PHY 564
Advanced Accelerator Physics
Lectures 11 and 12
Linear accelerators
and RF systems for storage rings

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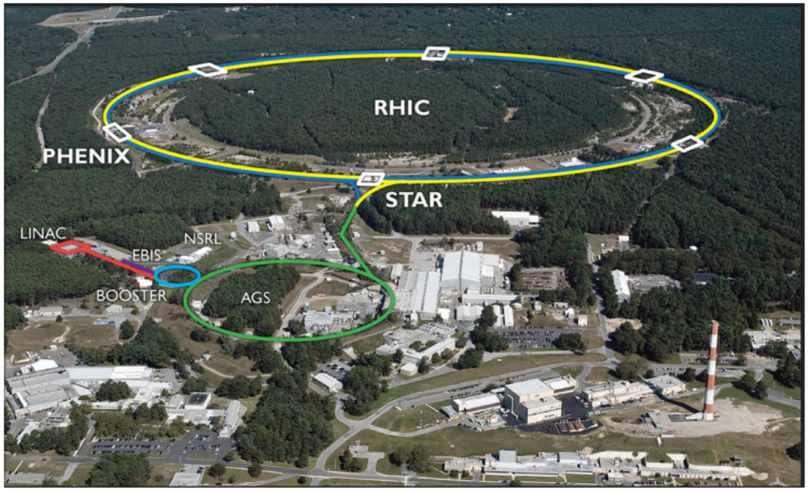
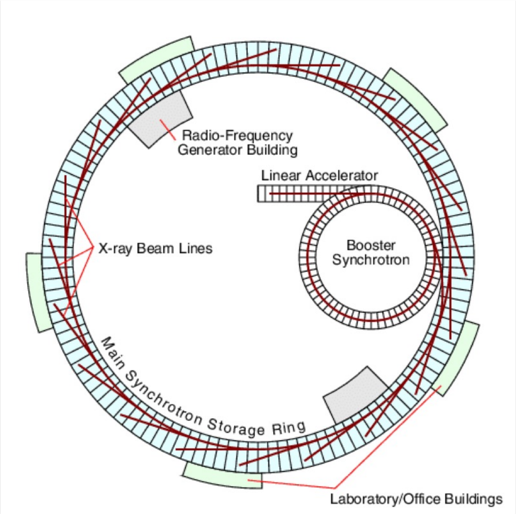
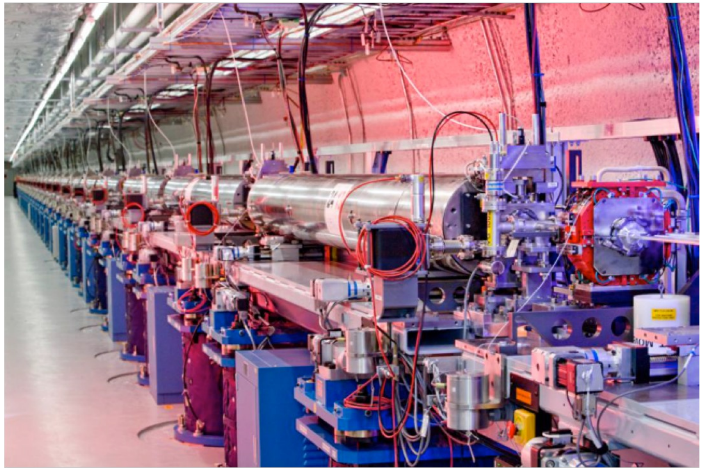
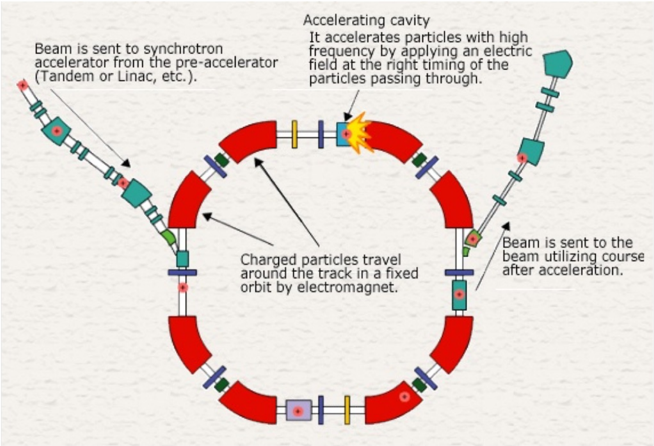
Today we will focus our attention other important elements for accelerating particles – RF cavities and linear accelerators. These devices are no longer have time independent fields – instead, they generate oscillating EM field and need proper set of Maxwell equations. There is a lot of literature related to these important devices, but today we will focus on EM field structure important for this course, leaving a lot on the side: https://sites.google.com/view/srfsbu2023/home

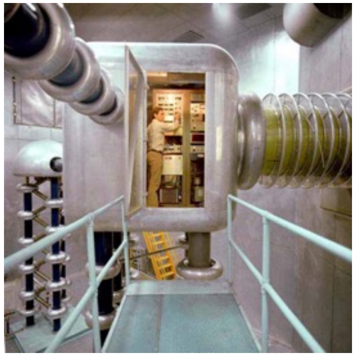


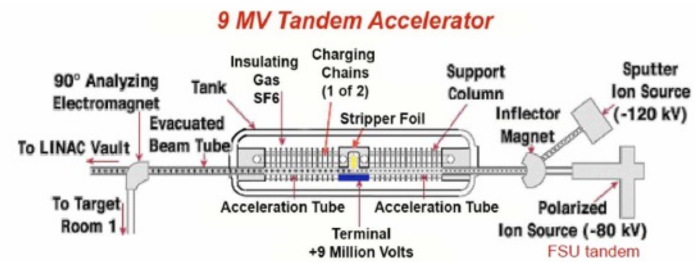


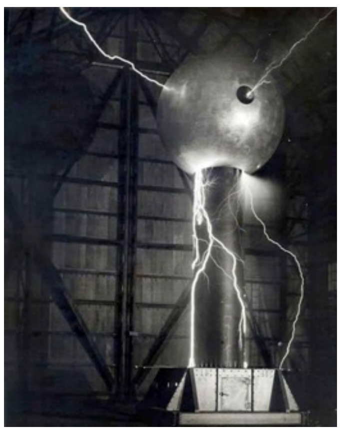


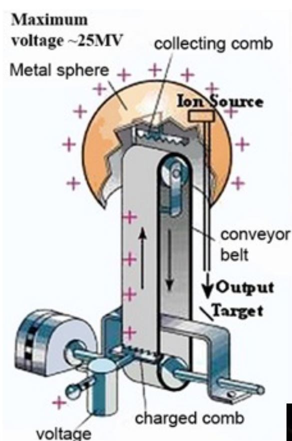


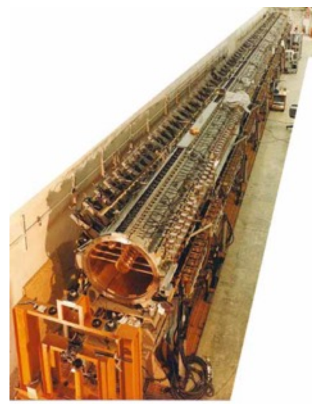
Linear accelerators: from electrostatic to RF

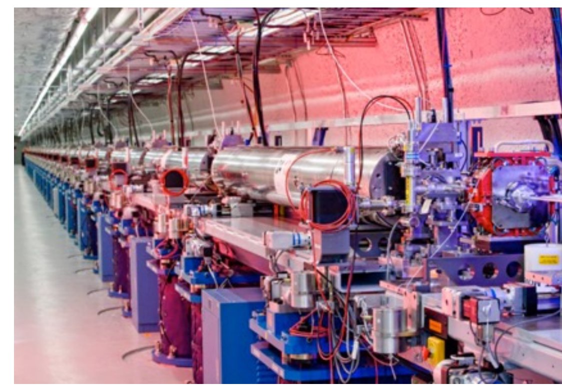


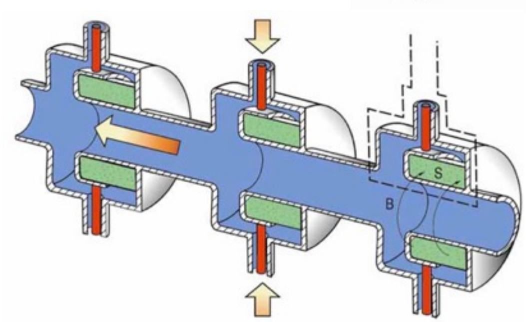










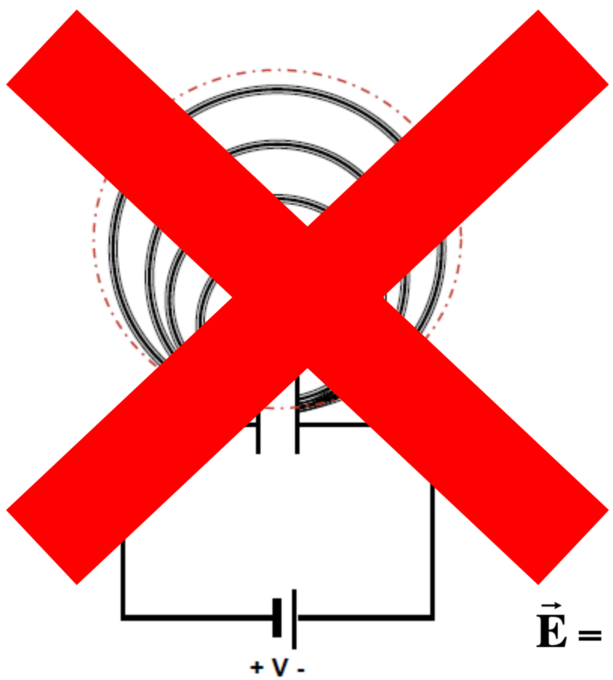


Can one gain the energy again and again by passing through a DC accelerating gap?

Electrostatics: what is the limit ?

Maxwell equations and energy conservation law!

$$\Delta E = e\oint \vec{\mathbf{E}}\cdot d\vec{l} = -\frac{e}{c}\frac{\partial}{\partial t}\left(\int \vec{\mathbf{H}}\cdot d\vec{s}\right)$$



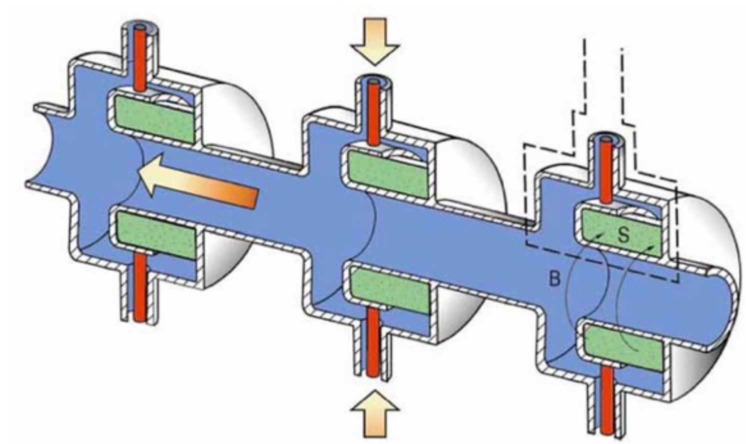
DC

$$\Delta E = e\oint \vec{\mathbf{E}}\cdot d\vec{l} = 0$$

$$\vec{\mathbf{E}} = -\vec{\nabla}\varphi \quad \rightarrow \quad E(\vec{r}) = E(0) - e\varphi(\vec{r})$$

Can not cheat the Maxwell equations

Induction linacs: linear betatrons

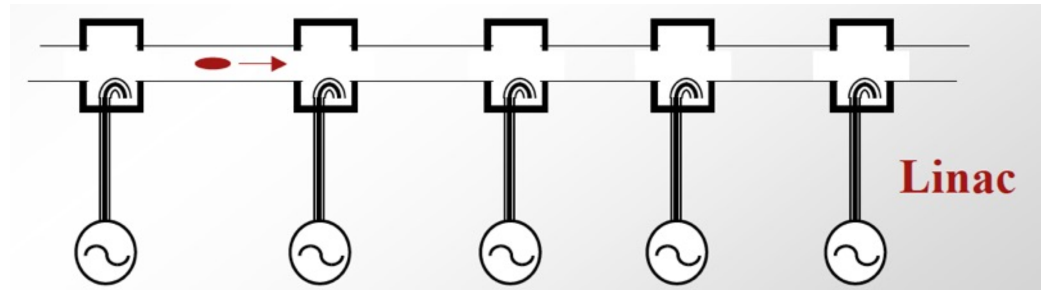


$$\oint \vec{\mathbf{E}} \cdot d\vec{l} = -\frac{\partial}{\partial t}\left(\int \vec{\mathbf{H}} \cdot d\vec{s} \right)$$

- Useful for high power and high current beams
- Have limited accelerating field
- By nature are pulsed, with relatedly low rep-rate (kHz)

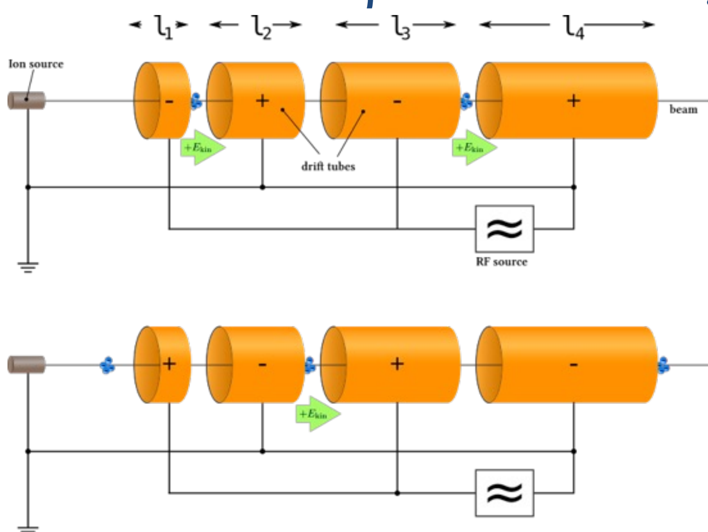
How RF accelerator works

- It has oscillating (typically sinusoidal in time) longitudinal (along the particle's trajectory) eclectic field
- It also has longitudinal structure (cells) which alternates the direction of the filed
- When particle propagates through the RF accelerator, the field direction in each cell is synchronized with the particle arrival and the effect from all cells is added coherently

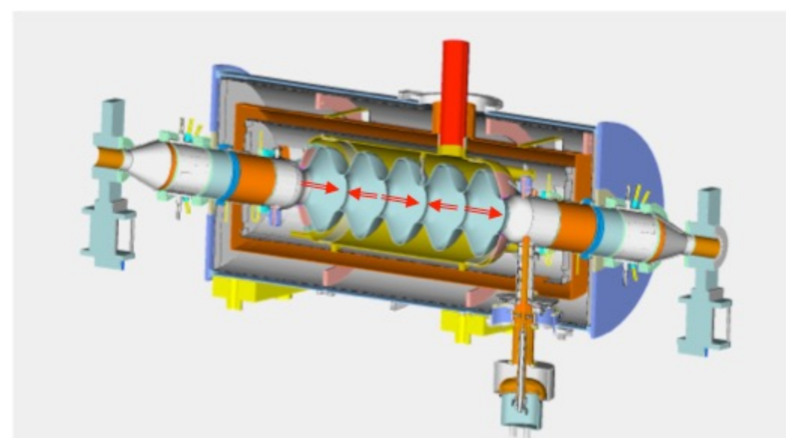


$$\frac{dE}{dt} = e\vec{\mathbf{E}}\cdot\vec{\mathrm{v}} \quad \rightarrow \mathrm{sign}(\vec{\mathbf{E}}\cdot\vec{\mathrm{v}})=\mathrm{const}$$

