{dt} = e\left(\vec{E}\left(\vec{r}_o(t),t\right) + \frac{1}{c}\cdot\left[\vec{\mathrm{v}}_o(t)\times\vec{B}\left(\vec{r}_o(t),t\right)\right]\right);$$

and we need to look at each of three components of this equation. The easiest is in y direction, which contestant unit vector (to torsion): both y, p_y and there derivatives bust stay zero for the reference particle to stay in the plane:

$$\vec{\mathrm{v}}_o(t) = \vec{\tau}\cdot\frac{ds}{dt} \Rightarrow \frac{ds}{dt} = \left|\frac{d\vec{r}_o(t)}{dt}\right| \equiv \left|\vec{\mathrm{v}}_o(t)\right|;\ \mathrm{v}_{xo}(t) = \mathrm{v}_{yo}(\mathrm{t}) \equiv 0$$

$$\mathrm{v}_{yo}(\mathrm{t}) = \frac{dy}{dt} = \frac{p_y}{\gamma_o mc} = 0 \Rightarrow p_y = 0;$$

$$\frac{dp_y}{dt} = e\left(E_y\left(\vec{r}_o(t),t\right) + \frac{\left|\vec{\mathrm{v}}_o(t)\right|}{c}\vec{b}\cdot\left[\vec{\tau}\times\vec{B}\left(\vec{r}_o(t),t\right)\right]\right) = 0;$$

$$\vec{b}\cdot\left[\vec{\tau}\times\vec{B}\right] = \vec{B}\cdot\left[\vec{b}\times\tau\right] = -\vec{B}\cdot\vec{n} = -B_x;\ \frac{dp_y}{dt} = e\left(E_y - \frac{\left|\vec{\mathrm{v}}_o\right|}{c}\cdot B_x\right) = 0$$

i.e. the condition identical to that in our lectures $E_y = \frac{\left|\vec{\mathrm{v}}_o\right|}{c}\cdot B_x$.

Evolution of the energy is also trivial

$$\mathrm{E} = \sqrt{m^2c^4 + \vec{p}^{\,2}c^2} = \gamma mc^2 \Rightarrow \frac{d\mathrm{E}}{dt} = \frac{d\vec{p}}{dt}\cdot\frac{\vec{p}c^2}{\mathrm{E}} = \vec{\mathrm{v}}\cdot\frac{d\vec{p}}{dt};\ \vec{\mathrm{v}}\cdot\left[\vec{\mathrm{v}}\times\vec{B}\right] \equiv 0;$$

$$\frac{d\mathrm{E}_o}{dt} = e\vec{\mathrm{v}}_o\cdot\vec{E}\left(\vec{r}_o(t),t\right) = e\mathrm{v}_o E_s \Rightarrow \frac{dE_o}{ds} = \frac{dE_o}{v_o dt} = eE_s.$$

And now it time to address curvature of the orbit

$$\vec{p}_o(t) = \vec{\tau}(t)\cdot p_o(t);\ \frac{d\vec{\tau}}{ds} = -K\cdot\vec{n};\ \frac{d\vec{\tau}(t)}{dt} = \mathrm{v}_o(t)\frac{d\vec{\tau}}{ds} = -\mathrm{v}_o(t)\cdot K(t)\cdot\vec{n}(t);$$

$$\frac{d\vec{p}_o(t)}{dt} = p_o(t)\frac{d\vec{\tau}(t)}{dt} + \vec{\tau}(t)\frac{dp_o(t)}{dt} = -K\cdot\vec{n}\cdot\mathrm{v}_o\cdot p_o + \vec{\tau}\frac{dp_o}{dt} = e\left(\vec{E} + \frac{\mathrm{v}_o}{c}\cdot\left[\vec{\tau}\times\vec{B}\right]\right);$$

$$-K\cdot\mathrm{v}_o\cdot p_o = e\vec{n}\cdot\left(\vec{E} + \frac{\mathrm{v}_o}{c}\cdot\left[\vec{\tau}\times\times\vec{B}\right]\right) = e\left(E_x + \frac{\mathrm{v}_o}{c}\vec{n}\cdot\left[\vec{\tau}\times\vec{B}\right]\right) = e\left(E_x + B_y\right);$$

$$K = \frac{1}{\rho} = -\frac{e}{pc}\left(B_y + \frac{c}{\mathrm{v}_o}E_x\right)\#$$

