Homework 8.

Problem 1. 10 points. In number of occasions, it is useful to stretch bunch using two frequency RF system. Consider a storage ring negative η_τ and the RF system operating at two frequencies, the fundamental and the 3rd harmonics:

$$\frac{dE\cdot}{ds} = \frac{e}{C}\left(V_1 \cdot \sin\left(h_{rf} \cdot k_o \cdot \tau\right) + V_3 \cdot \sin\left(3h_{rf} \cdot k_o \cdot \tau + \varphi_3\right)\right)$$

Find at what ratio between the voltages and phase of third harmonic the frequency of small oscillations turns into zero. For this case, find stationary points on the phase diagram, draw characteristic phase-space trajectories (approximately is fine) and show the direction of the motion by arrows.

Problem 2. 4x5 points.

For a single frequency RF system with Hamiltonian with α indicating an energy loss/gain,

$$\langle \mathrm{H}_s \rangle = \eta_\tau \frac{\pi_\tau^2}{2} + \frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf} \tau\right)}{k_o h_{rf}} + \alpha \cdot \tau; \quad \eta_\tau < 0.$$

1. Define the stationary points (RF phases) in the phase space and indicate level of α when stationary points are no longer exists.
2. Draw phase space trajectories for $\alpha = \frac{1}{2} \cdot \frac{1}{C}\frac{eV_{RF}}{p_o c}$. Show the direction of the motion by arrows.
3. Define the depth of the "RF bucket", e.g. the difference between the maximum and minimum π_τ staying within a single RF separatrix (e.g. being localized). Express it through the RF voltage, the slip factor and the value of stationary phase.
   Note – consider the central separatrix around $\tau = 0$.
4. Find period of the oscillation as function of $\langle \mathrm{H}_s \rangle$ inside the central separatrix (around $\tau = 0$).

Solution:

Problem 1. Adding the corresponding term into the longitudinal Hamiltonian gives:

$$\langle \mathrm{H}_s \rangle = \eta_\tau \frac{\pi_\tau^2}{2} - \frac{e}{p_o c \cdot C \cdot h_{rf} k_o}\left(V_1 \cos\left(h_{rf} k_o \tau\right) + V_3 \frac{\cos\left(3h_{rf} k_o \tau + \varphi_3\right)}{3}\right)$$

and we need to stationary points

$$\frac{d\tau}{ds} = \frac{\partial H_s}{\partial \pi_\tau} = \eta_\tau \cdot \pi_\tau = 0; \quad \frac{d\pi_\tau}{ds} = -\frac{\partial H_s}{\partial \tau} = \frac{e}{p_o c C}\left(V_1 \sin\left(h_{rf} k_o \tau\right) + V_3 \sin\left(3h_{rf} k_o \tau + \varphi_3\right)\right) = 0;$$

$$\pi_\tau = 0; \quad \frac{\sin\left(3\varphi_o + \varphi_3\right)}{\sin\left(\varphi_o\right)} = -\frac{V_3}{V_1}; \quad \varphi_o = h_{rf} k_o \tau_o$$

which clearly depend of the amplitude and phase of 3rd harmonic. The only trivial case is when φ_3=0, resulting in trivial $\varphi_0=n\pi$. Let's expand Hamiltonian to the second (oscillator) order near stationary point:

$$\langle H_s \rangle = \eta_\tau \frac{\pi_\tau^2}{2} + \frac{e}{p_o c \cdot C \cdot}\left\{\left(V_1 \sin\left(\varphi_o\right) + V_3 \sin\left(3\varphi_o + \varphi_3\right)\right) \cdot \left(\tau - \tau_o\right)\right\} +$$

$$+ h_{rf} k_o \cdot \left(V_1 \cos\left(\varphi_o\right) + 3V_3 \cos\left(3\varphi_o + \varphi_3\right)\right) \frac{\left(\tau - \tau_o\right)^2}{3} + const + O\left(\left(\tau - \tau_o\right)^3\right)$$

$$const = \left(V_1 \cos\varphi_o + V_3 \frac{\cos\left(3\varphi_o + \varphi_3\right)}{3}\right);$$