Wideröe's linac: β=v/c is changing

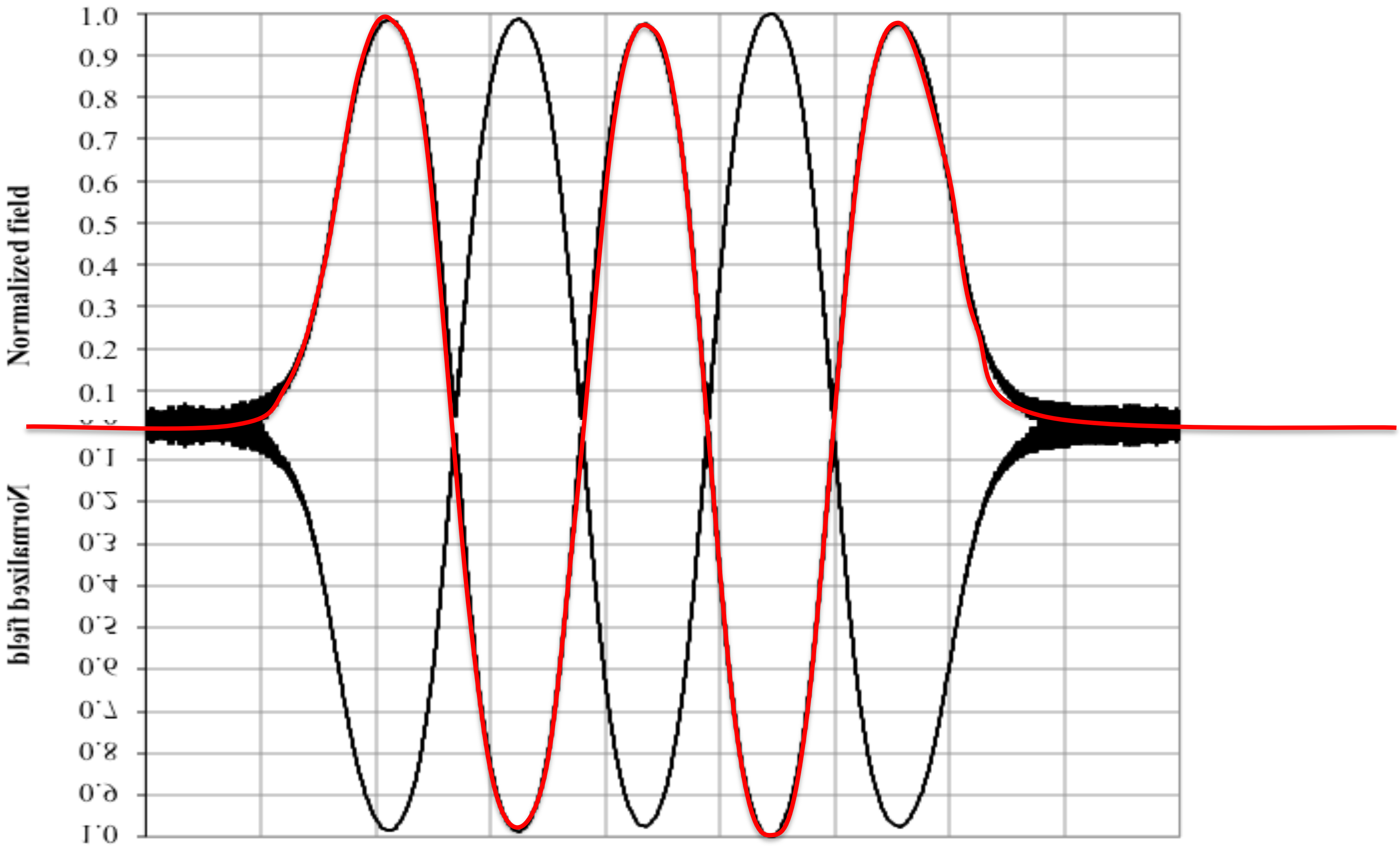


Electron linac

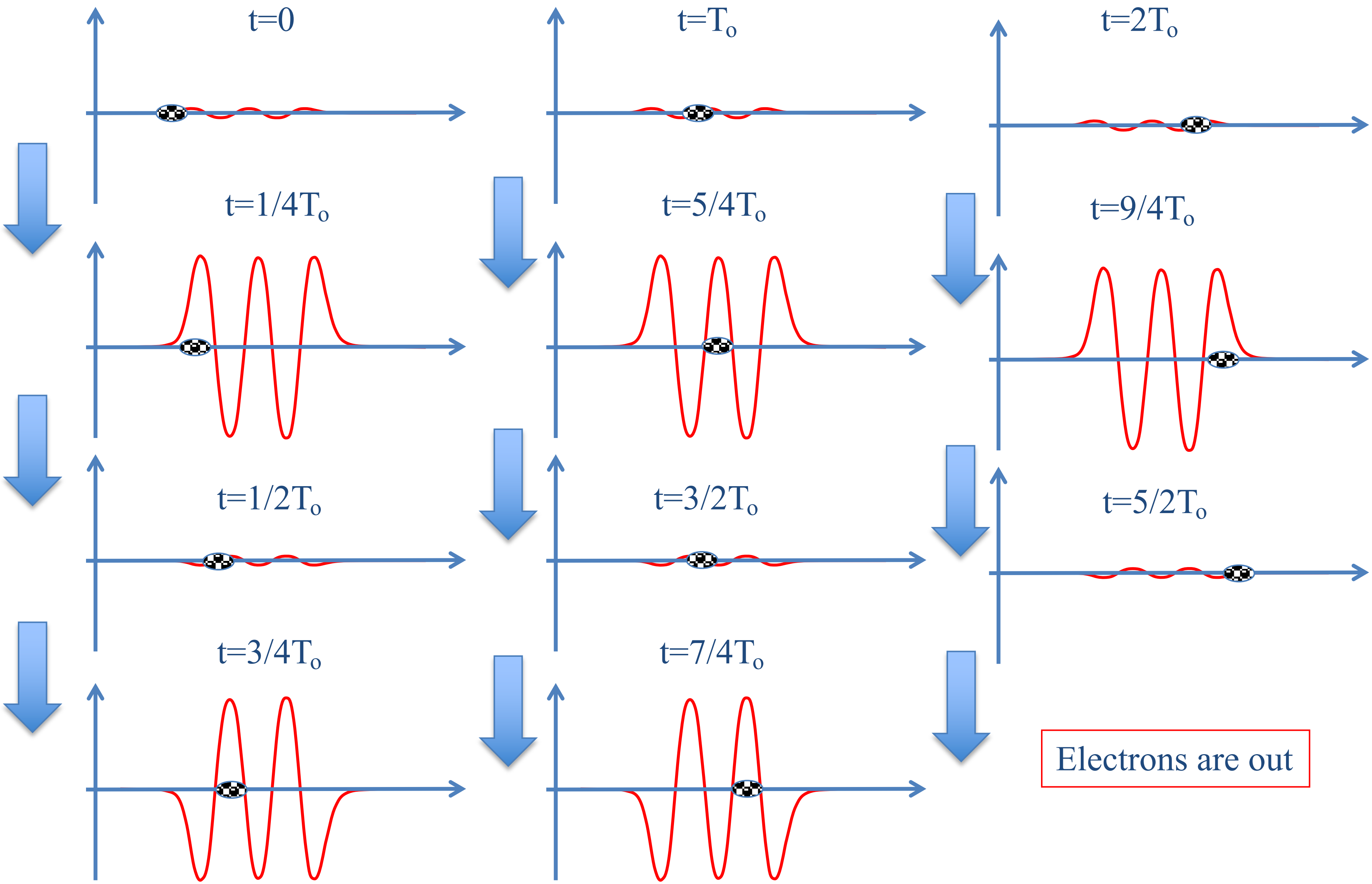


β=v/c ~1

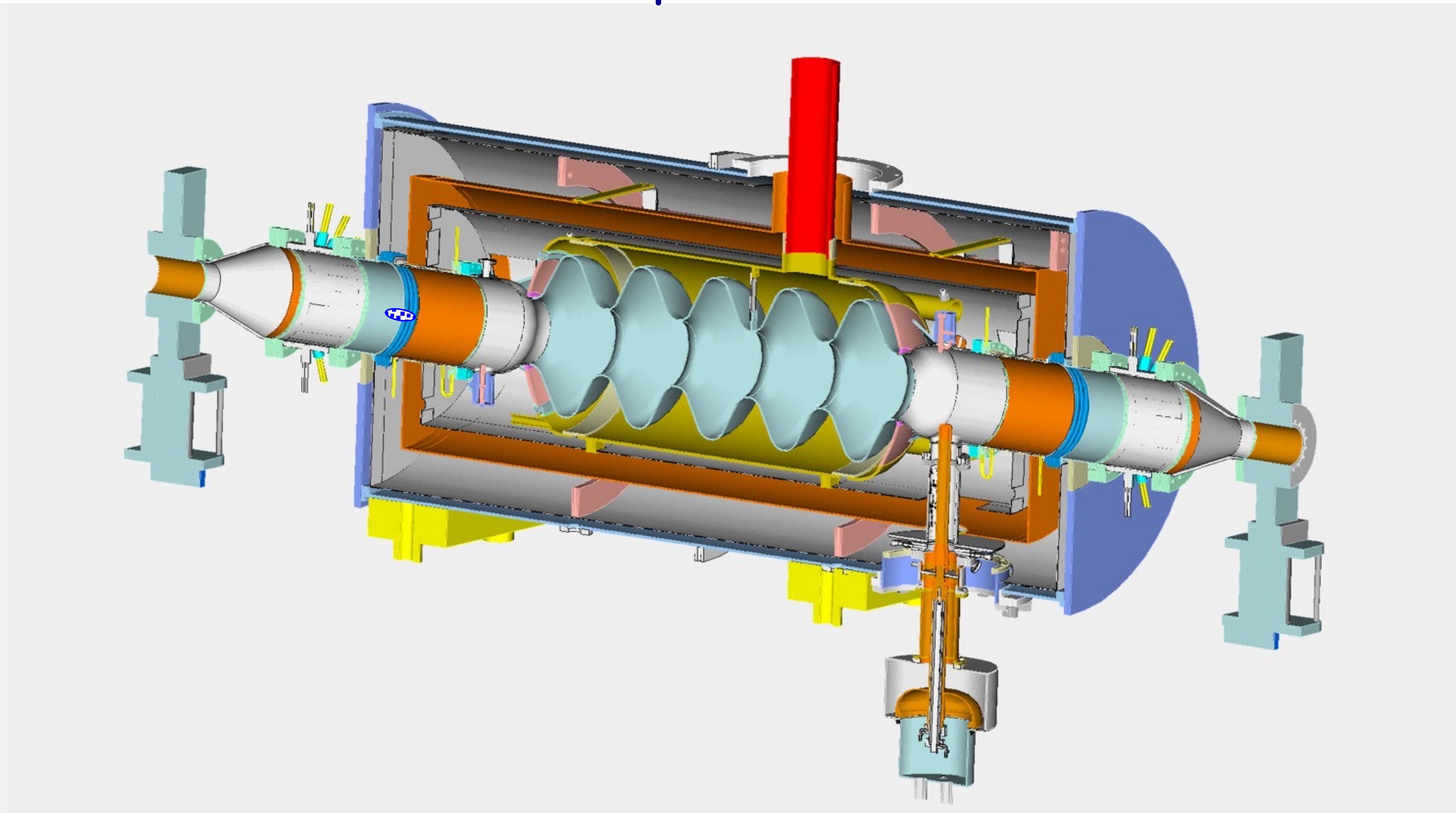
Wave-form in 5-cell cavity



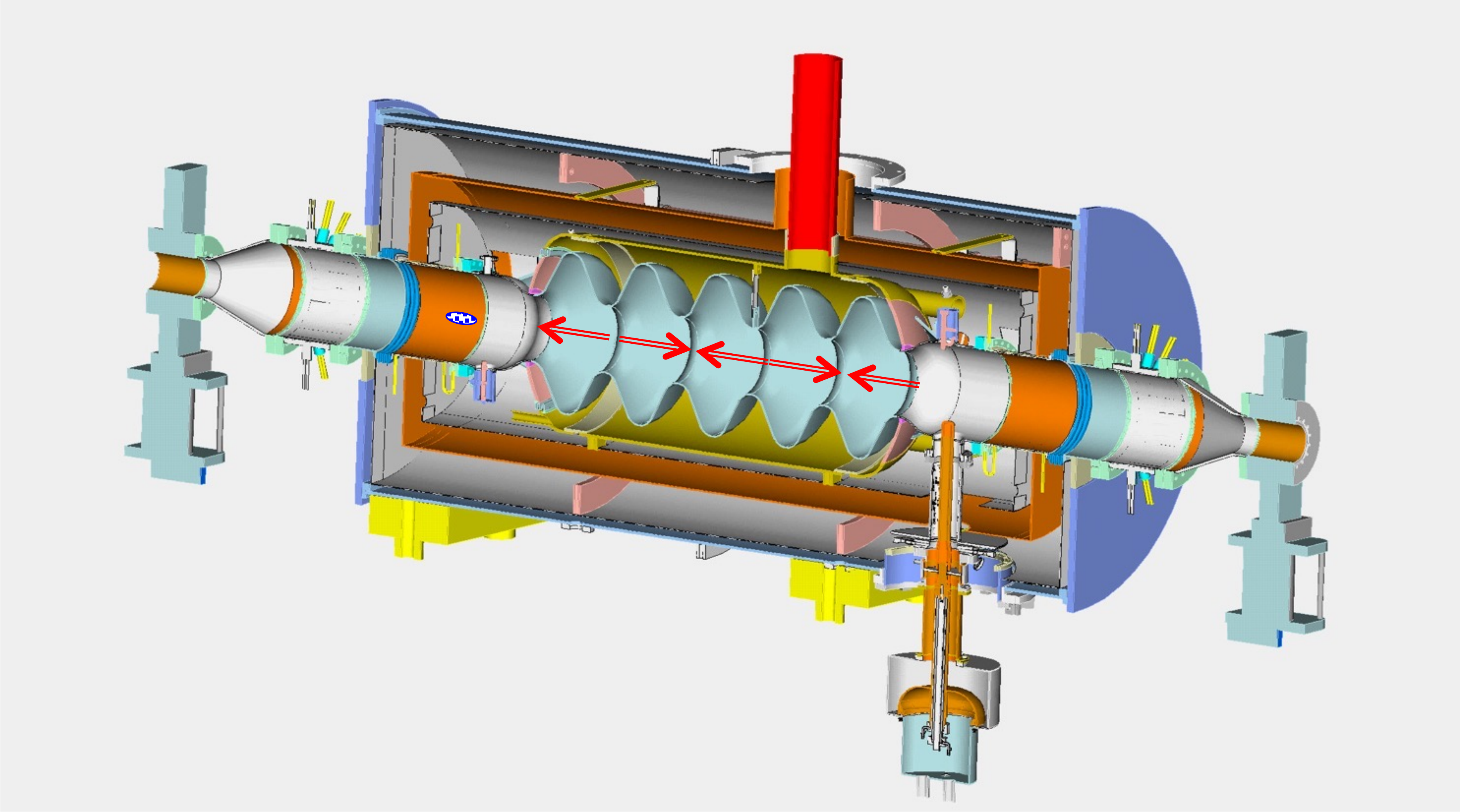
How β=1 RF linac works?
Example of 5-cell cavity