Problem 2. 6 points. Specific gauge for 4-potential

Use Lecture 3 equation (107) expressions for curl and gradient, and equation (105) for g^{ik}. Prove that 4-potential formulae in eq. (118) in Lecture 4 generate accurate expressions for all component of electric and magnetic fields from

$$\vec{B} = curl\vec{A},\ \vec{E} = -\frac{1}{c}\cdot\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi$$

This is so called STAR problem – I will increase points 5-fold if you find error in expansion (118).

Solution: Several things are important to solve this problem in relatively compact (not a dozen of pages) format. First, we need to clearly identify *x, y* expansion of any function as

$$f \equiv f(x,y,s,t) = \sum_{n,k=0}^{\infty} f^{nk}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}\ ;\ f^{nk} = \left.\frac{\partial^{n+k}f}{\partial_x^n\partial_y^k}\right|_{x,y=0};\ f^{00} = f(0,0,s,t);\ f^{10} = \left.\frac{\partial f}{\partial_x}\right|_{x,y=0};\ f^{01} = \left.\frac{\partial f}{\partial_y}\right|_{x,y=0}$$

with *x* and *y* integrals of this function

$$I_x(f) \equiv \int_0^x f(z,y,s,t)dz = \sum_{n,k=0}^{\infty} f^{nk}\cdot\frac{x^{n+1}}{(n+1)!}\cdot\frac{y^k}{k!}\ ;\ I_y(f) \equiv \int_0^y f(x,z,s,t)dz = \sum_{n,k=0}^{\infty} f^{nk}\cdot\frac{x^n}{n!}\cdot\frac{y^{k+1}}{(k+1)!}$$

It means that we can write compact expressions for A_1 and A_2 components of vector potential: Clearly identify contravariant component of magnetic field resulting from *curl* of the vector potential:

$$A_1 = \frac{1}{2}I_y\left(B_s\right);\ A_3 = -\frac{1}{2}I_x\left(B_s\right)$$

For compactness we can rewrite A_2 using two new notations:

$$b_y = (1+Kx)B_y + \kappa x B_s;\ b_x = (1+Kx)B_x - \kappa y B_s;$$

$$A_2 = \sum_{n=1}^{\infty}\left(b_y^{\ n0}\Big|\cdot\frac{x^n}{n!} - b_x^{\ 0n}\cdot\frac{y^n}{n!}\right) + \frac{1}{2}\sum_{n,k=0}^{\infty}\left(b_y^{\ n,k+1} - b_x^{\ n+1,k}\right)\cdot\frac{x^{n+1}}{(n+1)!}\cdot\frac{y^{k+1}}{(k+1)!}.$$

The same structures (b_x, b_y, B_s) are prominently appearing in curl and magnetic field:

$$\vec{B} = \frac{e^{ikl}}{\sqrt{g_o}}\vec{a}_i\frac{\partial A_l}{\partial q^k};B^1 = B_x - \frac{\kappa y}{1+Kx}\cdot B_s = \frac{b_x}{1+Kx};\ B^2 = \frac{1}{1+Kx}\cdot B_s;\ B^3 = B_y + \frac{\kappa x}{1+Kx}\cdot B_s = \frac{b_y}{1+Kx};$$

$$(1)\ B^1 = \frac{1}{1+Kx}\left(\frac{\partial A_3}{\partial s} - \frac{\partial A_2}{\partial y}\right) \Rightarrow \frac{\partial A_3}{\partial s} - \frac{\partial A_2}{\partial y} = b_x;$$

$$(2)\ B^2 = \frac{1}{1+Kx}\left(\frac{\partial A_1}{\partial y} - \frac{\partial A_3}{\partial x}\right) \Rightarrow \frac{\partial A_1}{\partial y} - \frac{\partial A_3}{\partial x} = B_s;$$

$$(3)\ B^3 = \frac{1}{1+Kx}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial sx}\right) \Rightarrow \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial s} = b_y;$$