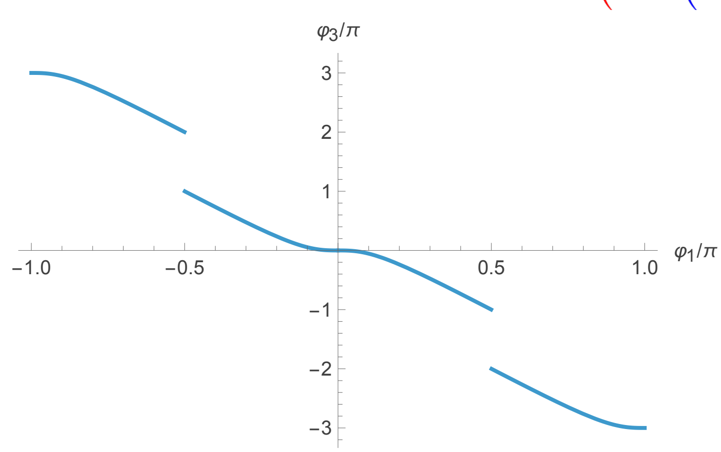
with linear term vanishing at stationary points we have second and higher order terms of τ. Eliminating second order term will result in zero rigidity of the oscillator and zero synchrotron tune at the stationary point we have:

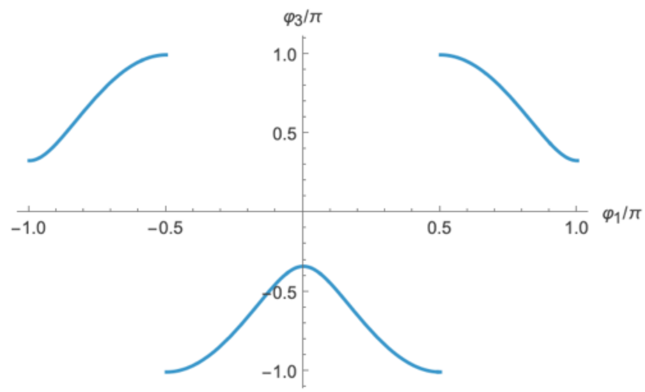
$$V_1 \sin\left(\varphi_o\right) = -V_3 \sin\left(3\varphi_o + \varphi_3\right); \quad V_1 \cos\left(\varphi_o\right) = -3V_3 \cos\left(3\varphi_o + \varphi_3\right) \Rightarrow$$

$$3 \cdot \tan\varphi_o = \tan\left(3\varphi_o + \varphi_3\right); \quad V_3 = -\frac{V_1 \sin\left(\varphi_o\right)}{\sin\left(3\varphi_o + \varphi_3\right)}.$$

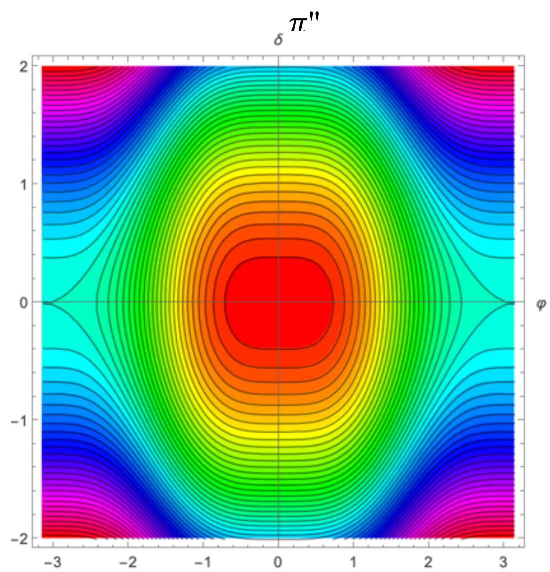
The easiest way is to solve first equation for φ_3

$$\theta\left(\varphi_o\right) = \varphi_3\left(\varphi_o\right) + 3\varphi_o = \tan^{-1}\left(\frac{\tan\varphi_o}{3}\right); \quad V_3 = -\frac{V_1 \sin\left(\varphi_o\right)}{\sin\left(\tan^{-1}\left(\frac{\tan\varphi_o}{3}\right)\right)}.$$

