PHY 554. Homework 1 solutions

Handed: September 6
Return by: September 13

HW 1.1 (3 points): Find available energy (so called C.M. energy) for a head-on collision of electrons and protons in two electron-hadron collisions:

(a) CEBAF collides 12 GeV electrons with protons at rest (the rest energy of proton is 0.938257 GeV);
(b) LHeC ep plans to collide 60 GeV electrons with 7 TeV protons.

Solution: we should use formula for available c.m. energy:

$$E_{cm} \equiv Mc^2 = c\sqrt{P_i P^i} = \sqrt{E^2 - \left(c\cdot\vec{p}\right)^2}; E = E_1 + E_2; \vec{p} = \vec{p}_1 + \vec{p}_2 \quad (1)$$

(a) Proton has zero momentum and energy of 0.938257 GeV and total energy is E=12.938257 GeV. Momentum of electron scaled by speed of light, *pc*, deviates from electron's energy by 11 eV – in other words, it is is the same as its energy up to the 10^{th} digit… Putting numbers in the exact formula gives E_{cm}=4.837 GeV. Approximate formula for beam heating target at rest that we derived in Lecture 1

$$E_{cm} \equiv Mc^2 \cong \sqrt{2E_1 \cdot m_2 c^2}$$

gives an approximation f E_{cm}~ 4.745 GeV, which is 2% lower.
(c) It is also true that *pc* for 60 GeV electrons is practically indistinguishable from its energy (till 12^{th} digit!). The *pc of* 7 TeV protons is also very close to 7 TeV – difference is only in 9^{th} digit! Hence, total energy E=(7+0.06) TeV and total momentum pc=(7-0.06) TeV, where we took into account that electrons and protons are moving in opposite direction! Putting numbers in exact formula (1) gives E_{cm}=1.296 TeV.

HW 1.2 (2 points): Electron ion collider (EIC) will be built in Brookhaven National Lab to collider 18 GeV electrons with 275 GeV protons and 100 GeV/u heavy ions. It will be located in RHIC tunnel with circumference of 3834 m.

(a) 1 point: What will be bending radius of 275 GeV protons in EIC dipole magnets with magnetic field of 3.8 T?
(b) 1 point: What magnetic field is required to turn 18 GeV electrons with the same radius?

Solution: we should use formula connecting beam rigidity, strength of magnetic field and the bending radius: :

$$B\rho = \frac{pc}{e} \Leftrightarrow \left\{ \begin{array}{c} B\rho\left[kGs \cdot cm\right] = \frac{pc[MeV]}{0.299792458} \cong \frac{pc[MeV]}{0.3} \\ B\rho\left[T \cdot m\right] = \frac{pc[GeV]}{0.299792458} \cong \frac{pc[GeV]}{0.3} \\ B\rho\left[T \cdot km\right] = \frac{pc[TeV]}{0.299792458} \cong \frac{pc[TeV]}{0.3} \end{array} \right\}$$

(a) The *pc* of 275 GeV protons is 274.998 GeV (again, very close to the energy!) and with 3.8 T magnetic field

$$\rho\left[m\right] = \frac{pc[GeV]}{0.299792458 \cdot B[T]} = 241.4m$$

(b) The *pc* of 18 GeV electrons is again parctically 18 GeV. Hence magnetic field requred to bet at radius of 241.4 m is 0.249 T. It can be calcuated from the formula

$$B[T] = \frac{18[GeV]}{0.299792458 \cdot 241.4\left[m\right]} = 0.249T$$

or simply scaled proportinally to the rigidity of the beams

$$\frac{B_e}{B_p} = \frac{pc_e}{pc_p} = 0.0655$$

HW 1.3 (2 points): For a classical microtron with orbit factor k=1 and energy gain per pass of 1.022 MeV and operational RF frequency 1.5 GHz (1.5 x 10^9 Hz) find required magnetic field. What will be radius of first orbit in this microtron?
Hint: Note that rest energy of electron with γ=1 is 0.511 MeV. This is energy gain per pass will define available n numbers in eq. (2.6)

