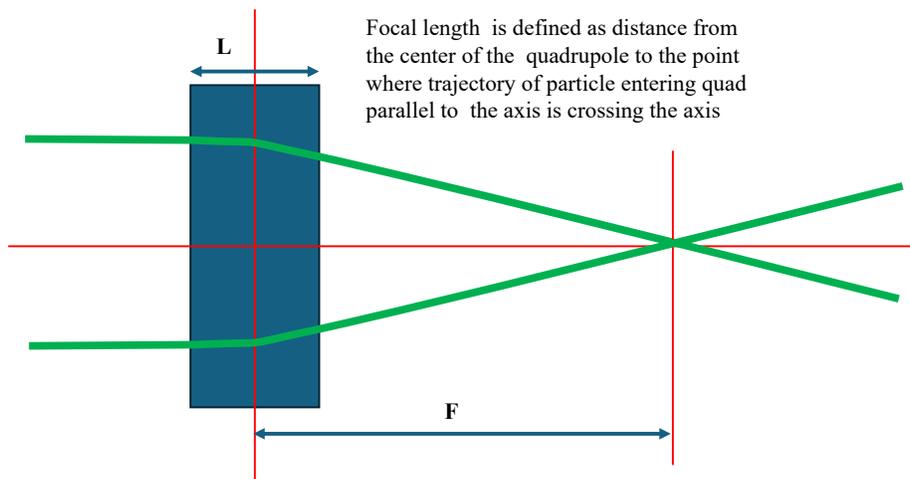
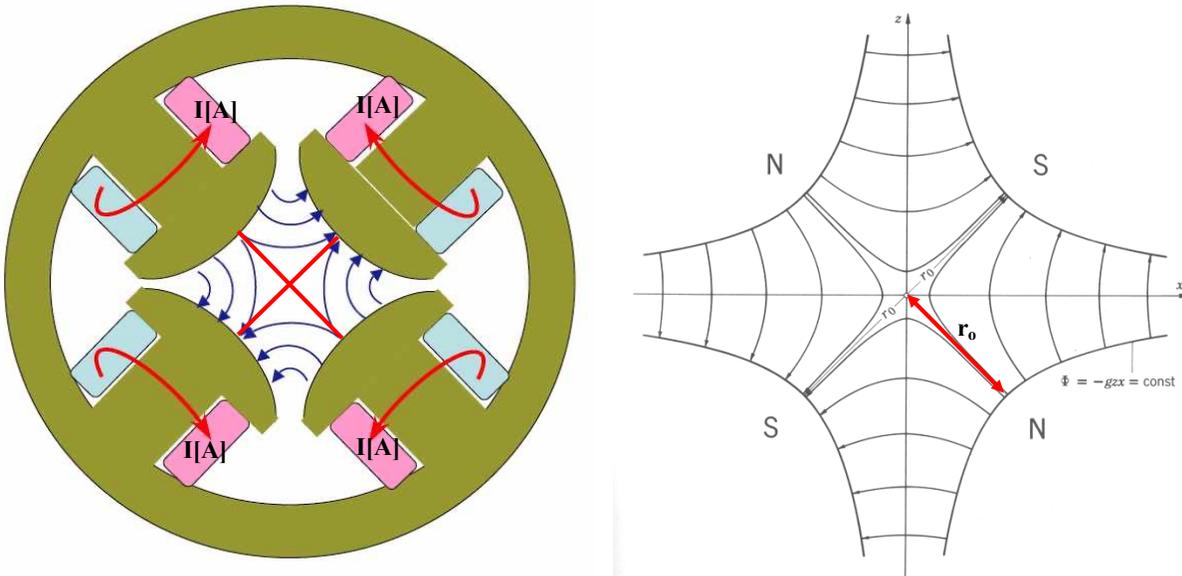


PHY 564 Midterm open-book, take home exam

Sent – evening Monday, March 23, 2026, Due -mid-night, Wednesday 25, 2026

Problem 1. Design a quadrupole for 3 GeV electron beam with pole radius of $r_o=2\text{ cm}$ (0.02 m), field gradient of 5 kGs/cm (50 T/m) and focal length of $F=1$ meter. See the figures and specific questions below. Total points: 40 points



- Fund value of field on the tip of the quadrupole pole ($r=2\text{ cm}$) – **3 points**
- What is the current in each of 4 quadrupole coils – **4 points**
- In so-called “short quadrupole” assumption, find necessary length of quadrupole needed for focal length of $F=1$ meter – **5 points**
- For a sick (real) quadrupole write equation for length L needed for $F=1$ meter – **10 points**
- Check what is actual focal length of sick quadrupole with length found in a “short-quad” approximation – **8 points**
- Attempt to solve (even approximately, or using Mathematica and other numerical tools) what is actual length needed for $F=1$ meter. – **10 points**

