Solutions for HW # 8

1. The longitudinal impedance is defined as

$$Z_{//}(\omega) = \frac{1}{c}\int_0^\infty w_{//}(s)e^{i\omega s/c}ds \ .$$ (1)

Taking the complex conjugate of eq. (1) and noticing that

$$w_{//}^*(s) = w_{//}(s) \ ,$$ (2)

it follows that

$$Z_{//}^*(\omega) = \frac{1}{c}\int_0^\infty w_{//}(s)e^{-i\omega s/c}ds = \frac{1}{c}\int_0^\infty w_{//}(s)e^{i(-\omega)s/c}ds = Z_{//}(-\omega) \ .$$ (3)

The transverse impedance is defined as

$$Z_\perp(\omega) = -\frac{i}{c}\int_0^\infty w_\perp(s)e^{i\omega s/c}ds \ .$$ (4)

Taking the complex conjugate of eq. (4) and noticing that

$$w_\perp^*(s) = w_\perp(s)$$ (5)

yield

$$Z_\perp^*(\omega) = \frac{i}{c}\int_0^\infty w_{//}(s)e^{-i\omega s/c}ds = -\left[-i\frac{1}{c}\int_0^\infty w_{//}(s)e^{i(-\omega)s/c}ds\right] = -Z_\perp(-\omega) \ .$$ (6)

2. Using the relation

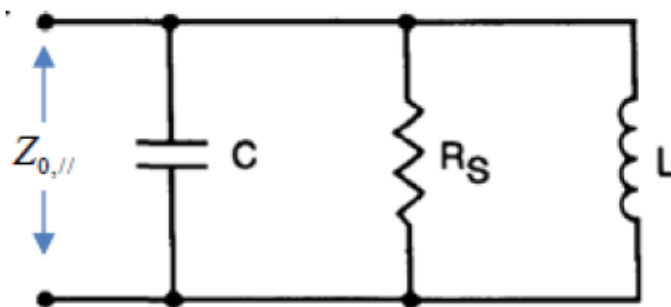$$F(z) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikz}\tilde{F}(k)dk \ ,$$

we obtain

$$\sum_{l=-\infty}^{\infty} F(lC) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iklC} \tilde{F}(k) dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \tilde{F}(k) \sum_{l=-\infty}^{\infty} e^{i2\pi l \cdot \frac{kC}{2\pi}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \tilde{F}(k) \sum_{p=-\infty}^{\infty} \delta\left(\frac{kC}{2\pi} - p\right) \qquad ,$$

$$= \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2\pi\delta\left(k - \frac{2\pi p}{C}\right)}{C} \tilde{F}(k) dk$$

$$= \frac{1}{C} \sum_{p=-\infty}^{\infty} \tilde{F}\left(\frac{2\pi p}{C}\right)$$

where we also used

$$\delta(g(x)) = \sum_{i} \frac{\delta(x - x_i)}{|g'(x_i)|}$$

with x_i being the roots of $g(x)$.

3.

The impedance is determined by

$$\frac{1}{Z_{0,//}} = \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C}$$

$$= \frac{1}{R_s} + \frac{1}{j\omega L} + j\omega C$$

$$= \frac{1 + jR_s\sqrt{\dfrac{C}{L}}\left(\omega\sqrt{LC} - \dfrac{1}{\omega\sqrt{LC}}\right)}{R_s} \quad , \tag{7}$$

$$= \frac{1 + jQ\left(\dfrac{\omega}{\omega_R} - \dfrac{\omega_R}{\omega}\right)}{R_s}$$

$$= \frac{1 + iQ\left(\dfrac{\omega_R}{\omega} - \dfrac{\omega}{\omega_R}\right)}{R_s}$$

i.e.

$$Z_{0,//} = \frac{R_s}{1 + iQ\left(\dfrac{\omega_R}{\omega} - \dfrac{\omega}{\omega_R}\right)}$$

where $j = -i$, $Q \equiv R_s\sqrt{\dfrac{C}{L}}$ and $\omega_R \equiv \dfrac{1}{\sqrt{LC}}$.