Hence, the easiest is to probe (2):

$$A_1 = \frac{1}{2}I_y\left(B_s\right);\ A_3 = -\frac{1}{2}I_x\left(B_s\right)$$

$$\partial_y A_1 - \partial_x A_3 = \sum_{n.k=0}^{\infty} B_s^{\ nk}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!} \equiv B_s(x,y,s,t)\#$$

Before going to deriving expressions for b_x and b_y, let's explore consequence of zero divergence of magnetic field, which does not depend on the sources of the field!

$$div\vec{B} = div\left(curl\vec{A}\right) \equiv 0;\ \ div\vec{B} = \frac{1}{\sqrt{g_o}}\frac{\partial}{\partial q^k}\left(\sqrt{g_o}B^k\right);\ \sqrt{g_o} = (1 + K\cdot x);\ q^k = (x,s,y)$$

$$\sqrt{g_o}B^1 = b_x;\ \sqrt{g_o}B^2 = B_s;\ \sqrt{g_o}B^3 = b_y;$$

$$\sqrt{g_o}\,div\vec{B} = \frac{\partial b_x}{\partial x} + \frac{\partial B_s}{\partial s} + \frac{\partial b_y}{\partial y} = 0;\ \ \frac{\partial B_s}{\partial s} = -\frac{\partial b_x}{\partial x} - \frac{\partial b_y}{\partial y}.$$

Let's now look at $b_{x,y}$:

$$b = \sum_{n.k=0}^{\infty} b^{nk}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!};\ \frac{\partial b}{\partial x} = \sum_{n.k=0}^{\infty} b^{nk}\cdot\frac{x^{n-1}}{(n-1)!}\cdot\frac{y^k}{k!};\frac{\partial b}{\partial y} = \sum_{n.k=0}^{\infty} b^{nk}\cdot\frac{x^n}{n!}\cdot\frac{y^{k-1}}{(k-1)!}$$

$$A_1 = \frac{1}{2}I_y\left(B_s\right);\ \ A_3 = -\frac{1}{2}I_x\left(B_s\right);\ \frac{\partial B_s}{\partial s} = -\frac{\partial b_x}{\partial x} - \frac{\partial b_y}{\partial y};$$

$$\frac{\partial A_3}{\partial s} = -\frac{1}{2}I_x\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) = \frac{1}{2}\left(\sum_{n,k}^{\infty} b_x^{\ nk}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!} + b_y^{\ nk}\cdot\frac{x^{n+1}}{(n+1)!}\cdot\frac{y^{k-1}}{(k-1)!}\right)$$

$$\frac{\partial A_1}{\partial s} = \frac{1}{2}I_y\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) = -\frac{1}{2}\left(\sum_{n,k}^{\infty} b_x^{\ nk}\cdot\frac{x^{n-1}}{(n-1)!}\cdot\frac{y^{k+1}}{(k+1)!} + b_y^{\ nk}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}\right)$$

$$A_2 = \sum_{n=1}^{\infty}\left(b_y^{\ n0}\Big|\cdot\frac{x^n}{n!} - b_x^{\ 0n}\cdot\frac{y^n}{n!}\right) + \frac{1}{2}\sum_{n,k=0}^{\infty}\left(b_y^{\ n,k+1} - b_x^{\ n+1,k}\right)\cdot\frac{x^{n+1}}{(n+1)!}\cdot\frac{y^{k+1}}{(k+1)!};$$

$$b_x = \frac{\partial A_3}{\partial s} - \frac{\partial A_2}{\partial y} = +\sum_{n=1}^{\infty} b_x^{\ 0n}\cdot\frac{y^{n-1}}{(n-1)!} - \frac{1}{2}\sum_{n,k=0}^{\infty}\left(b_y^{\ n,k+1} - b_x^{\ n+1,k}\right)\cdot\frac{x^{n+1}}{(n+1)!}\cdot\frac{y^k}{k!}$$

$$+\frac{1}{2}\left(\sum_{n.k=0}^{\infty} b_x^{\ nk}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!} + b_y^{\ nk}\cdot\frac{x^{n+1}}{(n+1)!}\cdot\frac{y^{k-1}}{(k-1)!}\right) = \sum_{n.k=0}^{\infty} b_x^{\ nk}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}\ \ \#1$$