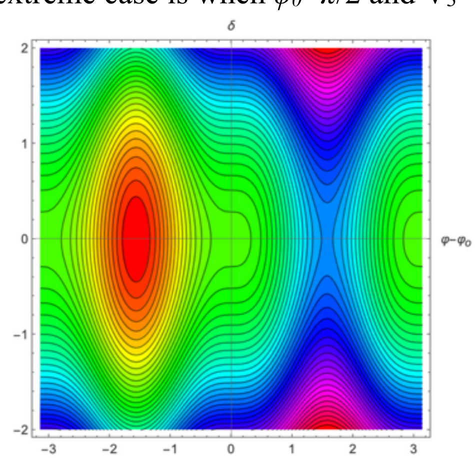


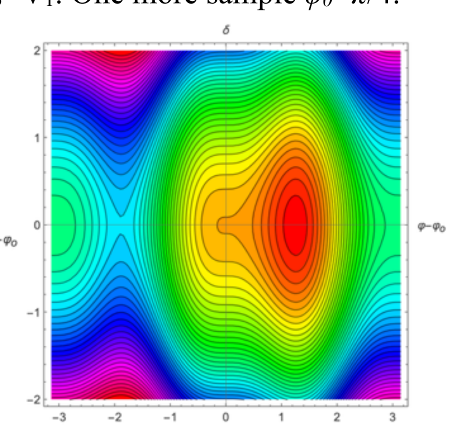


For demonstration of the typical cases, let's consider $\varphi_3=0$ giving us $\varphi_0=n\pi$; $V_3=\pm V_1/3$ and rectangular trajectories in the phase space (using negative V_1)

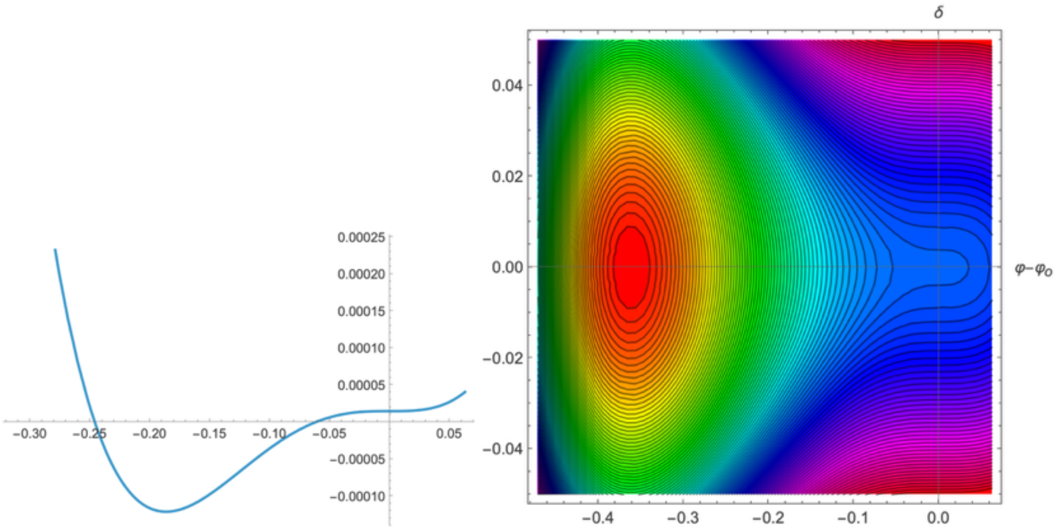


The other extreme case is when $\varphi_0=\pi/2$ and $V_3=V_1$. One more sample $\varphi_0=\pi/4$:





Only in the case φ_3=0 and φ_0=$n\pi$ we have a single separatrix with zero oscillation frequency in the center. In all other cases, the potential develops an additional stationary point with non-zero synchrotron frequency. In this sense, symmetric stretching only possible with φ_3=0 and φ_0=$n\pi$ and requires a careful handling of the voltages and phases of RF cavities.



Problem 2. With given Hamiltonian

$$\langle \mathrm{H}_s \rangle = \frac{\eta_\tau}{C}\frac{\pi_\tau^2}{2} + \frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos(k_o h_{rf}\tau)}{k_o h_{rf}} + \alpha \cdot \tau; \quad \eta_\tau < 0.$$

we can easily find stationary points:

$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau = 0 \Rightarrow \pi_\tau = 0; \quad \varphi = h_{rf} k_o \tau;$$

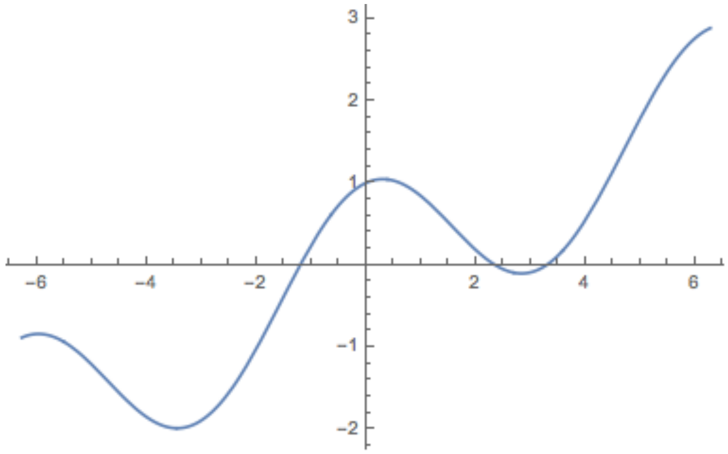
$$\frac{d\pi_\tau}{ds} = \frac{eV_o}{p_o c C}\sin\varphi - \alpha = 0 \Rightarrow \sin\varphi_o = l_f = C \cdot p_o c \left| \frac{\alpha}{eV_o} \right|.$$

$$\varphi_{o\pm} = \left\{ \begin{matrix} 2n\pi + \varphi_o \\ (2n+1)\pi - \varphi_o \end{matrix} \right\}; \ \varphi_o = \sin^{-1}(l_f);$$