How β=1 RF accelerator works?
In pictures



How β=1 RF accelerator works?
In pictures



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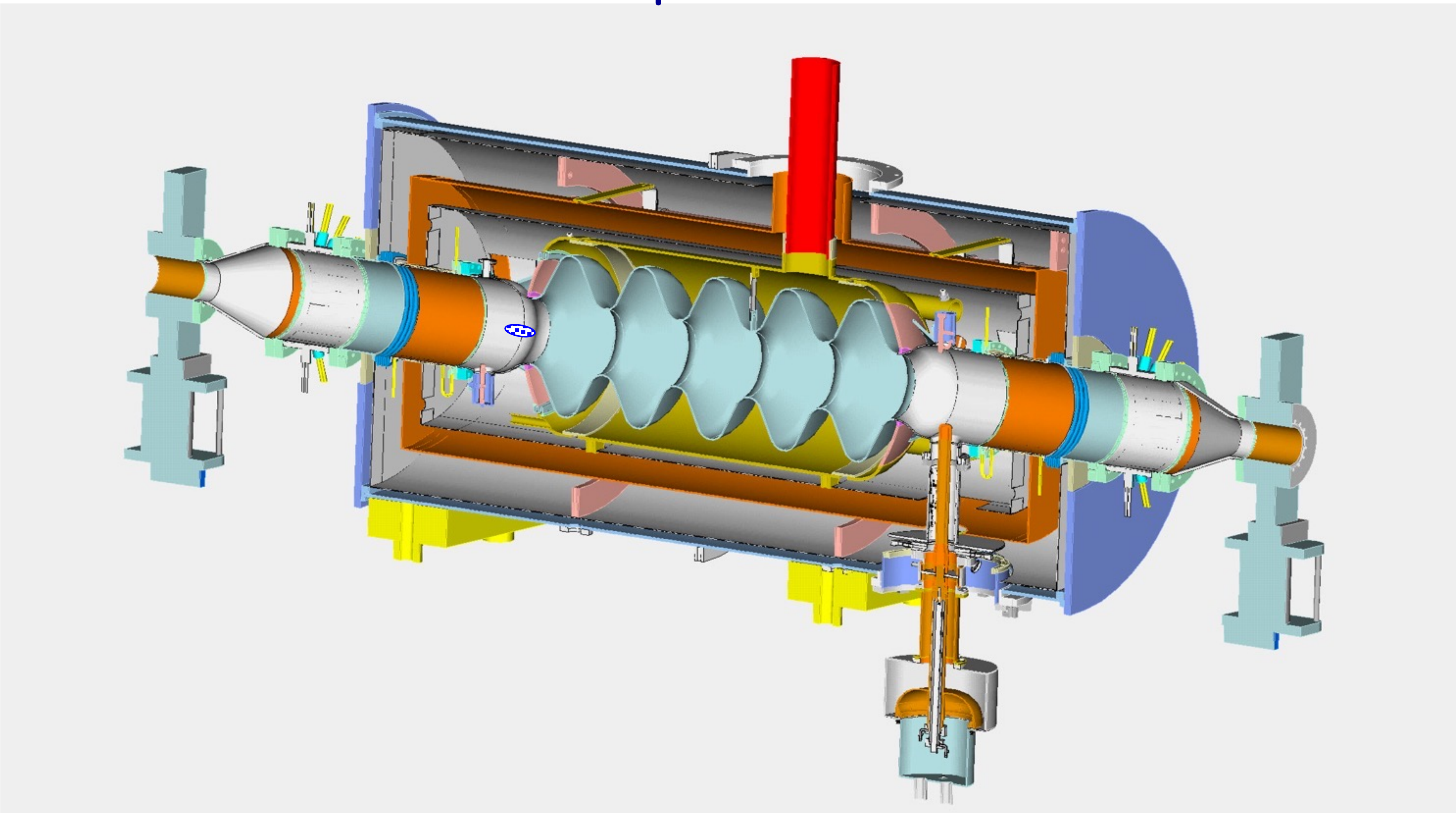
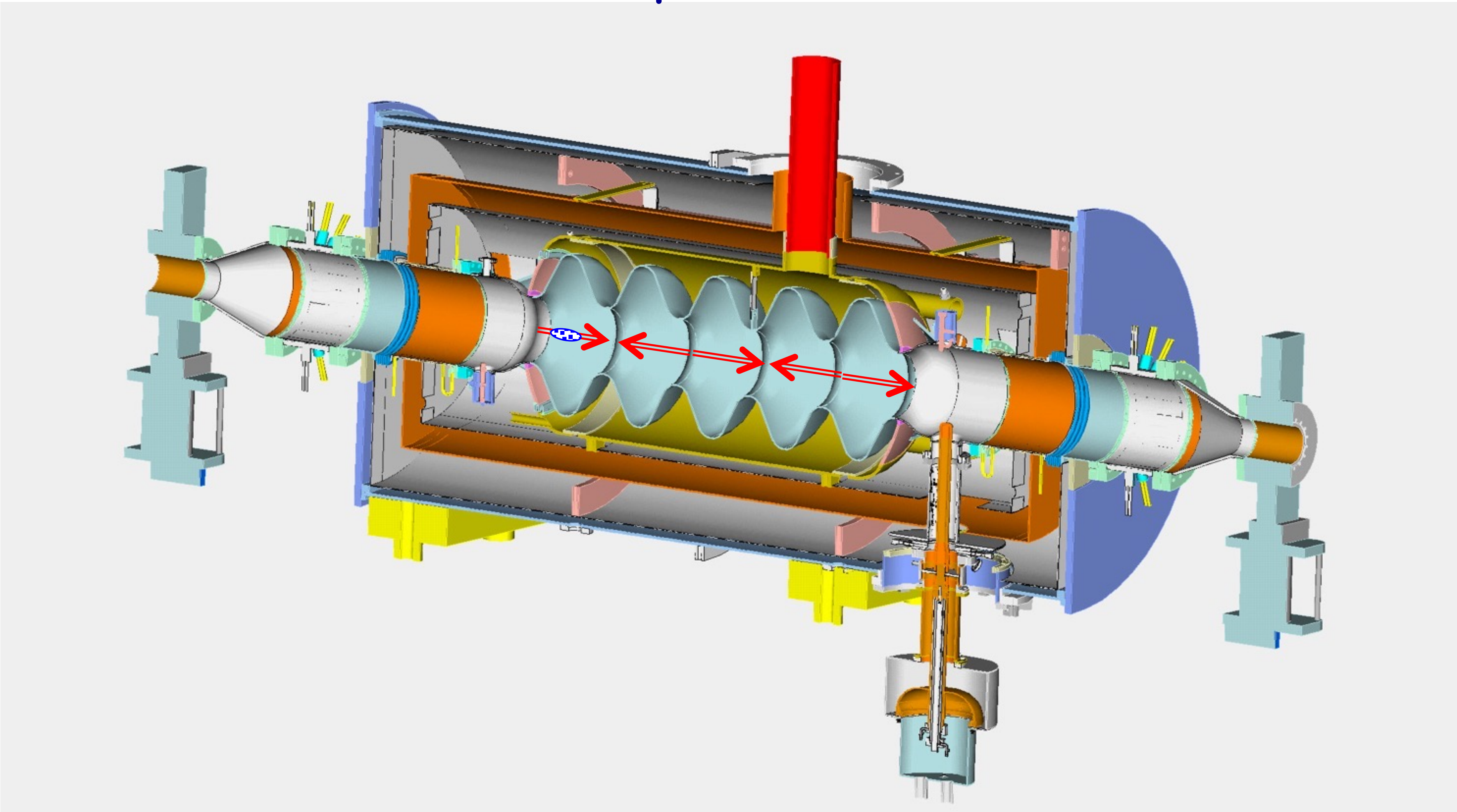


How β=1 RF accelerator works? In pictures



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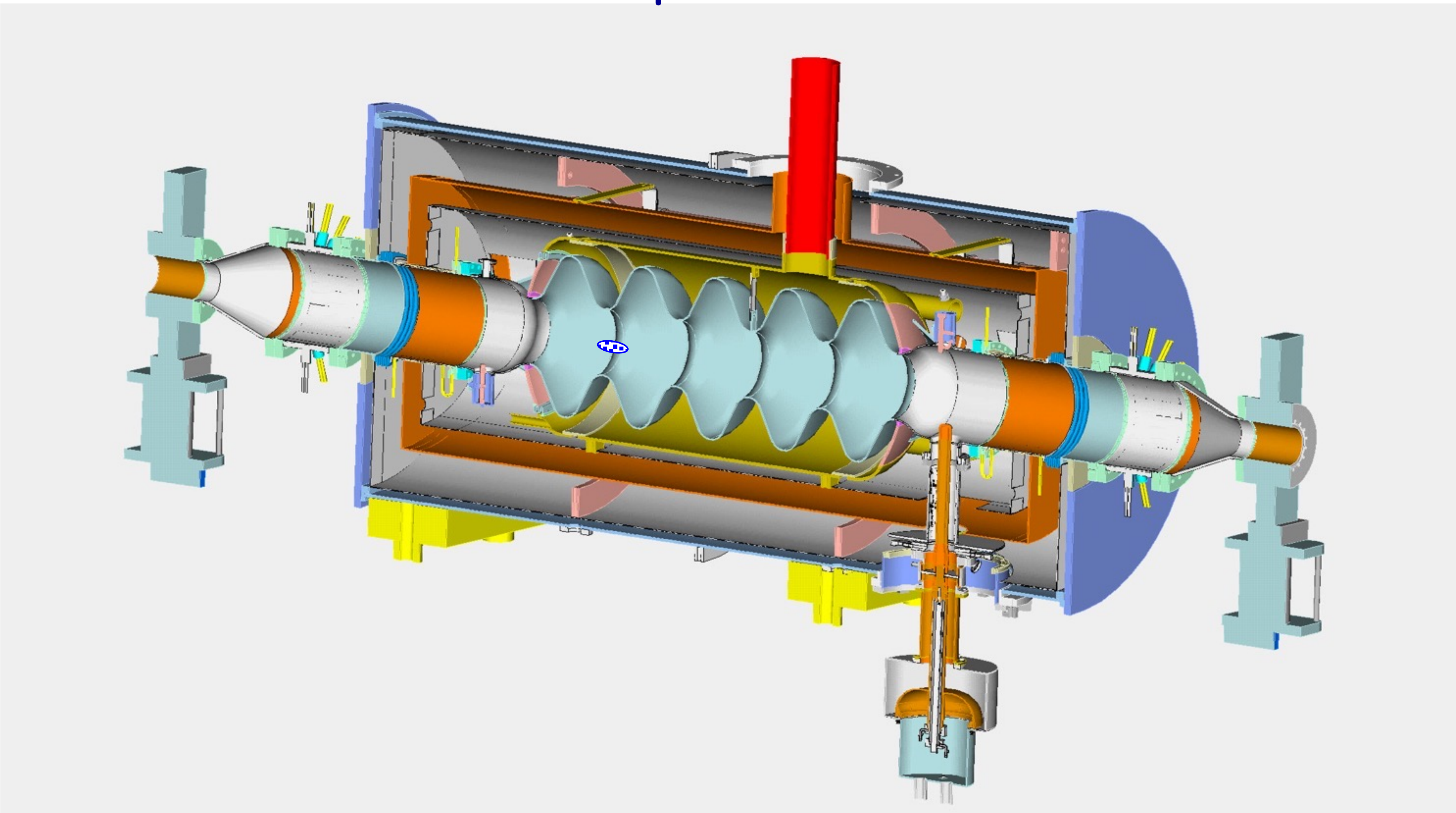
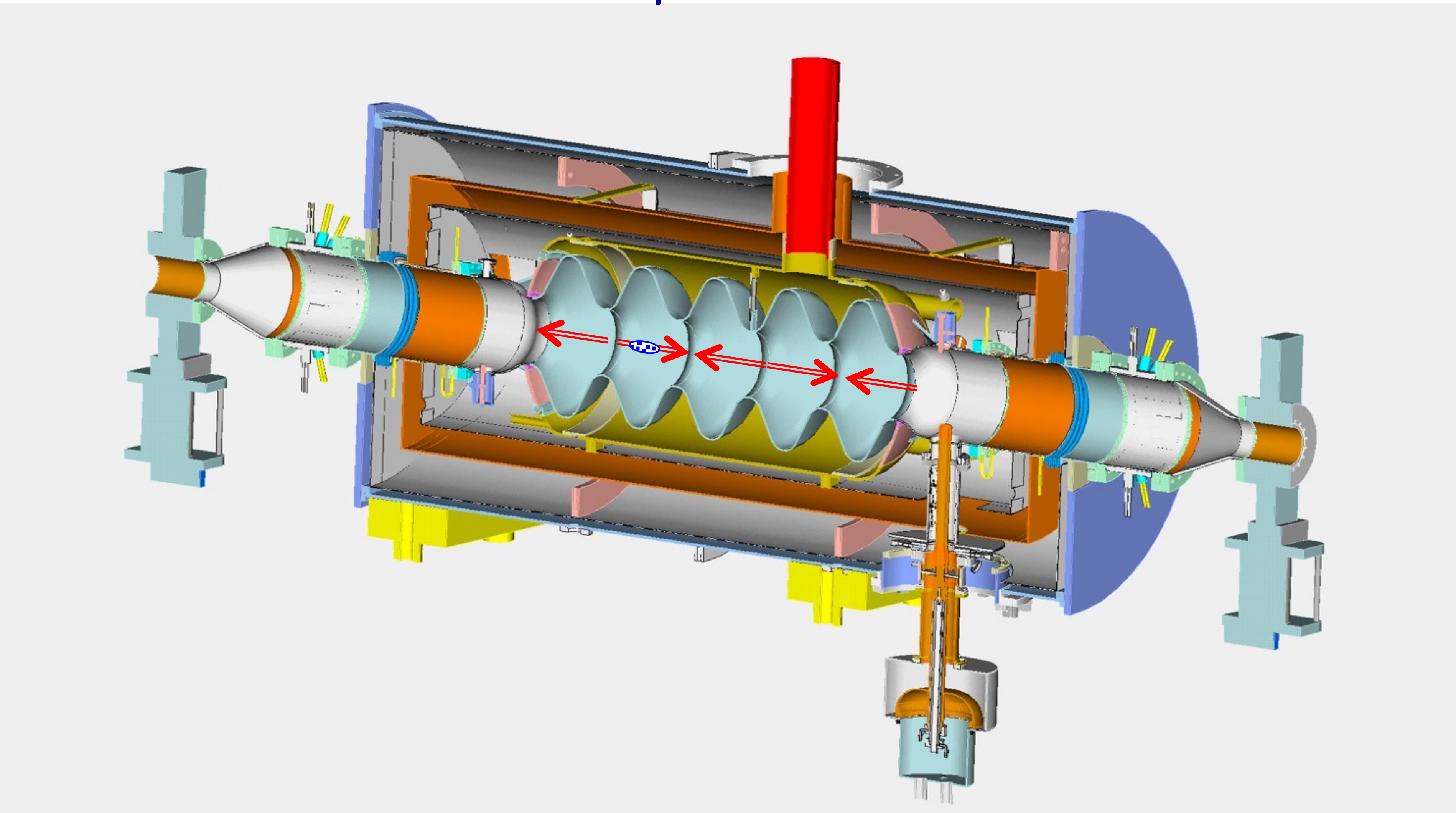


How β=1 RF accelerator works? In pictures



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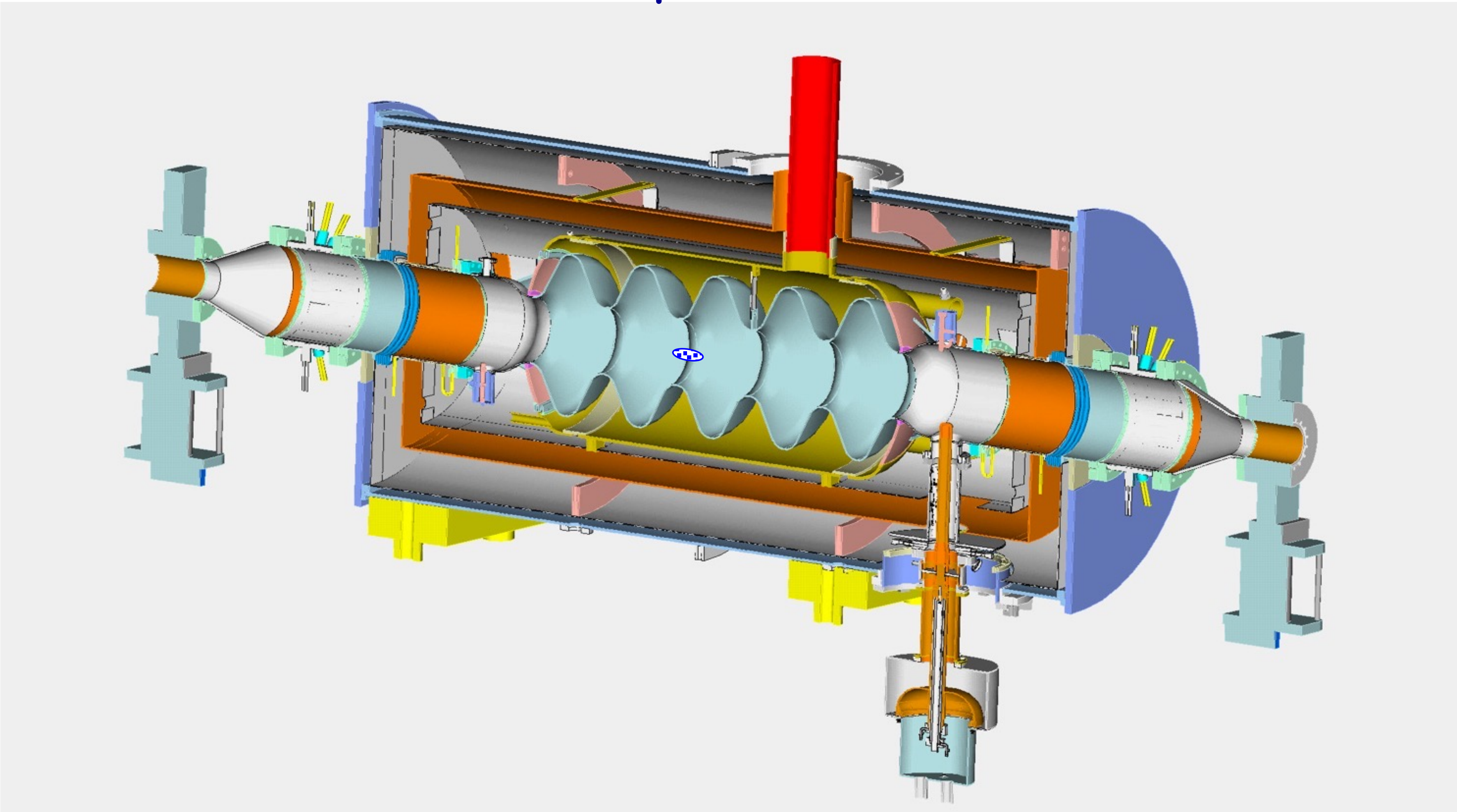