Solution: By design, the election's energy gain on the RF cavity is twice of the its rest energy – it means that electron at the first orbit has $\gamma=3$ and energy of 1.533 MeV. It also means that on the second orbit electrons will have $\gamma=5$, $\gamma=7$ at the third orbit, etc... With k=1 and integer γ and n, the resonance conditions (2.6)

$$\frac{2\pi mc \cdot f_{RF}}{eB} \cdot \gamma_n = n;$$

can be satisfied only if $j = \frac{2\pi mc \cdot f_{RF}}{eB}$ is an integer. Let's assume that $j=1$, then n takes numbers 3,5,7.. for first second and third orbits. For $j=2$, n starts at 6 and gain 4 at each turn…. It allows us to define required magnetic field

$$B = \frac{2\pi mc \cdot f_{RF}}{j \cdot e} = \frac{1}{j} \cdot \frac{mc^2}{e} \cdot \frac{2\pi f_{RF}}{c}$$

with mc^2=0.511 MeV, $\frac{mc^2}{e}$ gives us rigidity of 1.702 kGs cm. $\lambdabar_{RF} = \frac{c}{2\pi f_{RF}}$ = 3.18 cm is the RF wavelength divided by. 2π results in

$$B\left[kGs\right] = \frac{1}{j} \cdot \frac{1.703\left[kGs \cdot cm\right]}{\lambda_{RF}[cm]} = \frac{0.536}{j} kGs$$

At *E=1.533 MeV* electrons are still not moving at the speed of light and pc is slightly different from *E*:

$$pc = \sqrt{E^2 - (mc^2)^2} = 1.445 MeV$$

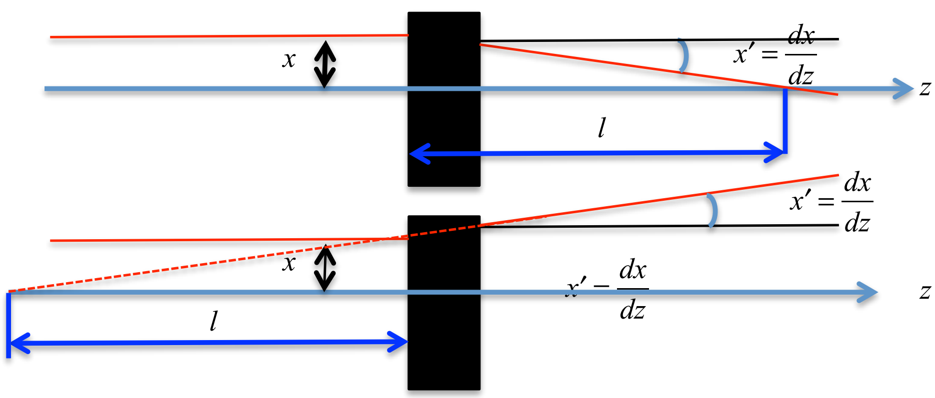
which correspond to radius of the trajectory of

$$\rho\left[cm\right] = \frac{pc[MeV]}{0.299792458 \cdot B[kGs]} = j \cdot 9cm$$

HW 1.4 (5 point): Let's first determine an effective focal length, *F*, of a paraxial (e.g. small angles!) focusing object (a black-box) as ratio between a parallel displacement of trajectory at its entrance to corresponding change of the angle at its exit (see figure below):

$$F = -\frac{x}{x'}; x' \equiv \frac{dx}{dz}$$

see figure below for



Let consider a doublet of two thin lenses: a focusing (*F*) and defocusing (*D*) lenses with equal but opposite in sign focal length F with center separated by distance L as in Fig. 1.

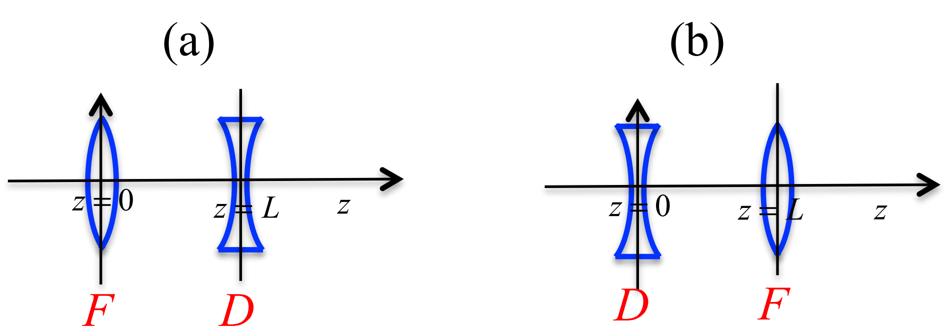


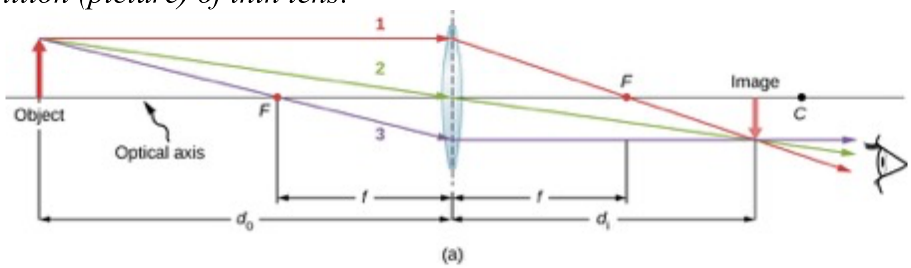
Fig.1. Two combinations of a doublet: *FD* and *DF*.