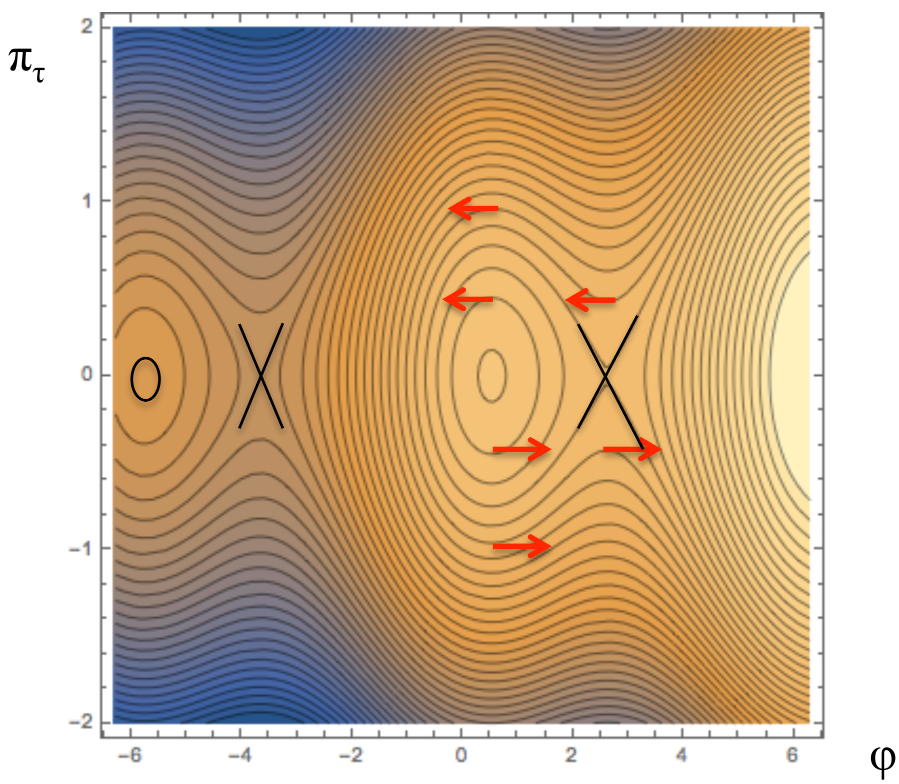
Stationary point do not exist (e.g. it requires impossible $\left|\sin\varphi_o\right| > 1$) if the energy loss (gain) per turn exceed the maximum energy change in the RF cavity:

$$|\Delta E| = |\alpha| C \cdot p_o c \geq \left| eV_o \right|$$

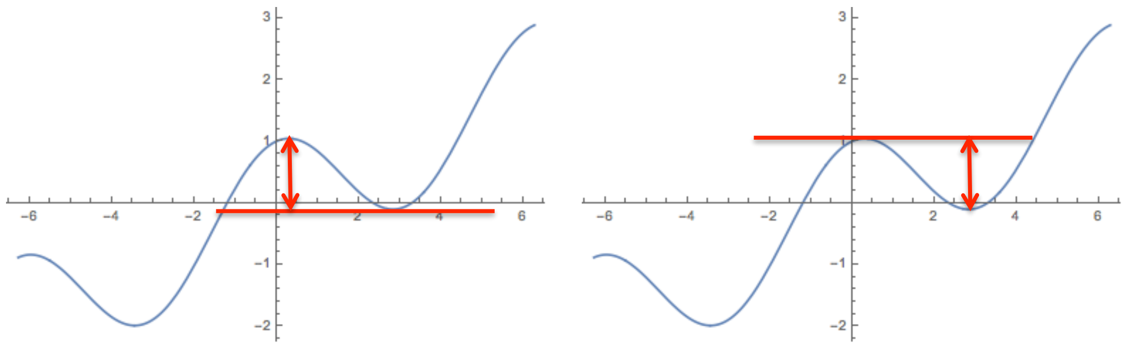
We now assume that the energy loss α is positive (typical for synchrotron radiation losses or acceleration), we can see that the potential in the Hamiltonian has maxima at $\varphi_o = 2n\pi + \sin^{-1}(l_f)$ and minima at $\varphi_o = (2n+1)\pi - \sin^{-1}(l_f)$. It means that for negative slip factor (typical for electron storage rings) $\varphi_o = 2n\pi + \sin^{-1}(l_f)$ will be stable.