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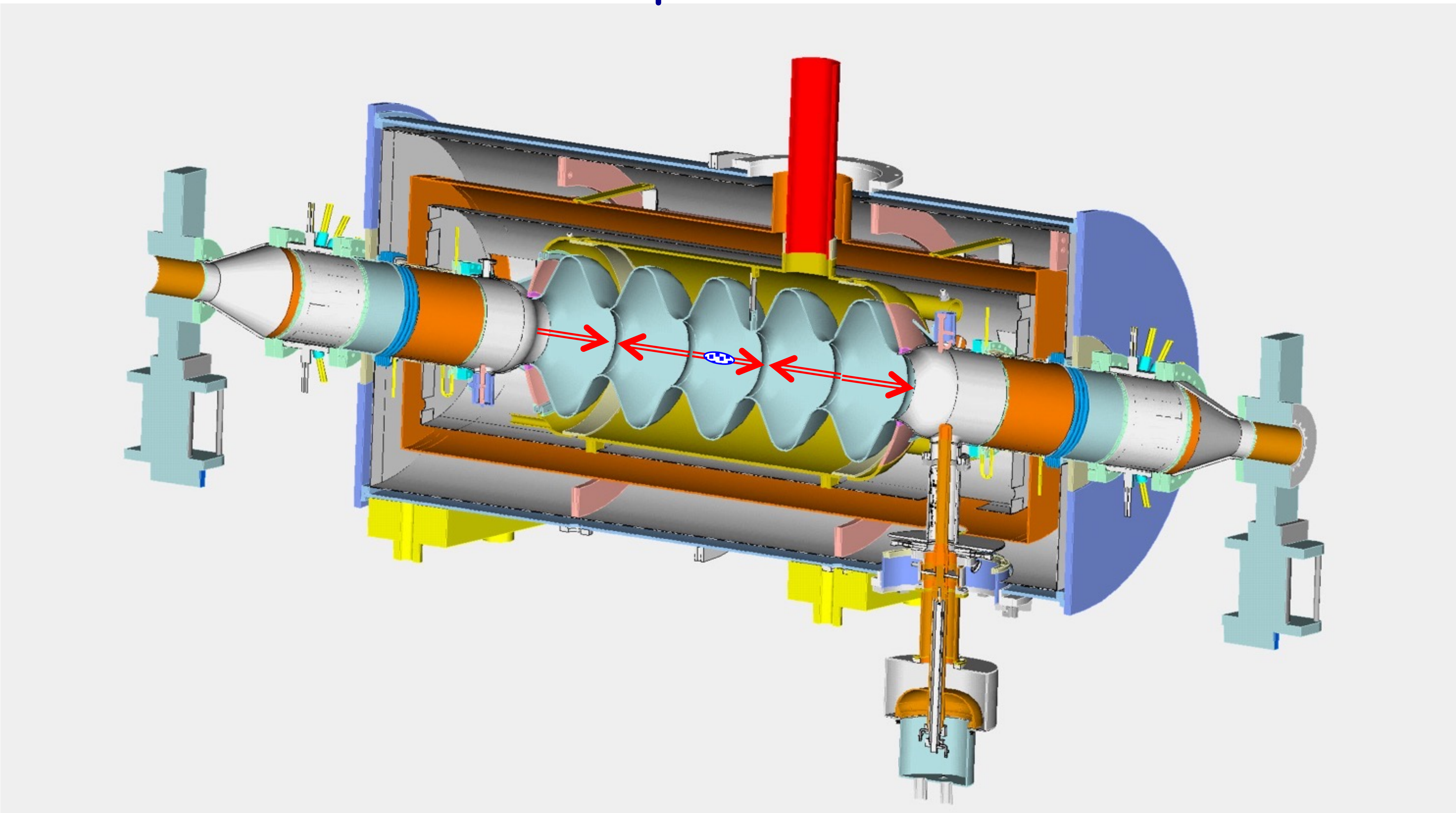
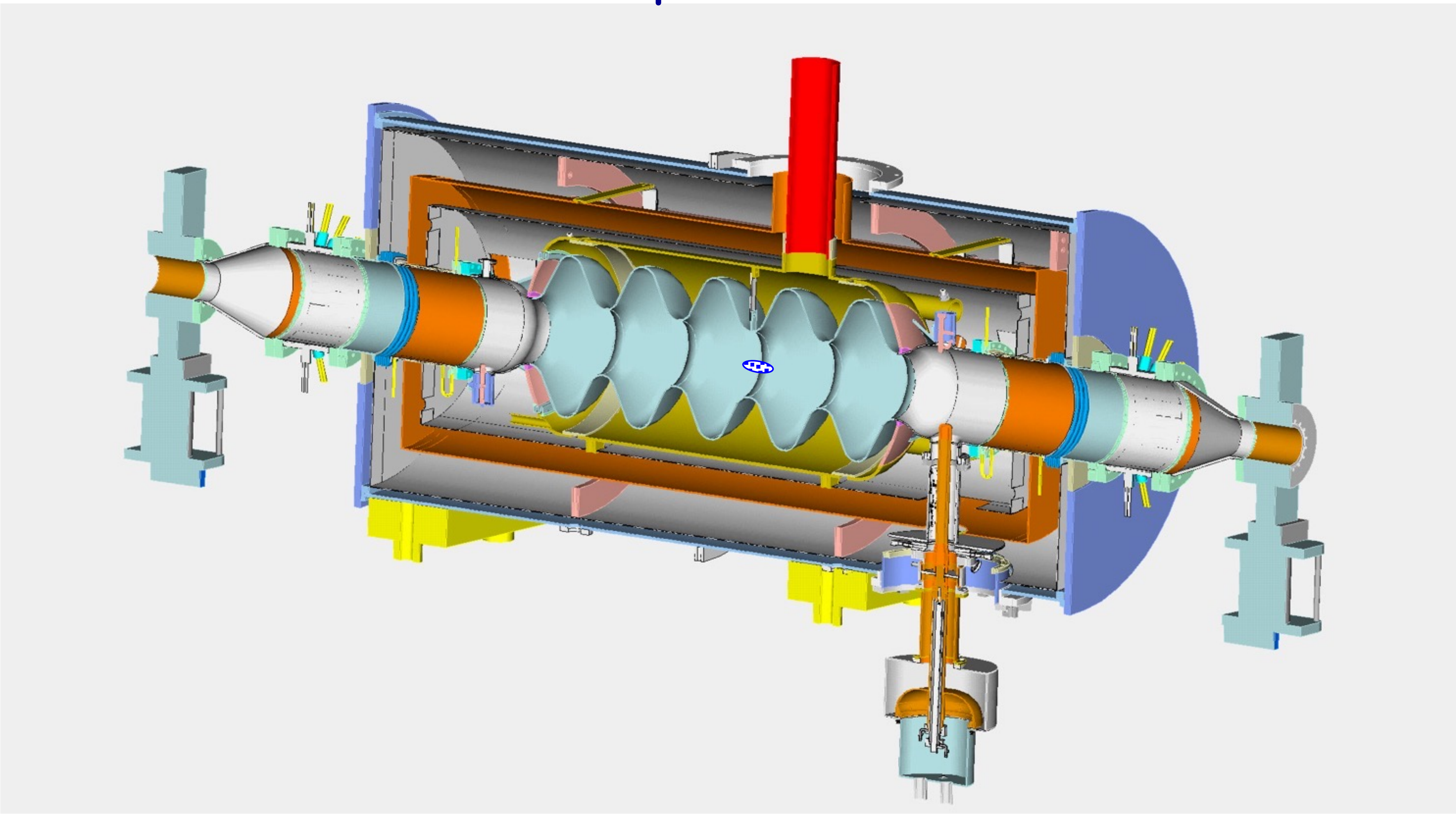
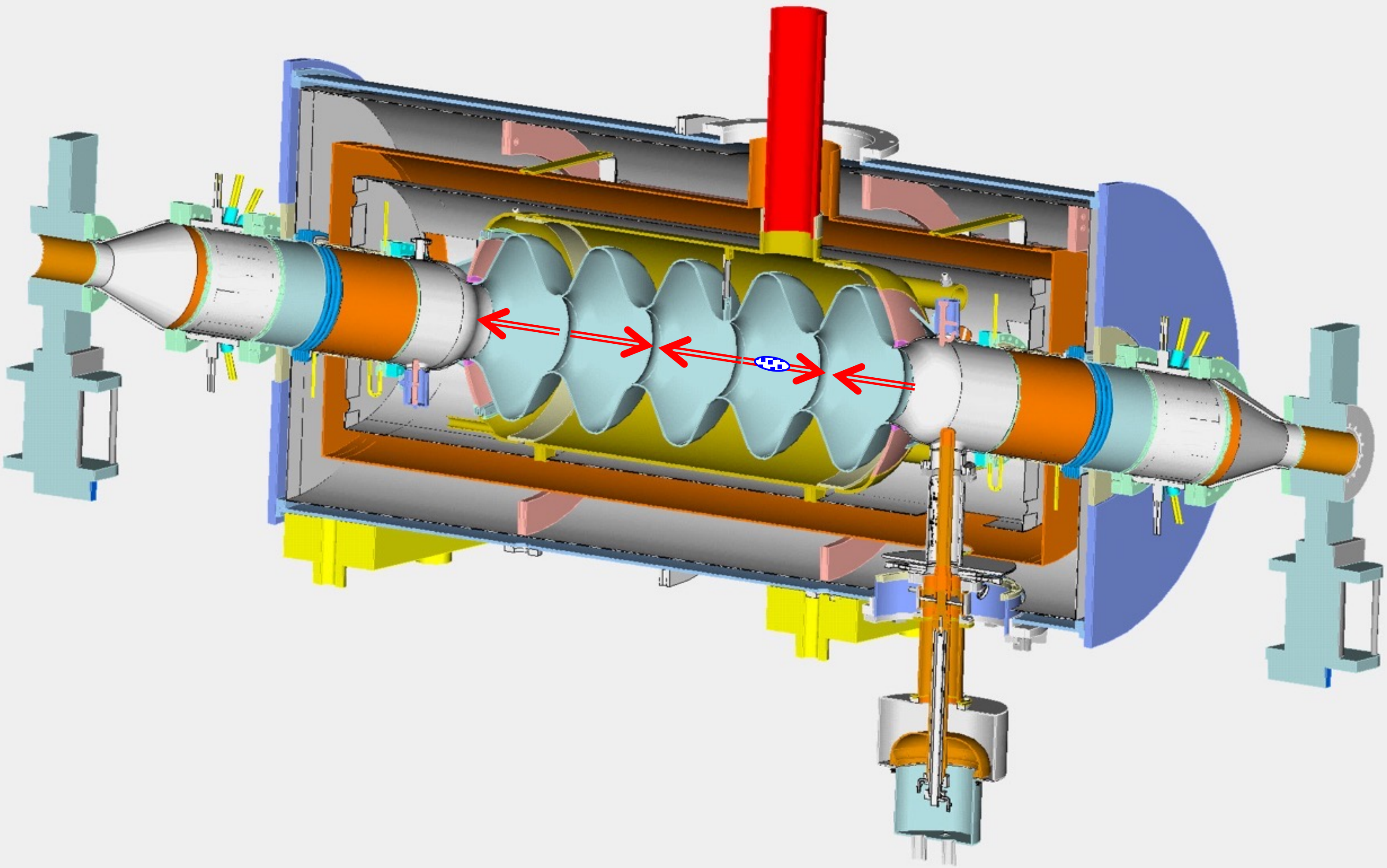
In pictures



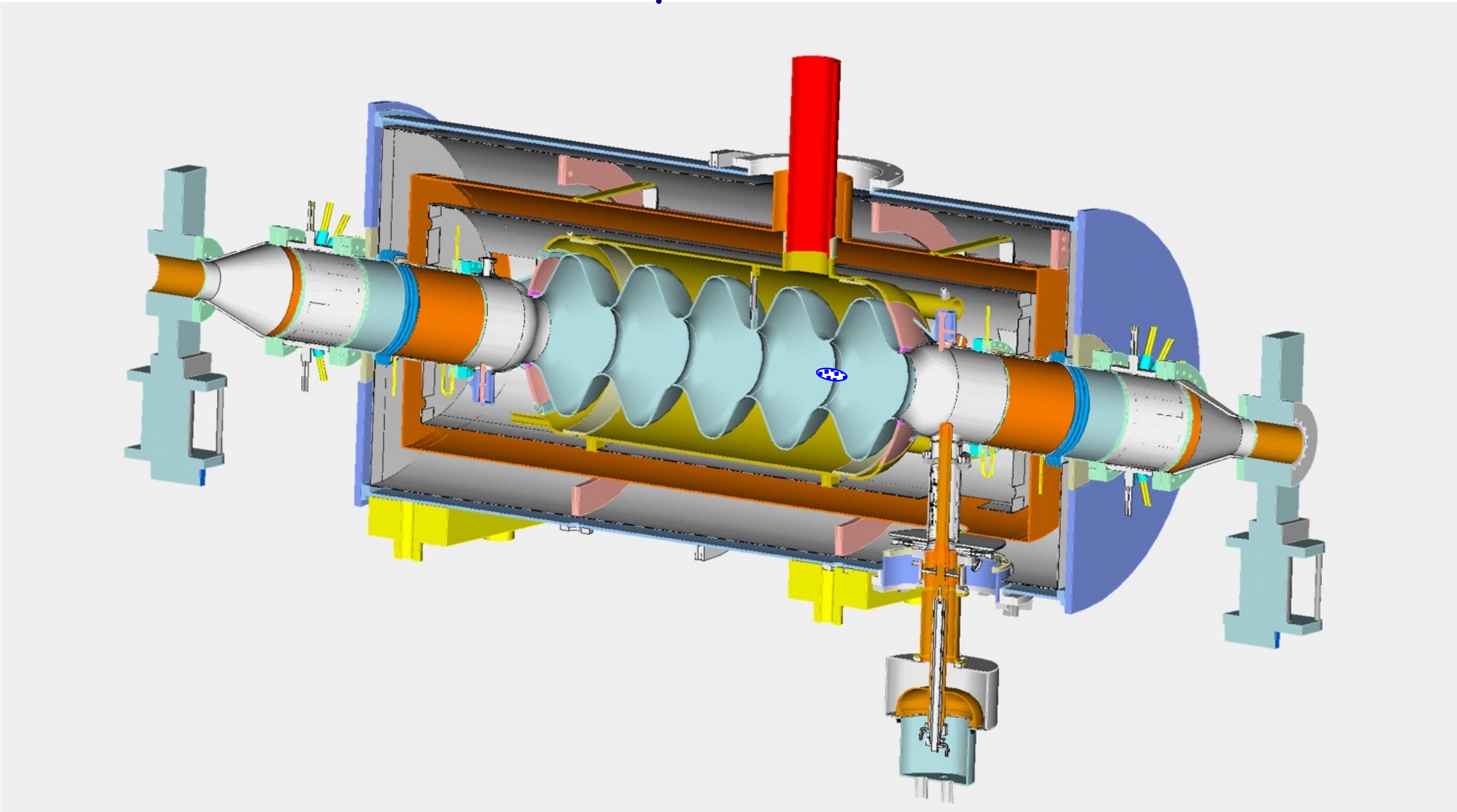
How β=1 RF accelerator works?
In pictures



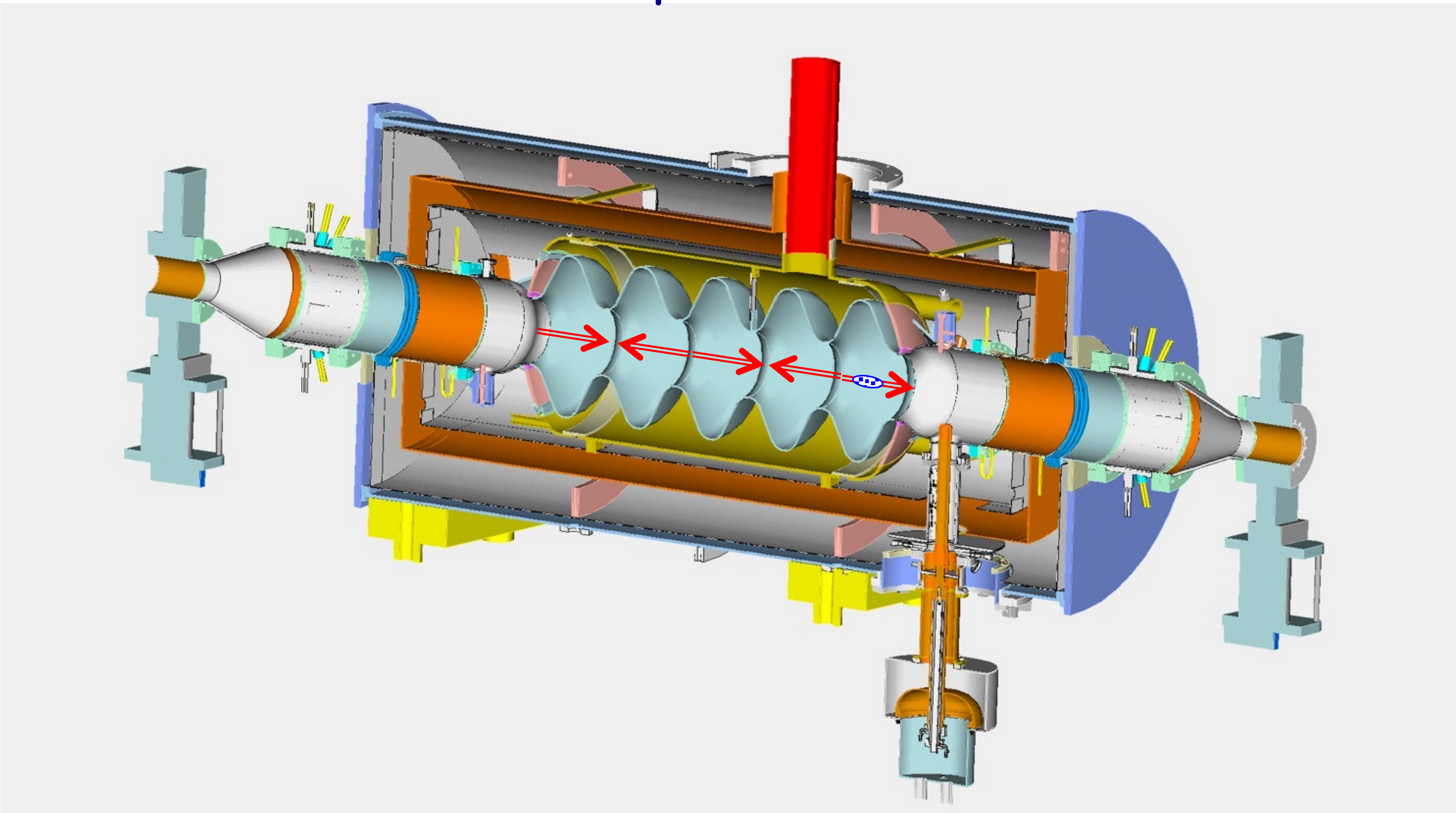
How β=1 RF accelerator works?
In pictures



