Problem 1

The maximal gain happens at the detuning satisfying the following equation.

$$\frac{d}{d\hat{C}}\left[\hat{C}^{-3}\left(1-\cos\hat{C}-\frac{1}{2}\hat{C}\sin\hat{C}\right)\right]=\hat{C}^{-4}\left[3\cos\hat{C}+2\hat{C}\sin\hat{C}-\frac{1}{2}\hat{C}^2\cos\hat{C}-3\right]=0. \tag{1}$$

Numerically finding the solutions of

$$3\cos\hat{C}+2\hat{C}\sin\hat{C}-\frac{1}{2}\hat{C}^2\cos\hat{C}-3=0 \tag{2}$$

, which leads to maximal gain, gives

$$\hat{C}\approx 2.606. \tag{3}$$

Inserting the definition of $\hat{C}$ into eq. (3) leads to

$$Cl_w=\left(k_w+k-\frac{\omega}{v_z\left(\varepsilon_0+\Delta\varepsilon\right)}\right)l_w\approx\frac{\omega l_w}{\gamma_z^2 c}\frac{\Delta\varepsilon}{\varepsilon_0}=2.606 \tag{4}$$

, or

$$\frac{\Delta\varepsilon}{\varepsilon_0}=2.606\frac{\gamma_z^2\lambda_0}{2\pi l_w}=2.606\frac{\lambda_w}{4\pi l_w}=\frac{2.606}{4\pi}\frac{1}{N_w}, \tag{5}$$

where $N_w=l_w/\lambda_w$ is the number of wiggler period.

Problem 2

From the criteria of FEL saturation,

$$\Omega_p L_G = 1 \, , \tag{6}$$

we obtain

$$\Omega_p \equiv \sqrt{\frac{eE\theta_s\omega}{\gamma_z^2 cE_0}} = \frac{1}{L_G} = \sqrt{3}\Gamma \, . \tag{7}$$

Taking the fourth power of eq. (7) yields

$$\left(\frac{eE\theta_s\omega}{\gamma_z^2 cE_0}\right)^2 = 9\Gamma^4 = 9\Gamma\Gamma^3 = 9\Gamma \frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma I_A} \, , \tag{8}$$

where we used

$$\Gamma \equiv \left[\frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma I_A}\right]^{1/3} \, . \tag{9}$$

Eq. (8) can be rewritten into

$$E^2 = 9\Gamma \frac{c\gamma_z^2}{\omega} \frac{\pi j_0}{e^2 \gamma I_A} E_0^2 = 9\rho \frac{\pi j_0}{e^2 \gamma I_A} E_0^2 \, . \tag{10}$$

Inserting the definition of Alfven current,

$$I_A = \frac{4\pi\varepsilon_0 mc^3}{e} \, , \tag{11}$$

into eq. (10) yields

$$E^2 = 9\rho \frac{\pi j_0}{e\gamma 4\pi\varepsilon_0 mc^3} E_0^2 = \frac{9}{4}\rho \frac{j_0}{e\varepsilon_0 c} E_0 \, , \tag{12}$$

where we used

$$E_0 = m\gamma c^2 \, . \tag{13}$$

Consequently, the radiation power at saturation is

$$P_{sat} = \varepsilon_0 cE^2 A = \frac{9}{4}\rho \frac{j_0 A}{e} E_0 = \frac{9}{4}\rho \frac{E_0}{e} I_e \, , \tag{14}$$

i.e. $\chi = \dfrac{9}{4}$.