For positive slip factor, $\varphi_o = (2n+1)\pi - \sin^{-1}(l_f)$ will be stable point. For l_f =1/2 $\varphi_{o+} = \sin^{-1}(0.5) = \frac{\pi}{6}$, e.g. 30 degrees. For a negative slip factor the phase-space trajectories will be as shown below.



The depth of the RF bucket depends on the difference between the maxima and minima of the potential – see fig. below.



Potential depth for negative (left) and positive (right) slip factor – it is obvious that the just different by sign. Particles outside of the RF "bucket" are not limited in the motion and slip either toward positive or negative infinity…

Finding the difference is straightforward:

$$\frac{1}{Ck_o h_{rf}} \frac{|eV_{RF}|}{p_o c} \left| \Delta\left(\cos(\varphi) + \sin\varphi_o \cdot \varphi\right)\Big|_{\varphi_{o-}}^{\varphi_{o+}} \right| = \frac{1}{Ck_o h_{rf}} \frac{|eV_{RF}|}{p_o c} \left(2\cos\varphi_o + \sin\varphi_o \cdot (2\varphi_o - \pi)\right);$$

and the maximum π_τ can be calculated by simply equating the depth of the potential well with kinetic energy $|\eta_\tau| \frac{\pi_\tau^2}{2C}$:

$$\pi_{\tau acc} = \left| \frac{2}{\eta_\tau k_o h_{rf}} \frac{eV_{RF}}{p_o c} \left(2\cos\varphi_o + \sin\varphi_o \cdot \left(2\varphi_o - \pi\right)\right) \right|^{1/2}$$

It can be rewritten in many forms. Without energy loss the RF bucket depth is maximums and equal to

$$\pi_{\tau acc} = 2\left| \frac{1}{C\eta_\tau k_o h_{rf}} \frac{eV_{RF}}{p_o c} \right|^{1/2}$$

To find the small amplitude oscillation frequency, we need to expand the potential around the stationary point $\varphi_{o\pm}$ to the second order. Since $\alpha \cdot \tau$ is a linear function, it does not contribute to the second order ter. Hence, noticing that

$$\cos\left(\varphi_o + \delta\varphi\right) = \cos\varphi_o \cos\delta\varphi - \sin\varphi_o \sin\delta\varphi = \cos\varphi_o \left(1 - \frac{\delta\varphi^2}{2}\right) + \sin\varphi_o \delta\varphi + O\left(\delta\varphi^2\right)$$

we can simply conclude that our result will be different from what we derive in class by coefficient $\cos\varphi_o$:

$$\Omega = \frac{1}{C} \left| \eta_\tau k_o h_{rf} \frac{eV_{RF}}{p_o c} \cos\varphi_o \right|^{1/2}$$

We used here the fact that $\cos\varphi_{o-} = -\cos\varphi_{o+}$. For large amplitude oscillations, we can use the fact that the Hamiltonian is invariant and

$$\frac{\eta_\tau}{C}\frac{\pi_\tau^{\,2}}{2}+\frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}}+\alpha\cdot\tau=H_o=inv$$

$$\pi_\tau=\eta_\tau\frac{d\tau}{ds}=\pm\sqrt{H_o-\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}}-C\alpha\cdot\tau};$$

$$ds=\eta_\tau\frac{d\tau}{\pm\sqrt{H_o-\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}}-C\alpha\cdot\tau}};$$

$$s=s_o\pm\eta_\tau\int_{\tau_1}^{\tau_2}\frac{d\tau}{\sqrt{H_o-\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}}-C\alpha\cdot\tau}}$$

where $\tau_{1,2}$ are stopping points defined by $\pi_\tau=0$:

$$\frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau_{1,2}\right)}{k_o h_{rf}}+\alpha\cdot\tau_{1,2}=H_o.$$

Since the period of oscillations comprises of travel back and forth, we have:

$$\frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau_{1,2}\right)}{k_o h_{rf}}+\alpha\cdot\tau_{1,2}=H_o;$$

$$P=2\left|\eta_\tau\right|\int_{\tau_1}^{\tau_2}\frac{d\tau}{\sqrt{H_o-\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}}-C\alpha\cdot\tau}}.$$