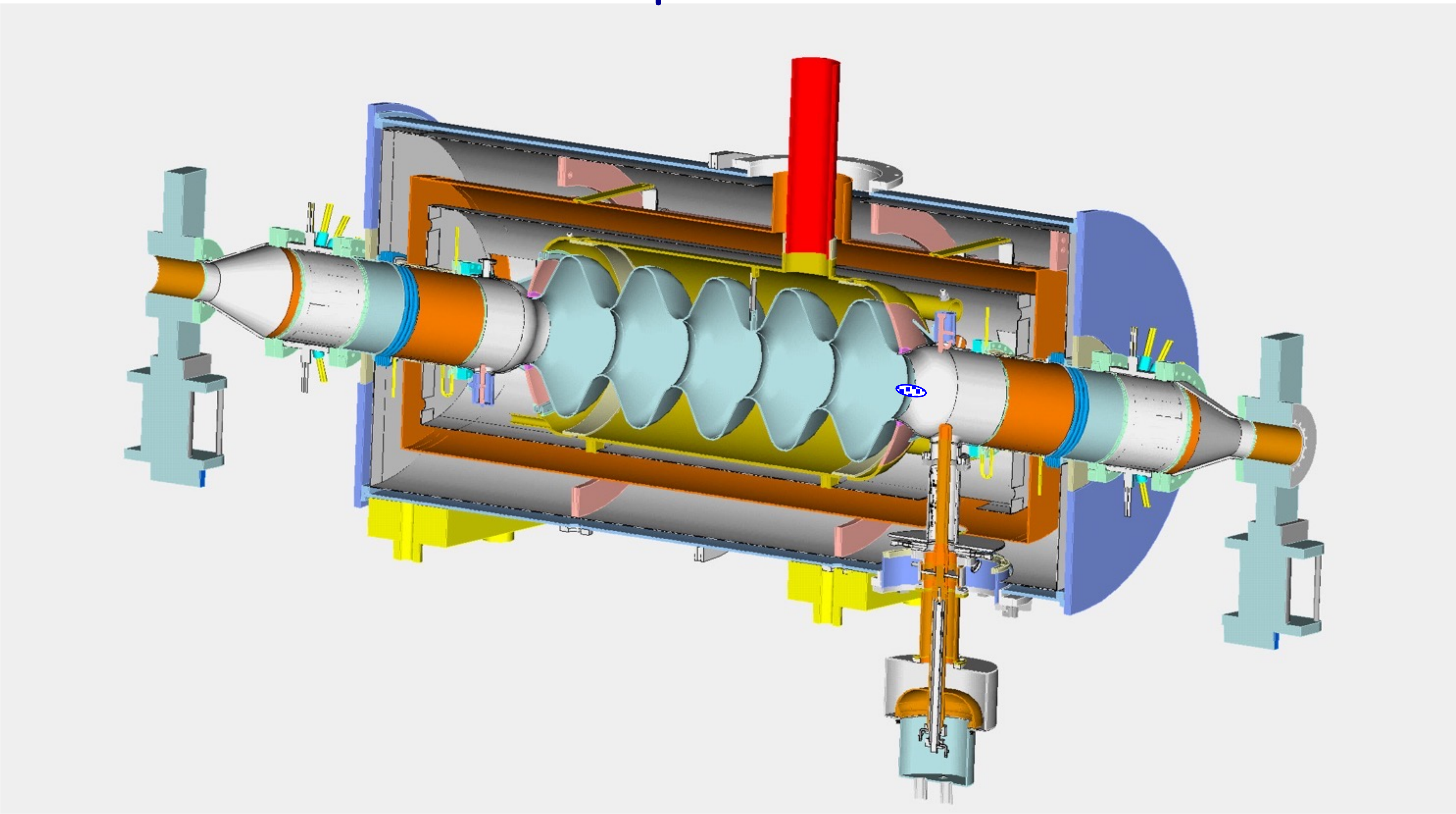
How β=1 RF accelerator works? In pictures



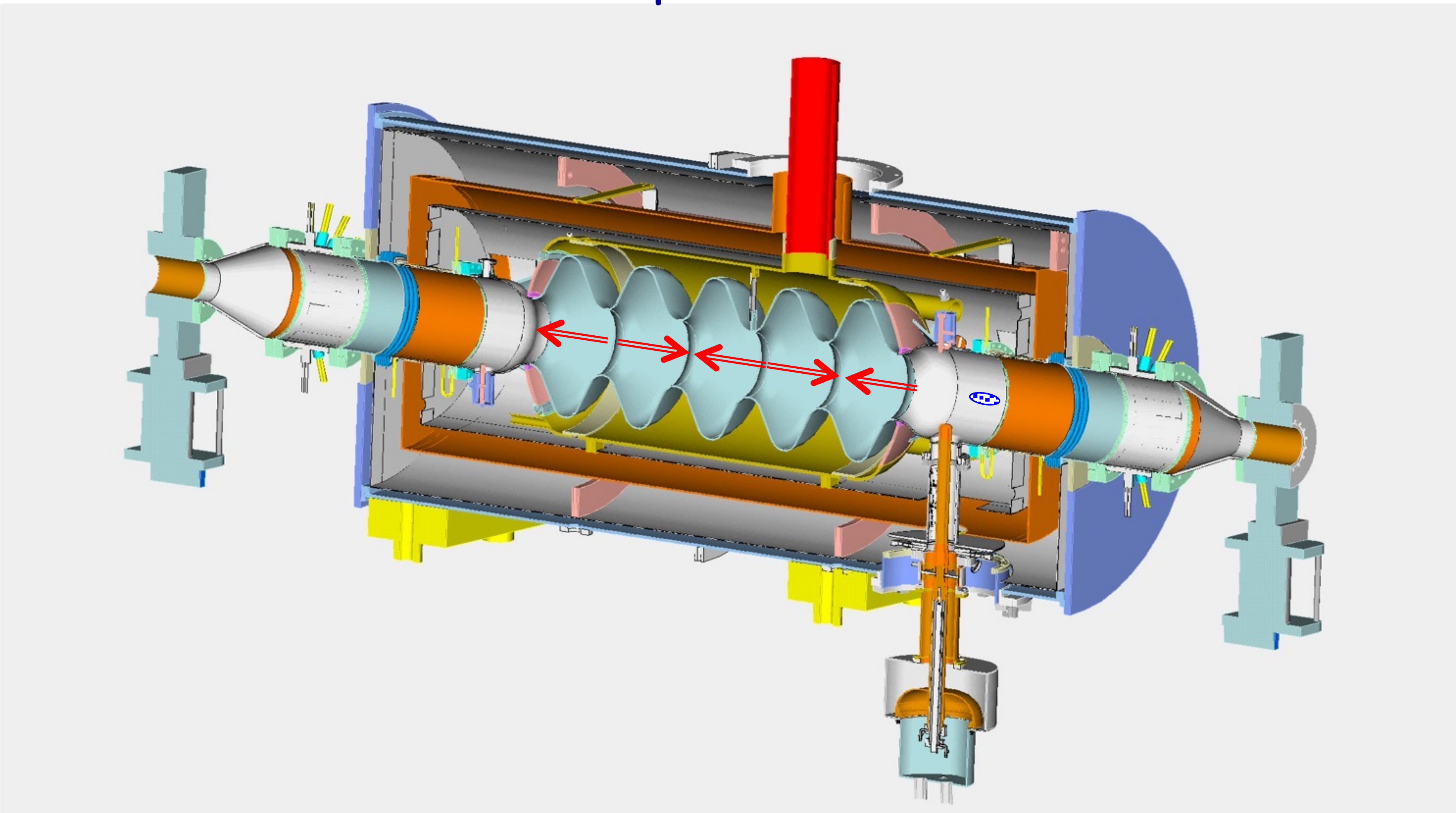
How β=1 RF accelerator works? In pictures



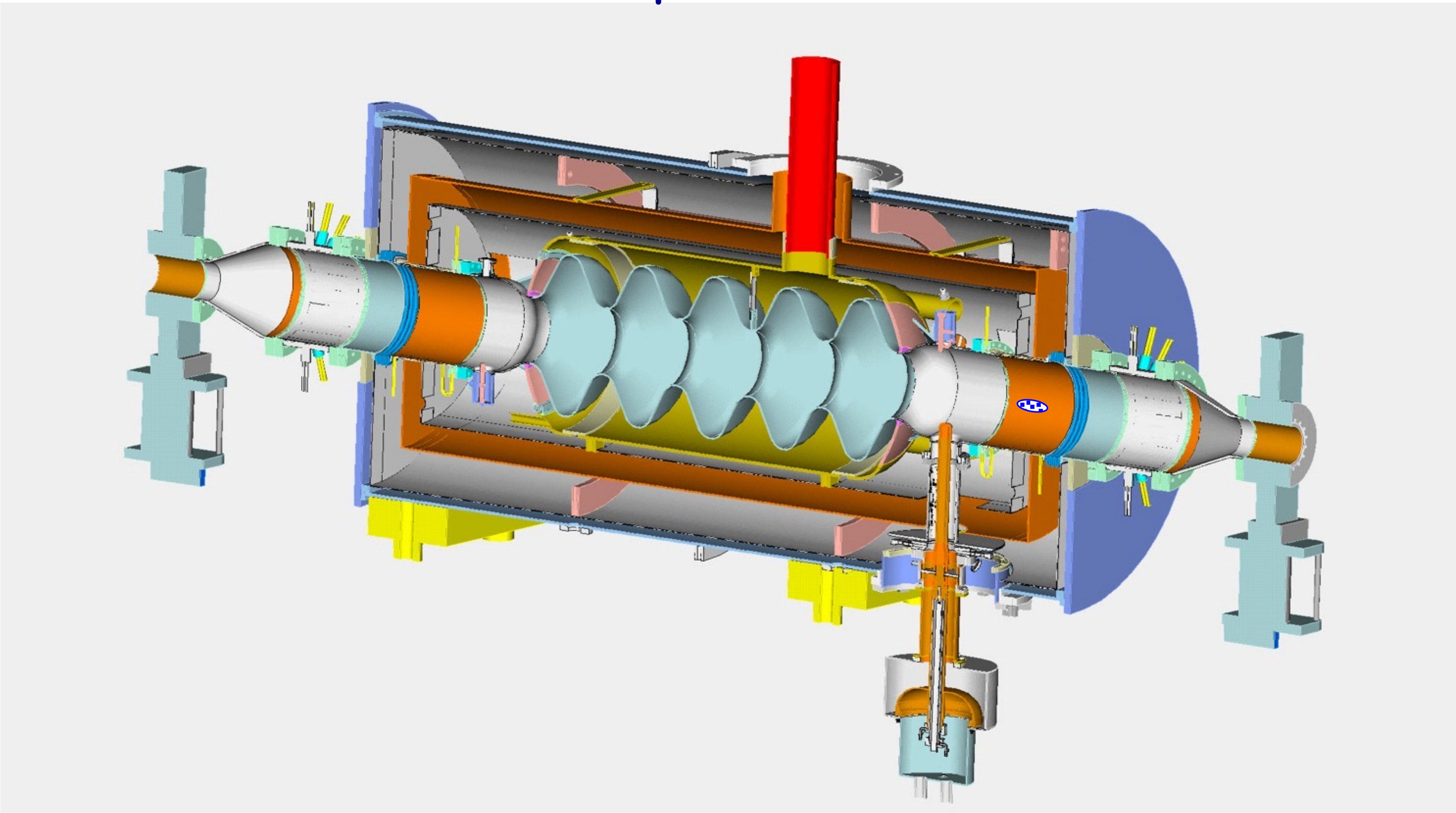
How β=1 RF accelerator works?
In pictures



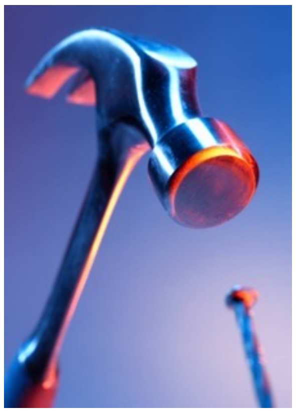
How β=1 RF accelerator works?
In pictures



How β=1 RF accelerator works?
In pictures



Simple things to remember



- Acceleration in DC electrostatic is limited to the difference in terminal potential (e.g. voltage between the ground and the cathode)
- RF linear accelerators (RF linacs or simply linacs) are not limited in beam energy
- In RF linacs, the coherent addition/subtraction of the energy gain from cell to cell happens by design: period of the electric field oscillation is matched to the travel time of electron between the cells.
- Accurate synchronization of RF linac is important task for any linear accelerator

A bit of EM and conducting media

$$\vec{j} = \sigma \vec{E};$$

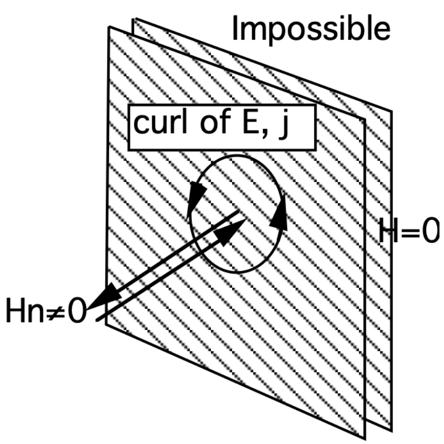
- Assuming oscillating field we can use Coulomb gauge for EM field

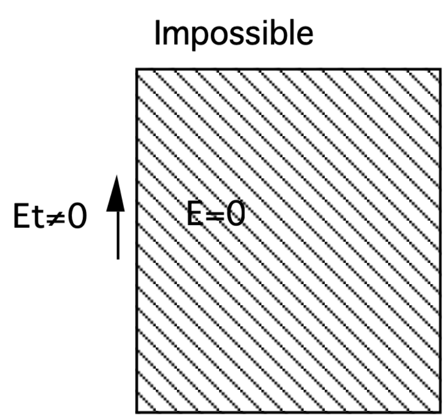
$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r})\exp(i\omega t)\right\}; \varphi = 0;$$

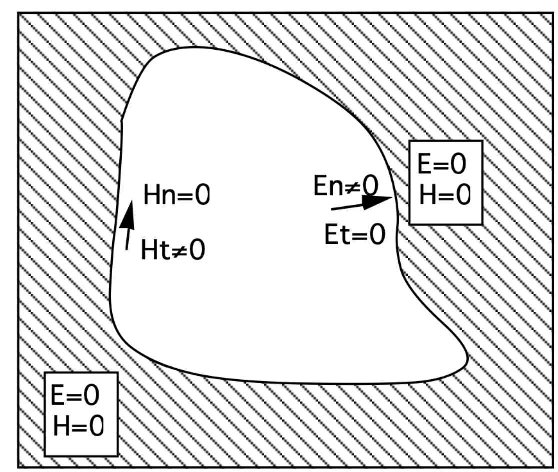
$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = curl\vec{A}.$$

$$\left|\vec{H}\right| \propto \left|\frac{(\alpha + i\beta)}{k_o}\right| \left|\vec{E}\right| = \left|\sqrt{1 + \frac{4\pi i\sigma}{\omega}}\right| \left|\vec{E}\right|$$

$$\sigma \to \infty$$





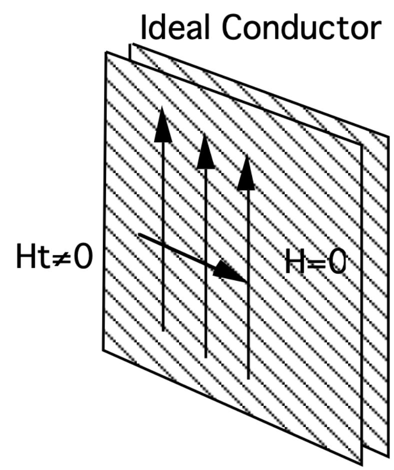


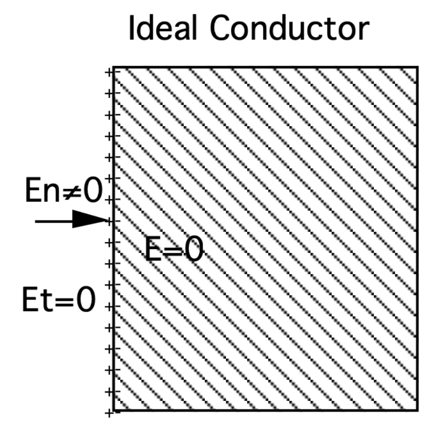
Boundary conditions

- We are considering oscillating EM fields in RF structures
- RF structures are built from highly conducting material, both to contain EM filed inside and to provide low losses
- In first approximation we can consider an ideal boundary conditions and take finite conductivity as a perturbation later
- Q-factor: $Q_{\text{room temp}} \sim 10^4\text{-}10^5$, $Q_{SRF} \sim 10^9\text{-}10^{10}$

$$\vec{A} = \operatorname{Re}\left\{\vec{A}(\vec{r})\exp(i\omega t - \alpha t)\right\};$$

$$\alpha = \frac{2\pi\omega}{Q}$$

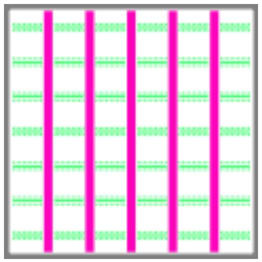


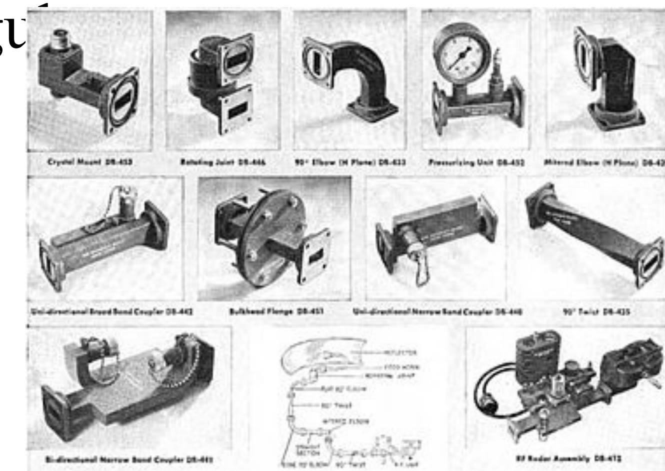


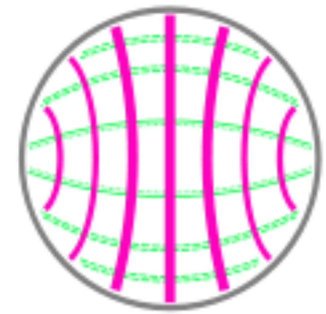
$$\vec{E} = \vec{n}\left(\vec{n}\vec{E}\right) + \vec{E}_{//}; \vec{B} = \vec{n}\left(\vec{n}\vec{B}\right) + \vec{B}_{//};$$