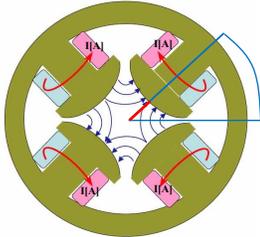
Solution:

a) Magnetic field in quadrupole is

$$\vec{B} = G \cdot (\hat{x} \cdot y + \hat{y} \cdot x); \quad x = y = \frac{r}{\sqrt{2}} \rightarrow B_r = G \cdot r$$

moving radially at 45 degrees at radius 2 cm we have radial field of 10 kGs (1 T).

b) Let's consider a contour of integral for Stock's theorem shown in the figure:



$$\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \cdot I$$

For ideal magnetic steel $H=0$ in the iron core and outside the magnet. At the horizontal part of the contour the field is perpendicular to the direction of pass. Hence, the only non-zero portion of the integral is coming from integral at 45-degrees from quad center to the pole tip:

$$\oint \vec{B} \cdot d\vec{l} = \int_0^r G \cdot r \cdot dr = \frac{G \cdot r_o^2}{2} \Rightarrow \frac{G \cdot r_o^2}{2} [Gs \cdot cm] = 0.4\pi \cdot I[A];$$

$$I[A] = \frac{G \left[\frac{Gs}{cm} \right] \cdot r[cm]_o^2}{0.8\pi} = 7,959 \text{ A}$$

c) Focal length of thin quadrupole is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -\frac{eG \cdot l}{pc} & 1 \end{bmatrix}; \quad F = -\frac{m_{11}}{m_{21}} = \frac{pc}{eG \cdot l}; \quad l_{sh} = \frac{pc}{eG \cdot F} = \frac{Bq}{G \cdot F};$$

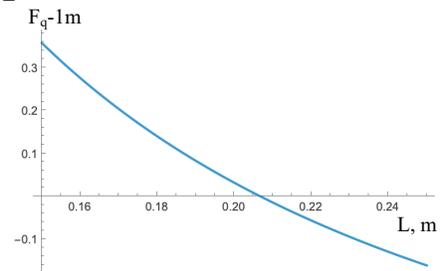
$$Bq [T \cdot m] = \frac{pc [GeV]}{0.299792458} = 10.0069 [T \cdot m]; \quad l_{sh} = 0.200138 \text{ m}$$

d) Matrix of real, thick, quadrupole and its focal length are:

$$\mathbf{M} = \begin{bmatrix} \cos\varphi & \frac{\sin\varphi}{\omega} \\ -\omega \cdot \sin\varphi & \cos\varphi \end{bmatrix}; \quad \omega = \sqrt{\frac{eG}{pc}} = \sqrt{\frac{G}{Bq}} = 2.23529 \text{ m}^{-1}; \quad \varphi = \omega \cdot L; \quad \varphi_{test} = \omega \cdot l_{sh} = 0.447368;$$

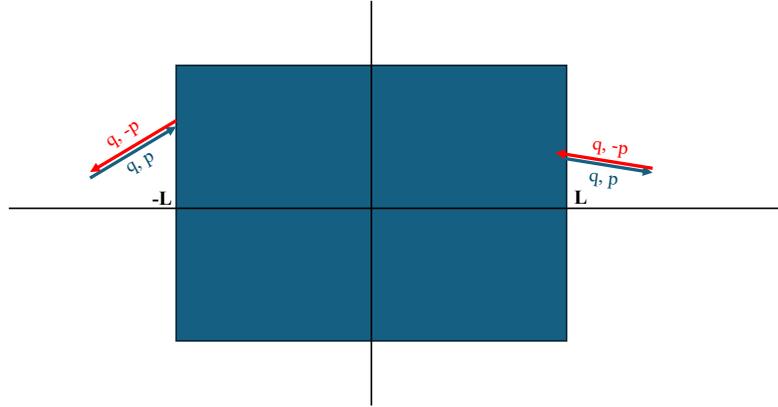
$$F_q = -\frac{m_{11}}{m_{21}} - \frac{L}{2}; \quad F_q(L=l_{sh}) = \frac{\cos\varphi_{test}}{\omega \cdot \sin\varphi_{test}} - \frac{l_{sh}}{2} = 1.03245 \text{ m}$$

e,f) Using Mathematica we get $L=0.207077$ m with a number of other roots like 2.2123 m and 4.0307 m, etc., which are longer than requested focal length. Hence, $L=0.207$ m is what we need for our design/



Problem 2. Mirror or bilaterally symmetric lattices. Total 45 points

Consider a linear transport system extending from $-L$ to $+L$ is s (value of L is not important!) and that it has a mirror (bilateral) symmetry $H(-s)=H(s)$. Derive condition on the transport matrix coefficient from $-L$ to $+L$.