$$b_y = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial s} = \sum_{n=1}^{\infty} b_y^{\ n0}\Big|\cdot\frac{x^{n-1}}{(n-1)!} + \frac{1}{2}\sum_{n,k=0}^{\infty}\left(b_y^{\ n,k+1} - b_x^{\ n+1,k}\right)\cdot\frac{x^n}{n!}\cdot\frac{y^{k+1}}{(k+1)!}$$

$$+\frac{1}{2}\left(\sum_{n,k}^{\infty} b_x^{\ nk}\cdot\frac{x^{n-1}}{(n-1)!}\cdot\frac{y^{k+1}}{(k+1)!} + b_y^{\ nk}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}\right) = \sum_{n.k=0}^{\infty} b_y^{\ nk}\cdot\frac{x^n}{n!}\cdot\frac{y^k}{k!}\ \ \#2$$

where we combined terms of the same orders to show that terms marked in red cancle each other and terms in green add (not forgetting that ½+½=1). So far, so good with magnetic field. What about electric field? Simple application of

$$\vec{E} = -\frac{1}{c}\cdot\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi\ \Rightarrow E_1 = E_x = -\frac{1}{c}\cdot\frac{\partial A_1}{\partial t} - \frac{\partial\varphi}{\partial x};\ E_3 = E_y = -\frac{1}{c}\cdot\frac{\partial A_3}{\partial t} - \frac{\partial\varphi}{\partial y};$$

$$E_2 = (1 + Kx)\cdot E_s + \kappa\left(y\cdot E_x - x\cdot E_y\right) = -\frac{1}{c}\cdot\frac{\partial A_2}{\partial t} - \frac{\partial\varphi}{\partial s};$$

does not directly generate E_x, E_y and E_s unused in the 4-potential expansion.

Since we did not made any assumptions about the sources of the EM field we can rely only on the first pair of Maxwell equations, including

$$cur\ \vec{E} = -\frac{1}{c}\cdot\frac{\partial \vec{B}}{\partial t};\ \frac{\partial E_3}{\partial s} - \frac{\partial E_2}{\partial x} = -\frac{1+Kx}{c}\cdot\frac{\partial B^1}{\partial t};$$

$$\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial s} = -\frac{1+Kx}{c}\cdot\frac{\partial B^3}{\partial t};\ \frac{\partial E_1}{\partial y} - \frac{\partial E_3}{\partial x} = -\frac{1+Kx}{c}\cdot\frac{\partial B^2}{\partial t};$$

or in *x, y, s* components

$$\frac{\partial E_y}{\partial s} - \frac{\partial\left((1+Kx)E_s + \kappa\left(E_x y - E_y x\right)\right)}{\partial y} = -\frac{1}{c}\cdot\frac{\partial\left((1+Kx)B_x - ky\cdot B_s\right)}{\partial t};$$

$$\frac{\partial\left((1+Kx)E_s + \kappa\left(E_x y - E_y x\right)\right)}{\partial x} - \frac{\partial E_x}{\partial s} = -\frac{1}{c}\cdot\frac{\partial\left((1+Kx)B_y + kx\cdot B_s\right)}{\partial t};\ \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = -\frac{1}{c}\cdot\frac{\partial B_s}{\partial t};$$

The last equation is the least simple and the most promising. Let's start with E_x and E_y:

$$\varphi = \varphi_o(s,t) - \sum_{n=1}^{\infty}\partial_x^{n-1}E_x\Big|_{ro}\frac{x^n}{n!} - \sum_{n=1}^{\infty}\partial_y^{n-1}E_y\Big|_{ro}\frac{y^n}{n!} - \frac{1}{2}\sum_{n.k=1}^{\infty}\left(\partial_x^{n-1}\partial_y^{k}E_x\Big|_{ro} + \partial_x^{n}\partial_y^{k-1}E_y\Big|_{ro}\right)\frac{x^n}{n!}\frac{y^k}{k!};$$

$$E_x = -\frac{1}{c}\cdot\frac{\partial A_1}{\partial t} - \frac{\partial\varphi}{\partial x};\ \frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y} + \frac{1}{c}\cdot\frac{\partial B_s}{\partial t};\ \partial_x^{n+1}\partial_y^{k-1}E_y = \partial_x^{n}\partial_y^{k}E_x + \partial_x^{n}\partial_y^{k-1}\cdot\frac{\partial B_s}{c\partial t}$$