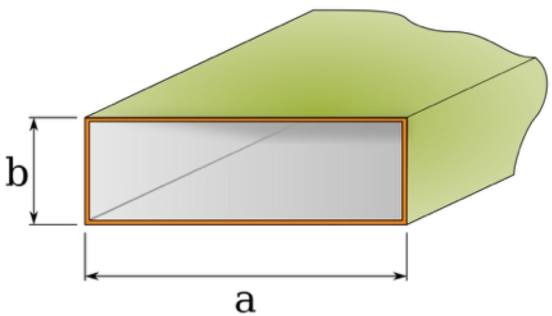
Waveguides

Rectangul Circular



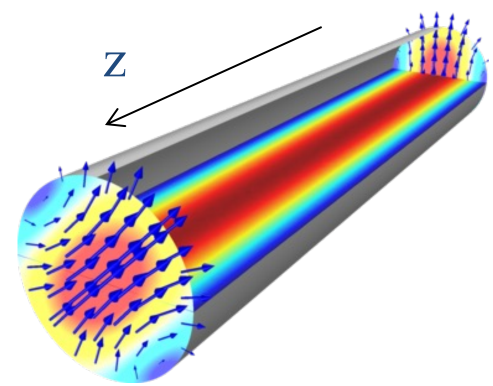






$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = 0; \quad \Delta \equiv \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2};$$

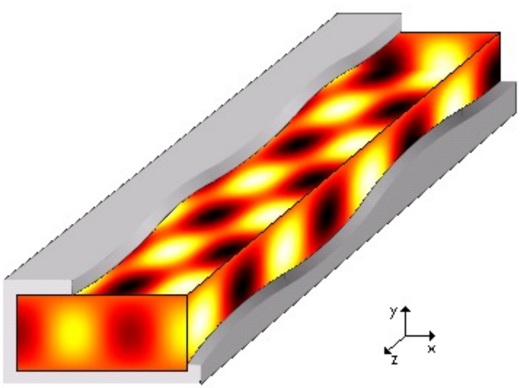
$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r}_\perp)\exp\left(i\left(\omega t - k_z z\right)\right)\right\};$$



$$\vec{\nabla}_\perp^2\vec{A} + \left(k_o^2 - k_z^2\right)\vec{A} = 0; \quad k_o = \frac{\omega}{c}.$$

$$\vec{\nabla}_\perp^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

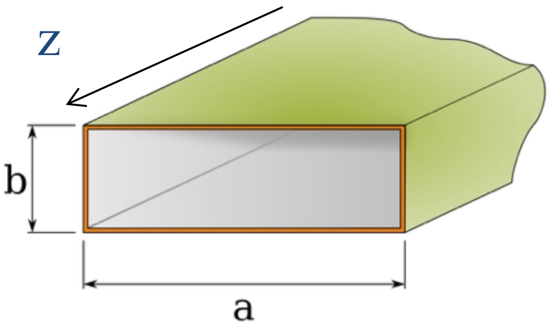
At the surfaces



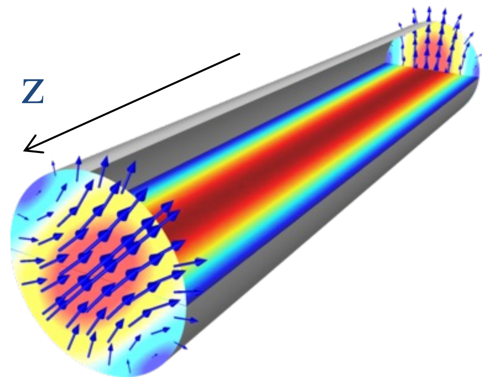
$$\vec{n}\times\vec{E}\Big|_s = 0; \ \vec{n}\cdot\vec{B} = 0 \rightarrow E_z\Big|_s = 0; \ \frac{\partial B_z}{\partial n}\bigg|_s = 0$$

TE and TM waves

Rectangular



Circular



- There is simplification
  - The modes are divided into two types: TE (transverse electric) and TM (transverse magnetic)

$$\vec{E} = \vec{E}_z + \vec{E}_\perp; \vec{B} = \vec{B}_z + \vec{B}_\perp; \ \vec{A}_z \equiv \hat{z} A_z;$$

$$\vec{\nabla} \times \vec{E} = ik_o \vec{B}; \ \vec{\nabla} \times \vec{B} = -ik_o \vec{E}; \ \Rightarrow$$

$$ik_z \vec{E}_\perp + ik_o \left[\hat{z} \times \vec{B}_\perp \right] = \vec{\nabla}_\perp \vec{E}_z;$$

$$ik_z \vec{B}_\perp - ik_o \left[\hat{z} \times \vec{E}_\perp \right] = \vec{\nabla}_\perp \vec{B}_z;$$

At the surfaces

$$\vec{n} \times \vec{E}\big|_s = 0; \vec{n} \cdot \vec{B} = 0 \rightarrow E_z\big|_s = 0; \ \frac{\partial B_z}{\partial n}\bigg|_s = 0$$

$$TM : B_z \equiv 0; \ E_z\big|_s = 0;$$

$$TE : E_z \equiv 0; \ B_z\big|_s = 0;$$

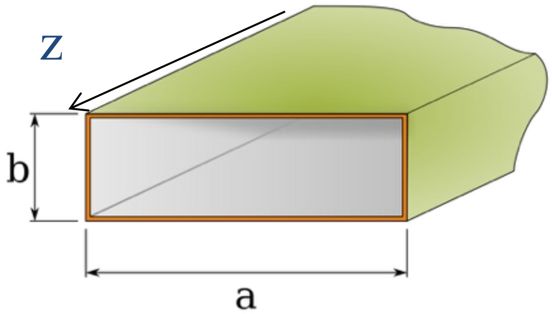
- Last two equations indicated that E_z and B_z fully determine transverse component of the EM field
- It means that we can always consider a linear combination of the fields with $E_z = 0$ *everywhere* (TE) and $B_z = 0$ *everywhere* (TM)
- Naturally, when we interested in accelerating particles, we will need TM mode with $E_z \neq 0$.

$$\vec{B}_\perp = \pm \frac{k_z}{k_o} \left[\hat{z} \times \vec{E}_\perp \right] \text{ for both TE and TM modes}$$

$$\text{TM: } \vec{E}_\perp = \vec{\nabla}_\perp \psi_1(\vec{r}_\perp); \ \text{TE: } \vec{B}_\perp = \vec{\nabla}_\perp \psi_2(\vec{r}_\perp);$$

Cut-off frequency

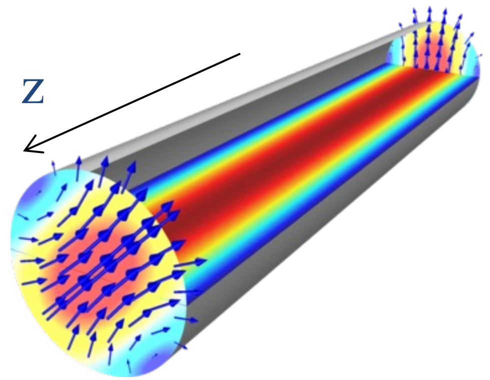
Rectangular | Circular



- EM field is a linear combination of modes with $E_z = 0$ *everywhere* (TE) and $B_z = 0$ *everywhere* (TM)

At the surfaces

$$\vec{n}\times\vec{E}\Big|_s = 0;\ \vec{n}\cdot\vec{B} = 0 \rightarrow E_z\Big|_s = 0;\ \frac{\partial B_z}{\partial n}\Big|_s = 0$$



$$\vec{B}_\perp = \pm\frac{k_z}{k_o}\left[\hat{z}\times\vec{E}_\perp\right]$$ for both TE and TM modes

$$TM: B_z \equiv 0;\ E_z\Big|_s = 0;$$

$$TE: E_z \equiv 0;\ B_z\Big|_s = 0;$$

TM: $\vec{E}_\perp = \vec{\nabla}_\perp\psi_1(\vec{r}_\perp)$; TE: $\vec{B}_\perp = \vec{\nabla}_\perp\psi_2(\vec{r}_\perp)$;

$$TM: \psi\Big|_s = 0;\ TE: \frac{\partial\psi}{\partial n}\Big|_s = 0.$$

$$\vec{\nabla}_\perp^2\psi + \left(k_o^2 - k_z^2\right)\psi = 0$$ + *boundary conditions*

Different boundary conditions for TE and TM modes
In general case we need to find eigen function (modes)

$$\vec{\nabla}_\perp^2\psi_\lambda + \gamma_\lambda{}^2\psi_\lambda = 0;\quad \lambda = 1,2,3,\ldots$$

Cut-off frequency

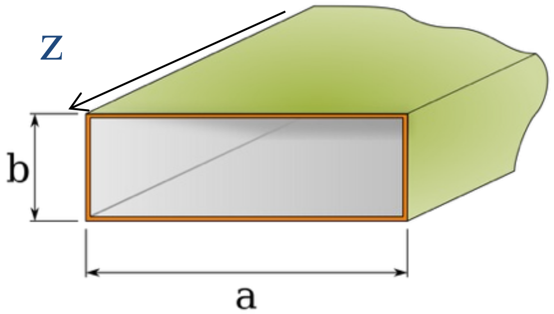
$$k_{z,\lambda}^2 = k_o^2 - \gamma_\lambda^2 > 0$$

$$k_{o\,\min} = \gamma_\lambda \rightarrow \omega_{cut-off} = c\gamma_\lambda$$

Below cut-off $\omega < \omega_{cut-off}$
evanescent wave: $k_z = \pm i\sqrt{\omega^2{}_{cut-off} - \omega^2} = \pm i\kappa_z$
Exp decay $\psi = \psi_0 e^{\pm\kappa_z z}$

Cut-off frequency

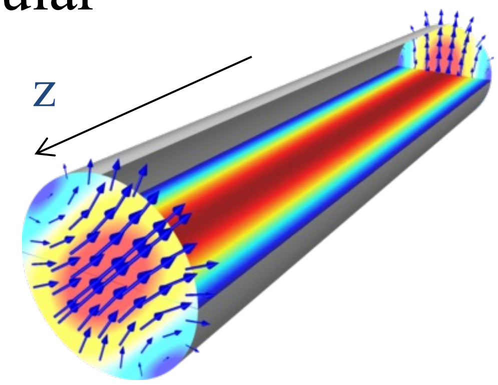
Rectangular



Different boundary conditions for TE and TM modes

$$TM: \psi\big|_s = 0; \quad TE: \left.\frac{\partial \psi}{\partial n}\right|_s = 0.$$

Circular



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + \gamma_{mn}^2 \psi = 0$$

$$TE: \psi^{TE}{}_{mn} = \psi_o \cos k_m x \cos k_n x; \quad m + n \geq 1;$$

$$TM: \psi^{TM}{}_{mn} = \psi_o \sin k_m x \sin k_n x; \quad m \geq 1; n \geq 1;$$

$$k_m = \pi \frac{m}{a}; k_n = \pi \frac{n}{b}; \gamma_{mn} = k_m{}^2 + k_n{}^2.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \gamma_{mn}^2 \psi = 0$$

$$\psi_{mn} = \varphi_{mn}(r) e^{in\theta} \Rightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial \psi_m}{\partial r}\right) + \left(r^2\gamma_{mn}^2 - n\right) = 0$$

$$\varphi_{mn} = J_n(\gamma_{mn} r)$$

Lowes cut-off frequency

Rectangular

$$TE: a > b; m = 1; n = 0; \quad \omega_{cut-off} = \frac{\pi c}{a};$$

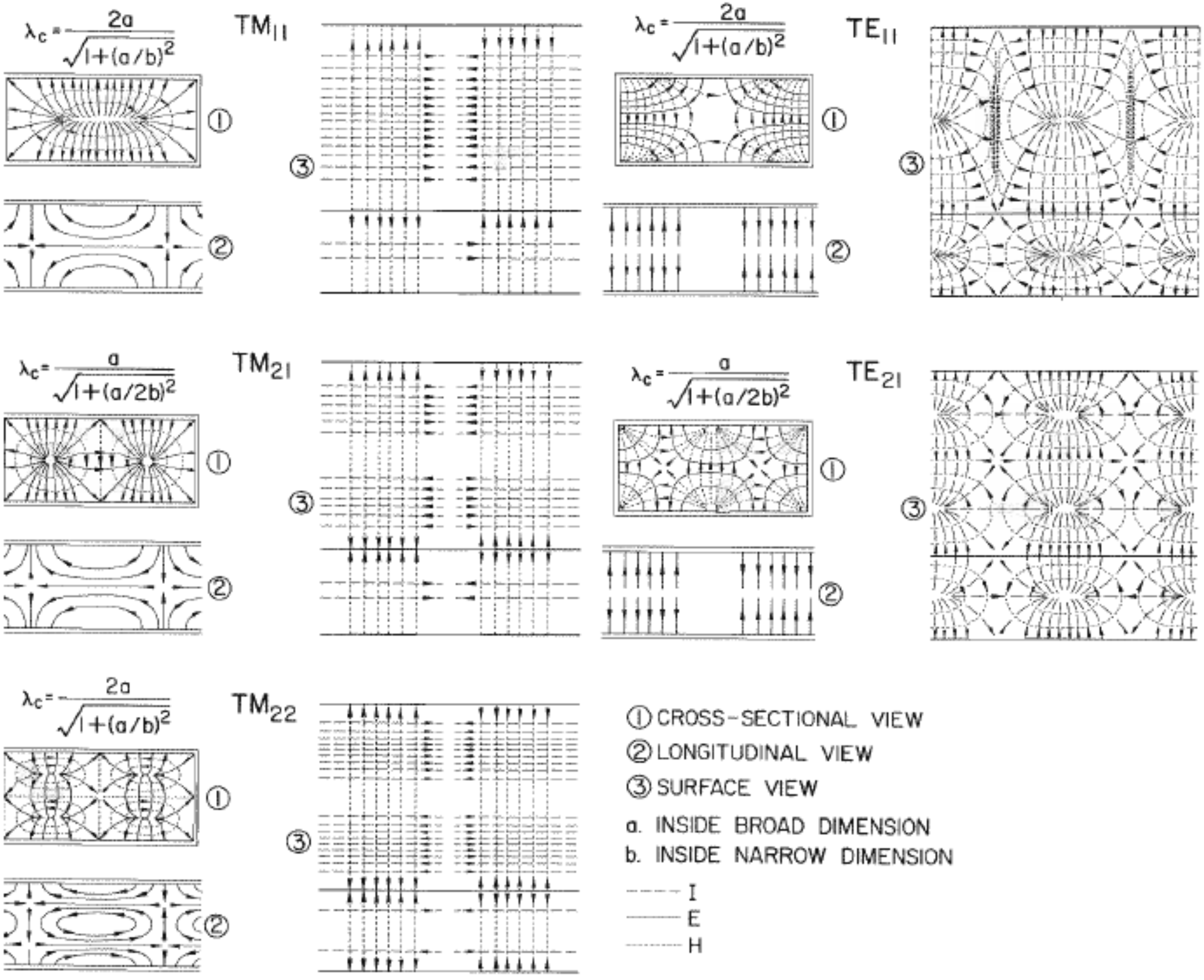
$$TM: m = 1; n = 1; \quad \omega_{cut-off} = \frac{\pi c}{a}\sqrt{1 + \frac{a^2}{b^2}}.$$