- Observe (discuss) the fact that reversed trajectory has the same matrix as that in the forward direction and that $q_{\text{rev}}(+L)=q_{\text{fov}}(+L)$ and $P_{\text{rev}}(+L)=-P_{\text{fov}}(+L)$ will result in $q_{\text{rev}}(-L)=q_{\text{fov}}(-L)$ and $P_{\text{rev}}(-L)=-P_{\text{fov}}(-L)$ – **5 points**
- Consider the fact that reverse motion is described by the same matrix, but we also can use inverse matrix to connect which can be found using symplecticity conditions – **5 points**
- Derive conditions for 1D case and 2×2 matrix – **15 points**
- Derive conditions for N dimensions and $2N \times 2N$ matrices. - **20 points**

Hint: While it does not matter for 1D case, I suggest that you use overscored (Alex Dragt's) convention for $2N \times 2N$ matrices.

Solution:

a) Additional explanation (below) of the problem below explains why reverse motion is described by the same map and therefore the same matrix:

$$\begin{bmatrix} Q \\ P \end{bmatrix}_L = \mathbf{M} \cdot \begin{bmatrix} Q \\ P \end{bmatrix}_{-L} \Rightarrow \begin{bmatrix} Q \\ -P \end{bmatrix}_{-L} = \mathbf{M} \cdot \begin{bmatrix} Q \\ -P \end{bmatrix}_{+L} \quad (0)$$

Additional explanation. It is important to implicitly specify the meaning of the bilateral symmetry. We know the Hamiltonian equations (Q and P have N components):

$$\frac{dQ}{ds} = \frac{\partial H(Q,P,s)}{\partial P}; \quad \frac{dP}{ds} = -\frac{\partial H(Q,P,s)}{\partial Q}; \quad (1)$$

Let's now use reverse coordinate $z=-s$:

$$\frac{dQ}{dz} = -\frac{dQ}{ds} = -\frac{\partial H(Q,P,-s)}{\partial P}; \quad \frac{dP}{dz} = -\frac{dP}{ds} = \frac{\partial H(Q,P,-s)}{\partial Q};$$

Let's also introduce new variable

$$\widehat{P} = -P;$$

$$\frac{dQ}{dz} = \frac{\partial H(Q, -\widehat{P}, -z)}{\partial \widehat{P}}; \quad \frac{d\widehat{P}}{dz} = -\frac{\partial H(Q, -\widehat{P}, -z)}{\partial Q}.$$

This set of questions will be identical to eq (1) if

$$H(Q, -P, -s) = H(Q, P, s) \quad (2)$$

or that in addition to bilateral symmetry the Hamiltonian is an even function of momenta. In this case map from $\{Q(-L), P(-L)\}$ to $\{Q(+L), P(+L)\}$ is identical to reverse map from $Q(+L), -P(+L)\}$ to $\{Q(-L), -P(-L)\}$.

In linear system eq. (2) means that there should be no $h_{qipj}(s)q_i P_j$ terms in the Hamiltonian or there sign is flipping with the sign of s : $h_{qipj}(s) = -h_{qipj}(-s)$. For example, a bilaterally symmetric system that has torsion or solenoidal field should have $\kappa(-s) = -\kappa(s)$; $B_s(-s) = -B_s(s)$

I hoped that this would be condition you get in part a) of the problem, but since I already have questions, I decided to add this explanation to the problem.

b) At the same time we know that we can inverse symplectic matrix to connect initial

$$\begin{bmatrix} Q \\ P \end{bmatrix}_{-L} = \mathbf{M}^{-1} \cdot \begin{bmatrix} Q \\ P \end{bmatrix}_L = -\mathbf{S} \cdot \mathbf{M}^T \cdot \mathbf{S} \cdot \begin{bmatrix} Q \\ P \end{bmatrix}_L \quad (3)$$