$$-\frac{\partial\varphi}{\partial x} = \sum_{n=0}^{\infty}\partial_x^{n}E_x\Big|_{ro}\frac{x^n}{n!} - \frac{1}{2}\sum_{n=0,k=1}^{\infty}\left(\partial_x^{n}\partial_y^{k}E_x\Big|_{ro} + \partial_x^{n+1}\partial_y^{k-1}E_y\Big|_{ro}\right)\frac{x^n}{n!}\frac{y^k}{k!}$$

$$-\frac{\partial\varphi}{\partial x} = \sum_{n,k=0}^{\infty}\partial_x^{n}\partial_y^{k}E_x\Big|_{ro}\frac{x^n}{n!}\frac{y^k}{k!} + \frac{1}{2}\sum_{n,k=0}^{\infty}\partial_x^{n}\partial_y^{k}\cdot\frac{\partial B_s}{c\partial t}\frac{x^{n+1}}{(n+1)!}\frac{y^k}{k!};$$

$$-\frac{\partial A_1}{c\partial t} = -\frac{1}{2}\sum_{n.k=0}^{\infty}\partial_x^{k}\partial_y^{n}\frac{\partial B_s}{c\partial t}\Bigg|_{ro}\frac{x^{k+1}}{(k+1)!}\frac{y^n}{n!};\ -\frac{\partial\varphi}{\partial x} - \frac{\partial A_1}{c\partial t} = E_x\#1$$

$$E_y = -\frac{1}{c}\cdot\frac{\partial A_3}{\partial t} - \frac{\partial\varphi}{\partial y};\ \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} - \frac{1}{c}\cdot\frac{\partial B_s}{\partial t};\ \partial_x^{n-1}\partial_y^{k+1}E_x = \partial_x^{n}\partial_y^{k}E_y + \partial_x^{n-1}\partial_y^{k}\cdot\frac{\partial B_s}{c\partial t}$$

$$-\frac{\partial\varphi}{\partial y} = \sum_{n,k=0}^{\infty}\partial_x^{n}\partial_y^{k}E_y\Big|_{ro}\frac{x^n}{n!}\frac{y^k}{k!} - \frac{1}{2}\sum_{n,k=0}^{\infty}\partial_x^{n}\partial_y^{k}\cdot\frac{\partial B_s}{c\partial t}\frac{x^{n}}{n!}\frac{y^{k+1}}{(k+1)!};$$

$$-\frac{\partial A_2}{c\partial t} = \frac{1}{2}\sum_{n.k=0}^{\infty}\partial_x^{n}\partial_y^{k}\frac{\partial B_s}{c\partial t}\Bigg|_{ro}\frac{x^{n}}{n!}\frac{y^{k+1}}{(k+1)!};\ -\frac{\partial\varphi}{\partial y} - \frac{\partial A_3}{c\partial t} = E_y\#1$$

Again, it was critical to use connection between electric and magnetic fields to prove that our 4-potential expansion is indeed generates correct transverse components of the EM field.

Now it is good place to discuss fortune of E_s! It can be calculated using with multiple terms in

$$E_s = -\frac{1}{1+Kx}\cdot\left(\frac{\partial A_2}{c\partial t} + \frac{\partial\varphi}{\partial s}\right) - \frac{\kappa}{1+Kx}\left(y\cdot E_x - x\cdot E_y\right);$$

with the only one term which does not involve two components of electric field (E_x, E_y) and three of the magnetic field:

$$E_s(0,0,s,t) = -\frac{\partial \varphi_o(0,0,s,t)}{\partial s}.$$

while other term in *x, y* expansion are connected to the rest of the field. The root is in the fact that EM fields are fully described by 4-potential with a gauge invariance. I am not aware of any way of including E_s into *x, y* expansion of 4-potentials at given *s* and *t*. Since there is no general solution for set of four Maxwell equations and with very rare exceptions EM fields are evaluated usng sophisticated computer programs, this is the best we can do using the gauge invariance and first two Maxwell equations. We will consider cases when analytical formulae can be used for EM fields and all complications of general case are no longer a problem.