Circular

$$TM: J_o(\gamma_{01} R) = 0 \rightarrow \gamma_{01} \cong \frac{2.40483..}{R}; \quad \omega_{cut-off} \cong \frac{2.40\, c}{R};$$

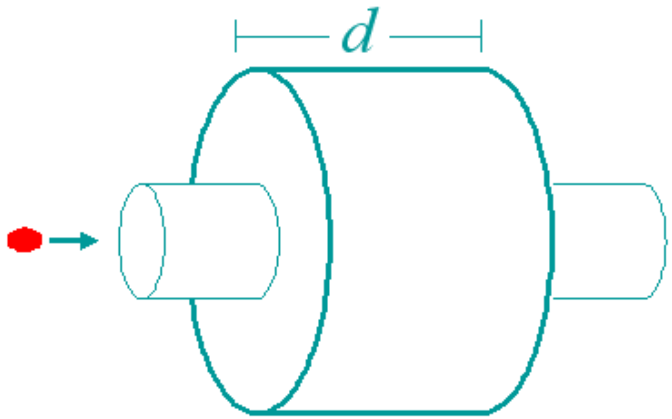
$$TE: J_1'(\gamma_{11} R) = 0 \rightarrow \gamma_{11} \cong \frac{1.84118....}{R}; \quad \omega_{cut-off} \cong \frac{1.84\, c}{R}.$$

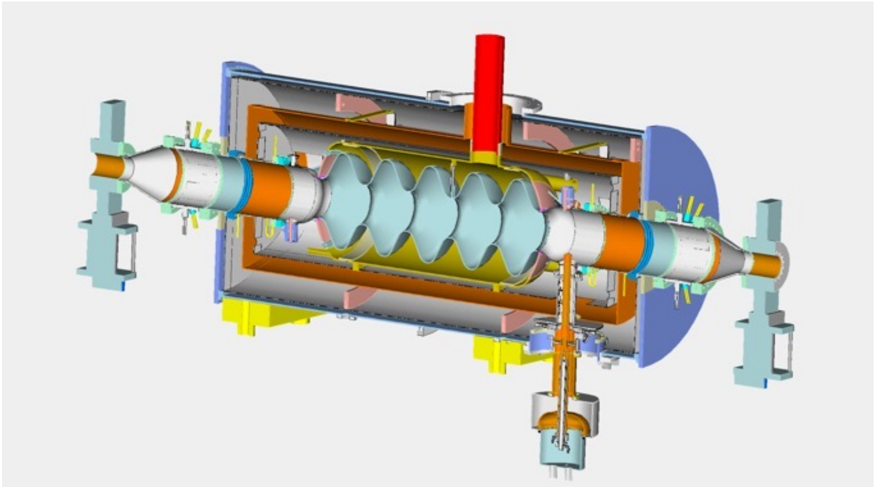
Modes in rectangular waveguide

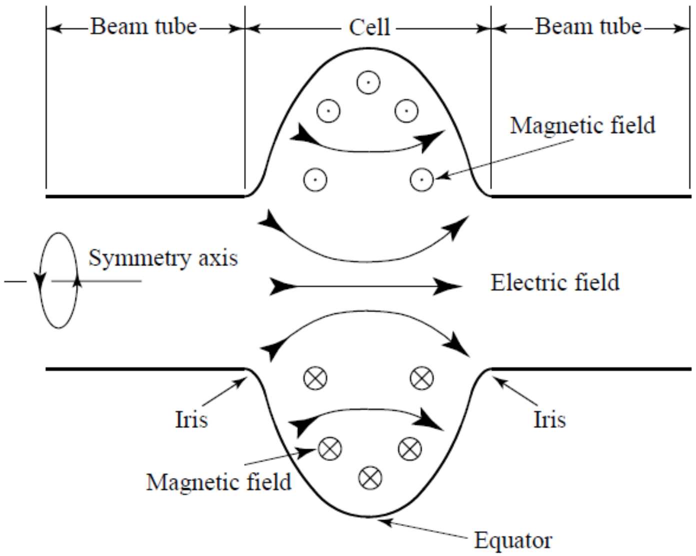


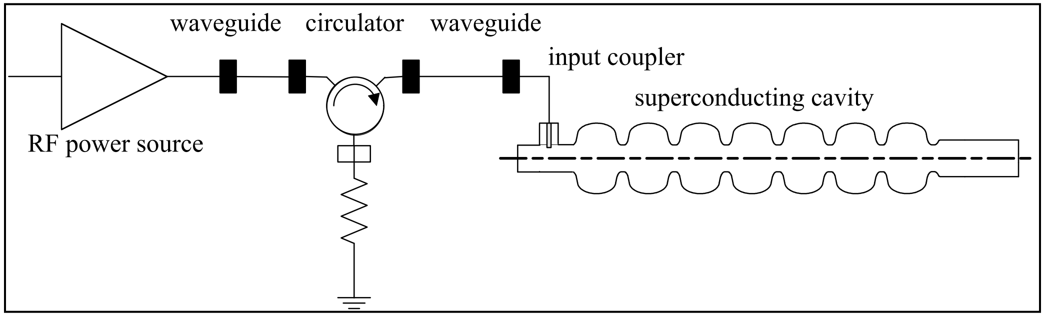
RF cavities

are designed to confine the EM field inside: It means that they operate at frequency below cut-off of the beam-pipes attached to them



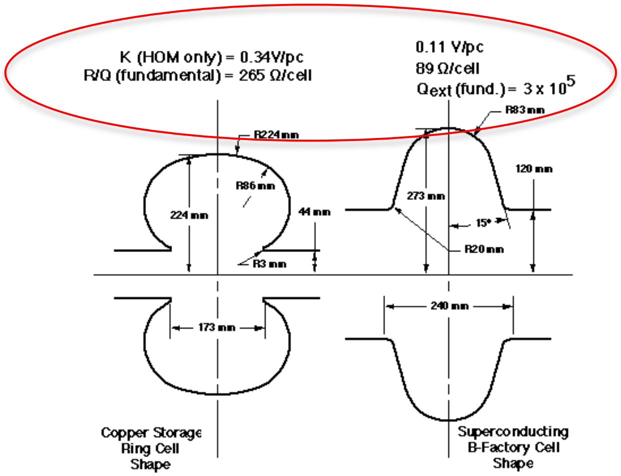


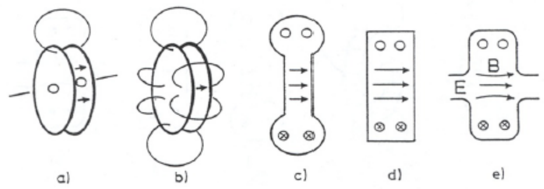


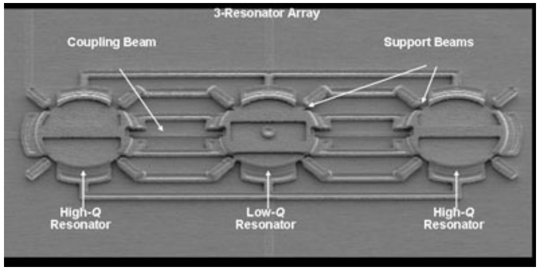


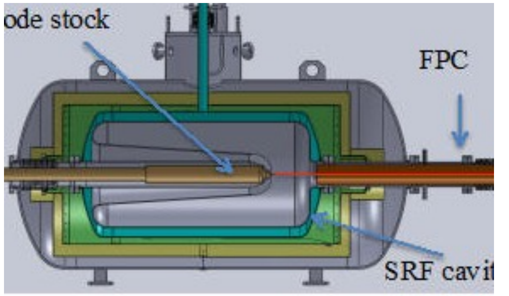
RF cavities come in many shapes, forms and sizes

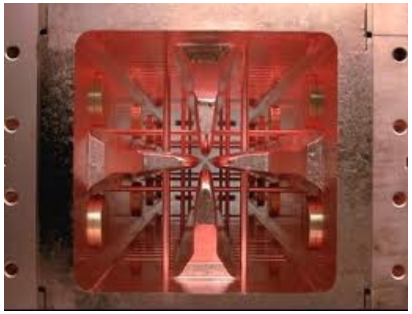
What these mean?

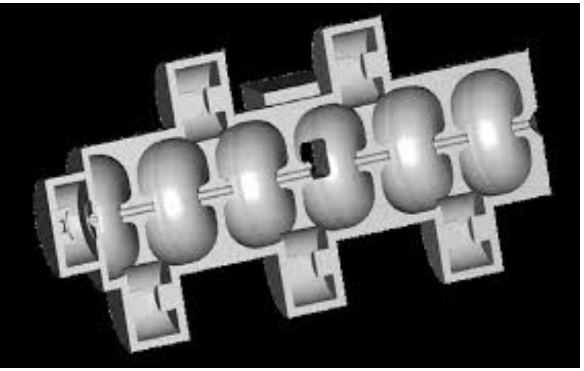


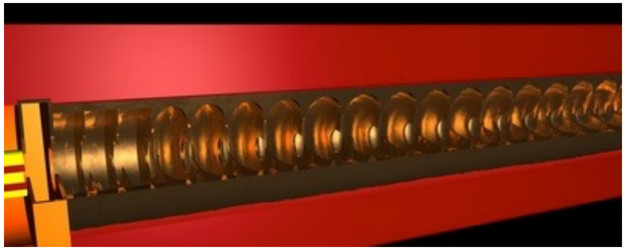


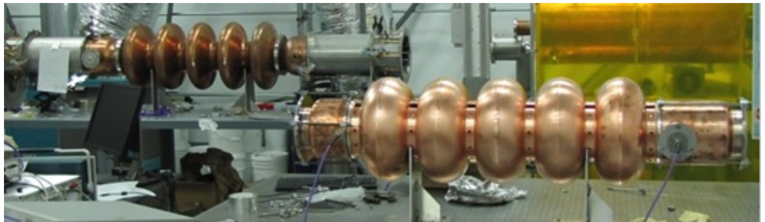












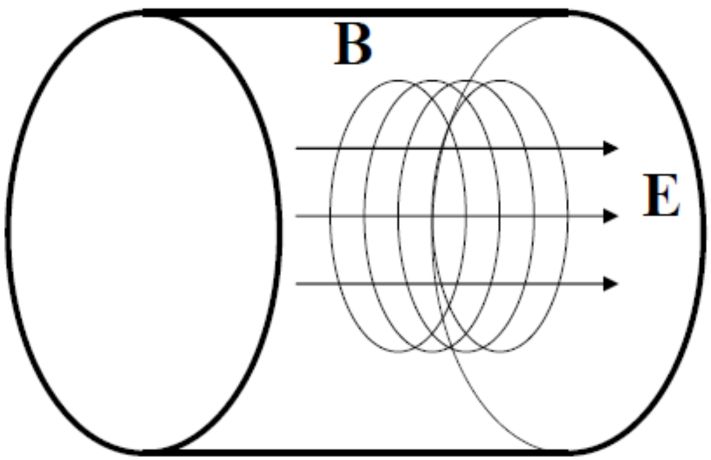
RF Cavity Modes:
the lowest accelerating is TM_{010} mode

- Fields in the cavity are solutions of the equation $$\left(\nabla^2 - \frac{1}{c}\frac{\partial^2}{\partial t}\right)\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} = 0$$
- Subject to the boundary conditions $\hat{n}\times\mathbf{E}=0,\ \hat{n}\cdot\mathbf{H}=0$
- Two extra surfaces (z=0 and z=d): but this is no problem for TM mode
- An infinite number of solutions (eigen modes) belong to two families of modes with different field structure and eigen frequencies: TE modes have only transverse electric fields, TM modes have only transverse magnetic fields.
- One needs longitudinal electric field for acceleration, hence the lowest frequency TM_{010} mode is used.
- For the pillbox cavity w/o beam tubes
- Note that frequency does not depend of the cavity length! But only its radius.

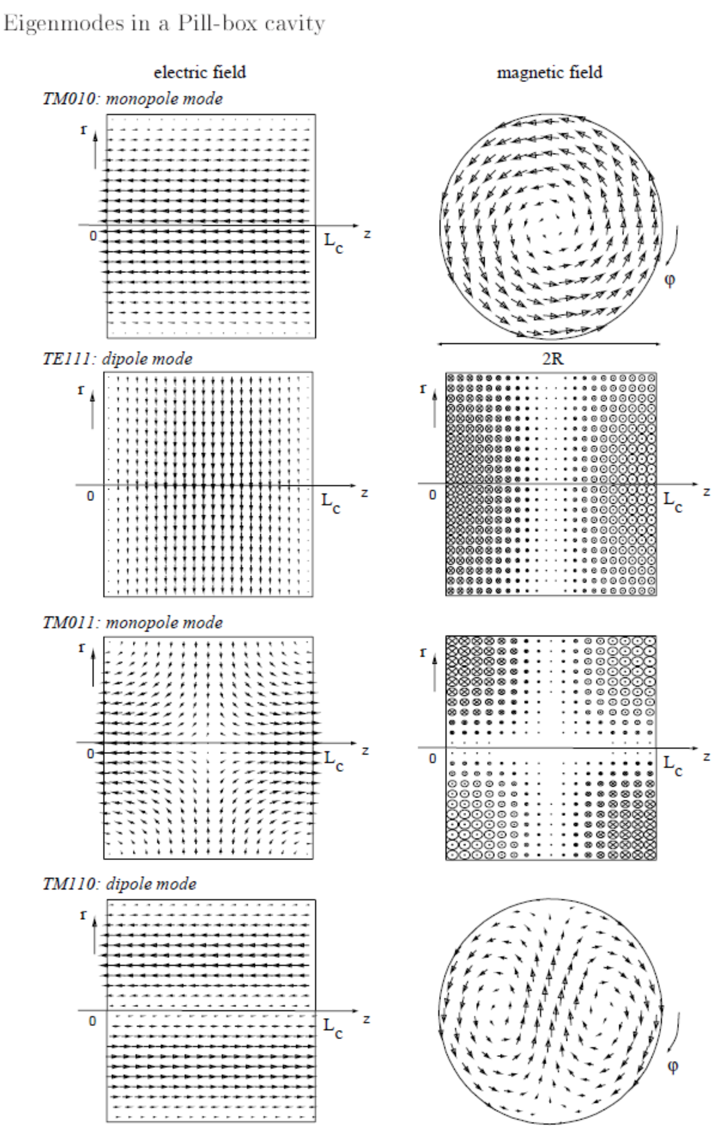
$$E_z = E_0 J_0\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

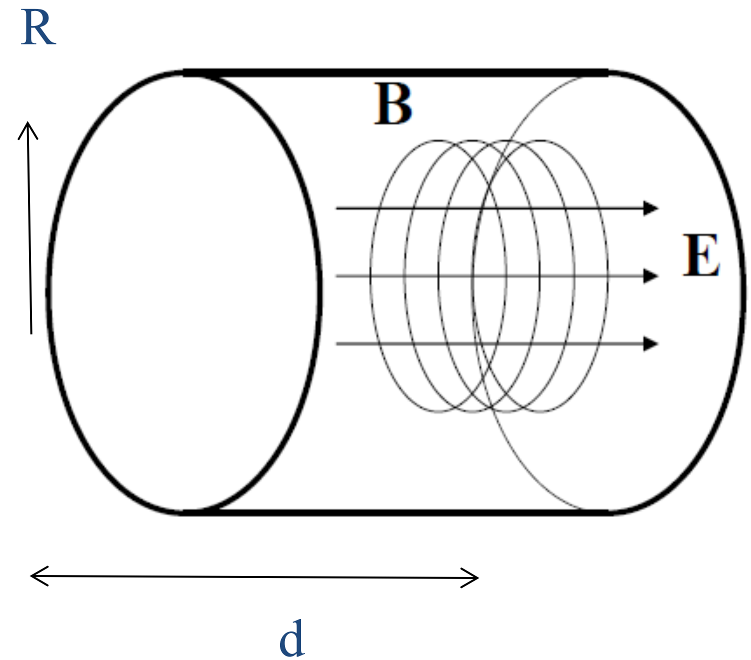
$$H_\phi = -iE_0 J_1\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

$$\omega_{010} = \frac{2.405c}{R},\quad \lambda_{010} = 2.61R$$



Fundamental and high order modes (HOMs)





$$TM: \varphi_{mnl} = J_n(\gamma_{mn} r)\cos k_{z,l} z;\; k_{z,l} = l\frac{\pi}{d};\; J_n(\gamma_{mn} R) = 0;$$

$$\omega_{res} = c\sqrt{\gamma_{mn}^2 + l^2\frac{\pi^2}{d^2}}; l = 0,1,2....$$

$$TE: \varphi_{mnl} = J_n(\kappa_{mn} r)\sin k_{z,l} z;\; k_{z,l} = l\frac{\pi}{d};\; J_n'(\kappa_{mn} R) = 0;$$

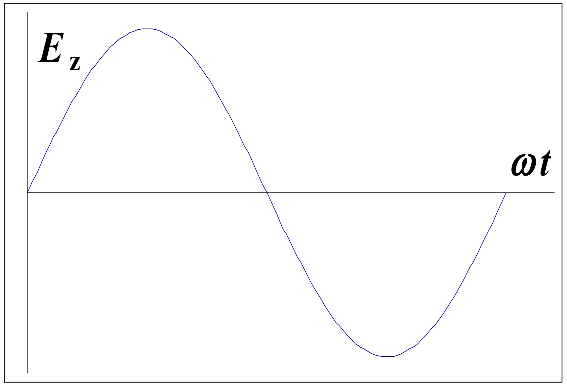
$$\omega_{res} = c\sqrt{\kappa_{mn}^2 + l^2\frac{\pi^2}{d^2}}; l = 1,2....$$