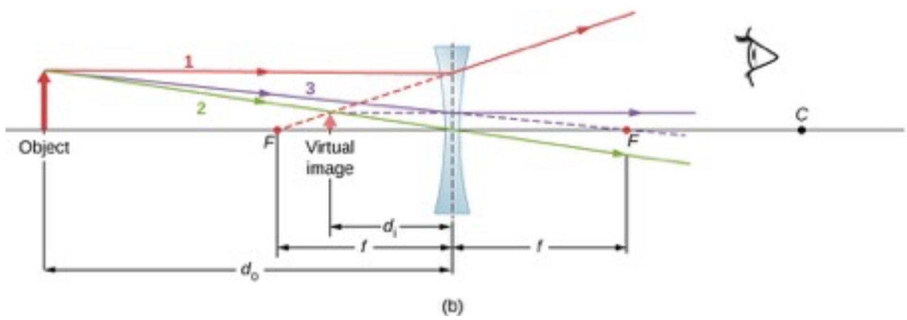
1. (3 points) Show through a calculation of the ray trajectory that the focal lengths of *FD* and *DF* doublets are equal and given by following expression:

$$F_{eff} = \frac{F^2}{L}$$

2. (2 points) The ray (trajectory) parallel to the axis is entering the FD or DF system of lenses. Using you calculation of the trajectories in *FD* and *DF* doublets, determine location of to the ray crossing the axis and find their difference between *FD* and *DF* doublets. Since a quadrupole focusing in horizontal plane is defocusing in vertical plane - and visa versa – by solving this your find astigmatism of a doublet built from two quadrupoles, i.e. difference between locations of the focal planes for horizontal and vertical direction of motion.

P.S. Definition (picture) of thin lens:





Solution: In both cases we start from initial conditions

$$x = x_o; x' = 0;$$

and apply following transformations:

$$F\,lens : x_{out} = x_{in}; x'_{out} = x'_{in} - \frac{x_{in}}{F};$$

$$D\,lens : x_{out} = x_{in}; x'_{out} = x'_{in} + \frac{x_{in}}{F};$$

$$Drift : x_{out} = x_{in} + Lx'_{in}; x'_{out} = x'_{in};$$

For FD case is gives us

$$x_1 = x_0; x_1' = -\frac{x_0}{F} \to x_2 = x_0 - L\frac{x_0}{F}; x_2' = -\frac{x_0}{F} \to$$
$$x_3 = x_0 - L\frac{x_0}{F}; x_3' = -\frac{x_0}{F} + \frac{1}{F}\left(x_0 - L\frac{x_0}{F}\right) = -L\frac{x_0}{F^2}; \tag{1}$$

and for DF case

$$x_1 = x_0; x_1' = +\frac{x_0}{F} \to x_2 = x_0 + L\frac{x_0}{F}; x_2' = +\frac{x_0}{F} \to$$
$$x_3 = x_0 + L\frac{x_0}{F}; x_3' = +\frac{x_0}{F} - \frac{1}{F}\left(x_0 + L\frac{x_0}{F}\right) = -L\frac{x_0}{F^2}; \tag{2}$$

with x_3' being the angle at the exit of the “black box” and x_o being the postion at its entrance, the answer for the first question is coming from $x_3' = -L\frac{x_0}{F^2}$ for both FD and DF cases.

The location of the ray crossing the z-axis coming from dividing the position at the exit of the second lens by the angle and adding L (distance from the starting point):

$$F:\ Z = L - \frac{x_3}{x_3'} = L + \frac{F^2}{L}\left(1 - \frac{L}{F}\right) = L - F + \frac{F^2}{L}$$
$$D:\ Z = L - \frac{x_3}{x_3'} = L + \frac{F^2}{L}\left(1 + \frac{L}{F}\right) = L + F + \frac{F^2}{L} \tag{3}$$

Hence, the astigmatism of FD set is equal to 2F.