and also can rewrite (0) as

$$\begin{bmatrix} Q \\ -P \end{bmatrix}_{-L} = \mathbf{M} \cdot \begin{bmatrix} Q \\ -P \end{bmatrix}_{+L} \Rightarrow \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} Q \\ P \end{bmatrix}_{-L} = \mathbf{M} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} Q \\ P \end{bmatrix}_{+L}; \quad \begin{bmatrix} Q \\ P \end{bmatrix}_{-L} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \cdot \mathbf{M} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} Q \\ P \end{bmatrix}_{+L} \Rightarrow$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \cdot \mathbf{M} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} = -\mathbf{S} \cdot \mathbf{M}^T \cdot \mathbf{S} \Rightarrow \mathbf{M} = -\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \cdot \mathbf{S} \cdot \mathbf{M}^T \cdot \mathbf{S} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}. \quad (4)$$

c) For 2x2 matrix is very simple

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} c & a \\ d & b \end{bmatrix} = \begin{bmatrix} d & b \\ c & a \end{bmatrix} \Rightarrow a = d \#$$

resulting in equal diagonal terms.

d) Using overscored conventions, we can simply continue with (4) to derive a bit more conditions:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = -\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{D}^T \end{bmatrix} \cdot \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{D}^T \end{bmatrix} \cdot \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix};$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{D}^T & \mathbf{B}^T \\ \mathbf{C}^T & \mathbf{A}^T \end{bmatrix} \Rightarrow \mathbf{A} = \mathbf{D}^T, \mathbf{B} = \mathbf{B}^T; \mathbf{C} = \mathbf{C}^T \#$$

With all block matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ being $N \times N$, the first equation $\mathbf{A} = \mathbf{D}^T$ imposed N^2 conditions and next two symmetry conditions impose $N(N-1)/2$ conditions each, hence the total of $2N^2 - N$ conditions: 1 for $N=1$, 6 for $N=2$ and 15 for $N=3$. Hence, such lattices are popular...

Problem 3. Muon collider – 15 points

Muons are unstable particles with energy of 105.66 MeV and 2.197 microseconds e-fold lifetime in rest frame. Physics community is considering a multi-TeV circular muon collider where muons circulate and collide for a number of turns. Consider a storage ring with average bending magnetic field of 10 T and derive number of turns (collisions) muon can conclude during it e-fold lifetime. Does this number depend on the muon beam energy?

Suggestions/Hints: consider ultra-relativistic muons moving with speed practically indistinguishable from the speed of light (indeed only such colliders are of interest). Try to understand

Solution: Let's consider a circular collider for muons with energy E and corresponding momentum

$$p = \beta \cdot \frac{E}{c}; \beta = \sqrt{1 - \gamma^{-2}}; \gamma = \frac{E}{mc^2}.$$

The curvature of the orbit is defined by the value of magnetic field and beam rigidity

$$K_o(s) = \frac{eB_y(s)}{pc}.$$

To make one turn each particle has to turn by 2π angle, i.e. we can connect ring circumference with average bending magnetic field:

$$\int_0^C K_o(s) ds = \frac{e}{pc} \int_0^C B_y(s) ds = \frac{e \cdot \langle B_y \rangle \cdot C}{pc} = 2\pi \Rightarrow C = \frac{2\pi \cdot pc}{e \cdot \langle B_y \rangle} = \frac{2\pi \cdot BQ}{\langle B_y \rangle}.$$

Time required for particle to make a turn is:

$$T_o = \frac{C}{v} = \frac{C}{\beta c} = \frac{2\pi \cdot pc}{\beta c \cdot e \cdot \langle B_y \rangle} = \frac{2\pi}{c} \cdot \frac{E}{e \langle B_y \rangle}.$$

Life time of the moving muon is boosted by factor γ

$$\tau_c = \gamma \cdot \tau_o = \frac{E}{mc^2} \cdot \tau_o.$$

and number of turns during e-fold reduction of muon beam intensity is

$$N = \frac{\tau_c}{T_o} = c \cdot \tau_o \cdot \frac{E}{mc^2} \cdot \frac{e \langle B_y \rangle}{2\pi \cdot E} = \frac{c \cdot \tau_o}{2\pi} \cdot \frac{e \langle B_y \rangle}{mc^2}$$

does not depend on the beam energy but is proportional to the average bending magnetic field. This is why there is a strong drive to develop superconducting dipole magnets with as high field as possible. For 10 T average magnetic field we have

$$N = \frac{c \cdot \tau_o}{2\pi} \cdot \frac{e \langle B_y \rangle}{mc^2} = 3,122$$

This number is very small when compared with millions and billions of turns stable particles such as protons and electrons can survive in colliders – hence the problem with generating, cooling and colliding muons is one of the major challenges in modern accelerator science.