Acceleration inside RF cavity

- Let's consider a cavity terminated by a vacuum pipe for particles to pass
- Let's also consider a charge particle passing on the axis of the cavity the cavity with constant velocity (e.g. either particle is ultra relativistic, or velocity change is very small)
  - Electric field on the axis depending both on z and time

$$\mathbf{E_z}(z,t) = \mathbf{E_o}(z)\cos(\omega_0 t + \varphi)$$

- Specific form of $E_o(z)$ depends on the cavity design
- Energy change of the particle with charge q passing through the cavity is:



$$\Delta E = q\int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos(\omega_0 t + \varphi)\,dz$$

$$t = \frac{z}{\mathrm{v}} \Rightarrow \Delta E = q\int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos\left(\omega_0 \frac{z}{\mathrm{v}} + \varphi\right)dz$$

$$\Delta E = qV_{RF}\cos(\varphi + \varphi_o)$$

$$V_{RF} = \sqrt{V_s^2 + V_c^2}; \tan(\varphi_o) = \frac{V_c}{V_s};\ \ V_c = \int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos\left(\omega_0 \frac{z}{\mathrm{v}}\right)dz;\ V_s = \int_{-\infty}^{\infty} \mathbf{E_o}(z)\sin\left(\omega_0 \frac{z}{\mathrm{v}}\right)dz$$

How it is done

$$t=\frac{z}{\mathrm{v}};\Delta E=q\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos(\omega_0 t+\varphi)dz=q\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\left(\omega_0\frac{z}{\mathrm{v}}+\varphi\right)dz$$

$$\cos\left(\omega_0\frac{z}{\mathrm{v}}+\varphi\right)=\cos(\varphi)\cos\left(\omega_0\frac{z}{\mathrm{v}}\right)-\sin(\varphi)\sin\left(\omega_0\frac{z}{\mathrm{v}}\right)$$

$$\Delta E=q\cos(\varphi)\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\left(\omega_0\frac{z}{\mathrm{v}}\right)dz-q\sin(\varphi)\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\sin\left(\omega_0\frac{z}{\mathrm{v}}\right)dz$$

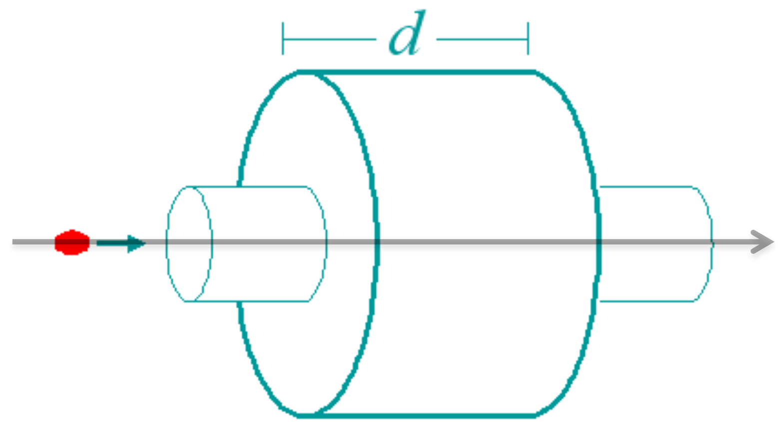
$$V_c=\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\left(\omega_0\frac{z}{\mathrm{v}}\right)dz;V_s=\int_{-\infty}^{\infty}\mathbf{E_o}(z)\sin\left(\omega_0\frac{z}{\mathrm{v}}\right)dz$$

$$V_{RF}=\sqrt{V_s^{\,2}+V_c^{\,2}};\tan(\varphi_o)=\frac{V_c}{V_s};$$

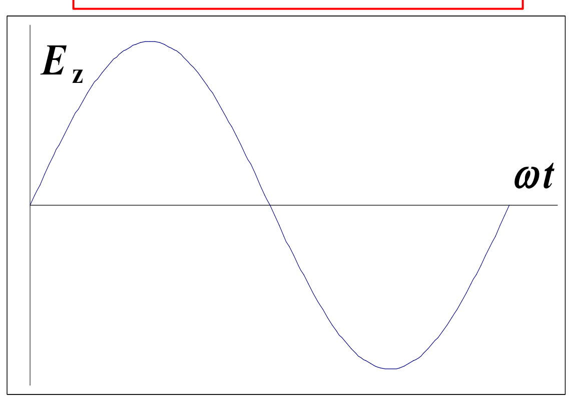
$$\Delta E=qV_{RF}\cos(\varphi+\varphi_o)$$

For particle moving with constant velocity all cavities are described by accelerating voltage and phase! Nothing else

- Now let's consider a pillbox cavity where E_z field is constant and extends from –*d/2* to +*d/2* with a small diameter vacuum pipes attached to it – the later required for particles to get through. With frequency of the RF cavity bellow the cut-off frequency of the vacuum piper, the RF field decays very fast in the pipe.

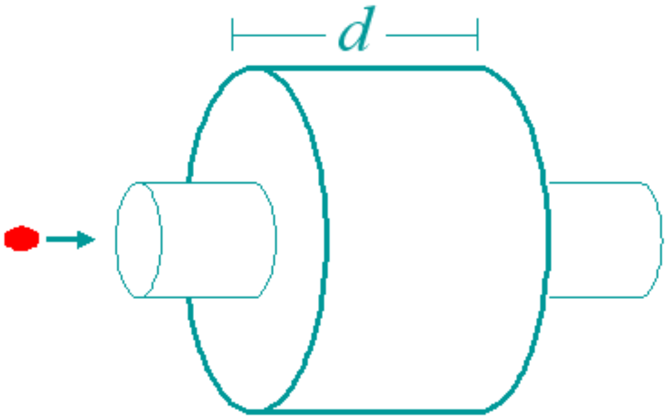


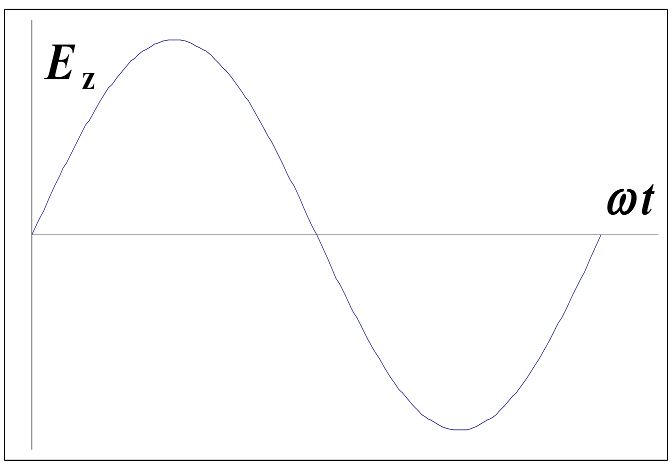
$$\Delta E = qV_{RF} \cdot \cos\varphi_o ; \varphi_o = \omega t$$



$$\mathbf{E_o}(z) = \begin{Bmatrix} \mathbf{E_o}, |z| \le d/2 \\ 0, |z| > d/2 \end{Bmatrix}$$

Accelerating voltage & transit time





- Assuming charged particles moving along the cavity axis, one can calculate accelerating voltage as

$$V_c = \left| \int_{-\infty}^{\infty} E_z(\rho = 0, z) e^{i\omega_0 z/\beta c} dz \right|$$

For the pillbox cavity one can integrate this analytically:

$$V_c = E_0 \left| \int_0^d e^{i\omega_0 z/\beta c} dz \right| = E_0 d \frac{\sin\left(\frac{\omega_0 d}{2\beta c}\right)}{\frac{\omega_0 d}{2\beta c}} = E_0 d \cdot T$$

where T is the transit time factor.

- To get maximum acceleration:

$$T_{transit} = t_{exit} - t_{enter} = \frac{T_0}{2} \Rightarrow d = \beta\lambda/2 \Rightarrow \quad V_c = \frac{2}{\pi} E_0 d$$

Thus for the pillbox cavity $T = 2/p$.

- The accelerating field E_{acc} is defined as $E_{acc} = V_c/d$. Unfortunately the cavity length is not easy to specify for shapes other than pillbox so usually it is assumed to be $d = bl/2$. This works OK for multi-cell cavities, but poorly for single-cell ones.

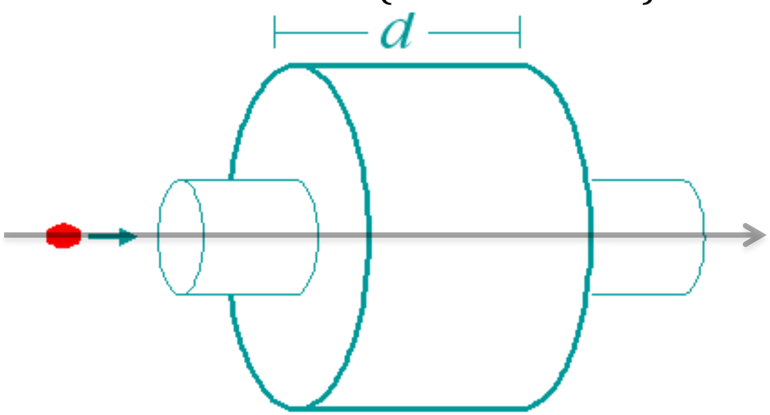
Acceleration inside RF cavity (cont..)

- Now let's consider a pillbox cavity where E_z field is constant and extends from –*d/2* to +*d/2*. Field decays very fact in the pipe

$$\mathbf{E_o}(z) = \begin{Bmatrix} \mathbf{E_o}, |z| \le d/2 \\ 0, |z| > d/2 \end{Bmatrix}$$

$$V_s = \mathbf{E_o} \int_{-d/2}^{d/2} \sin\left(\omega_0 \frac{z}{\mathrm{v}}\right) dz = 0$$

$$V_c = \mathbf{E_o} \int_{-d/2}^{d/2} \cos\left(\omega_0 \frac{z}{\mathrm{v}}\right) dz = \mathbf{E_o} \frac{2\mathrm{v}}{\omega_0} \cdot \sin\frac{\omega_0 d}{2\mathrm{v}} \Rightarrow V_c = \mathbf{E_o} d \cdot \frac{\sin X_t}{X_t}; X_t = \frac{\omega_0 d}{2\mathrm{v}}$$



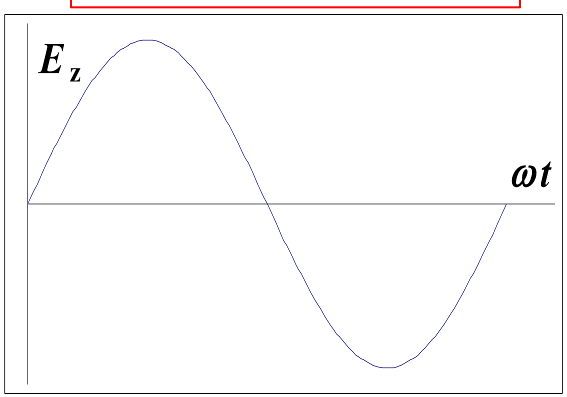
$$V_{RF} = |V_c|; \tan(\varphi_o) = 0;$$

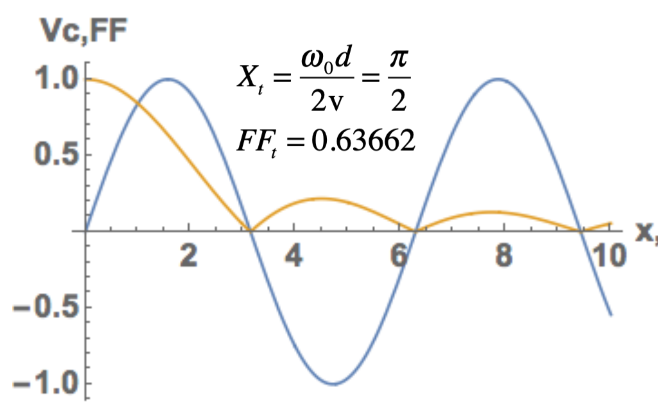
- Thus, the accelerating voltage differs from the ideal $E_o d$ by the transit time factor

$$\frac{V_{RF}}{\mathbf{E_o} d} = FF_t = \left|\frac{\sin X_t}{X_t}\right|; X_t = \frac{\omega_0 d}{2\mathrm{v}}$$

$$\Delta E = qV_{RF} \cdot \cos\varphi_o; \varphi_o = \omega t$$

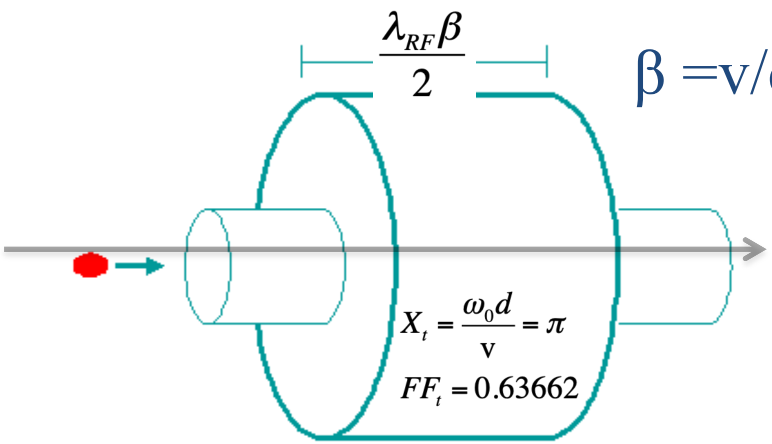
- Thus making cavity longer than the distance particle passed during ½ of the RF period makes no sense ($X_t = \pi/2$)





What are β=x cavities

- For heavy particles like protons, it takes a lot of RF cavities to accelerate to velocity comparable to speed of the light
- Hence, there are so called low-β cavities designed for slow particles
- You will see in literature β=0.1, β=0.5… cavities – it means that they are designed. For particle traveling nearly speed of light cavities called β=1.



β =v/c

$$\Delta E = qV_{RF} \cdot \cos\varphi_o; \varphi_o = \omega t$$

β =1 Pillbox

$$\frac{\omega_0 d}{c} = \pi$$

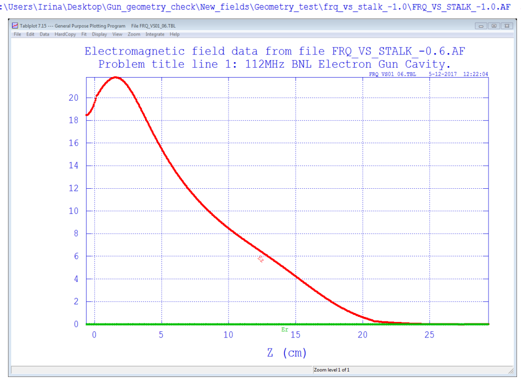
$$FF_t(\beta = 1) = 0.6366$$

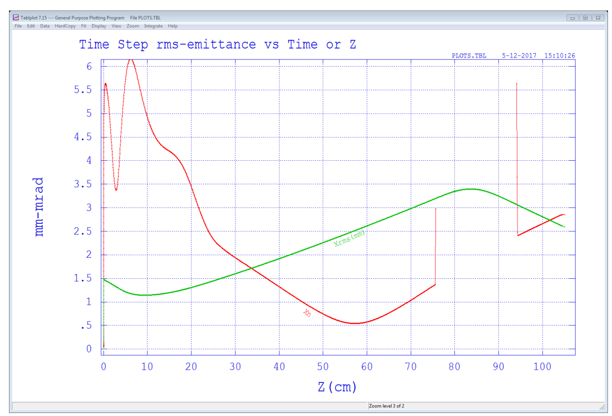
$$FF_t(\beta = 0.8) = 0.4705$$

$$FF_t(\beta = 0.5) = 0$$

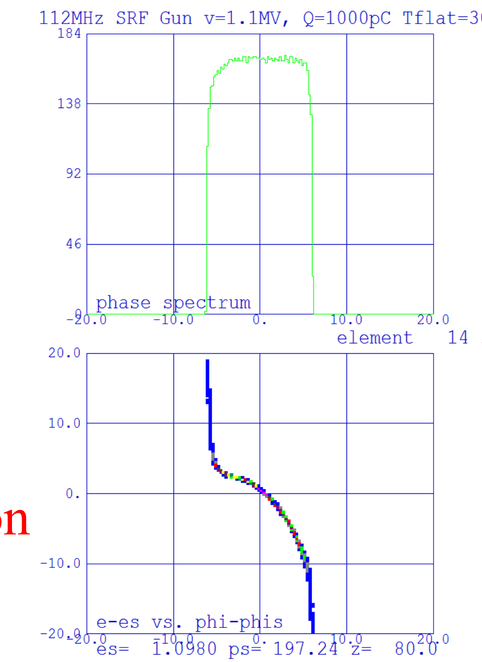
What about $\beta \neq$constant

- Typically we can use approximation that velocity of accelerating beam is nearly constant when it passing the cavity gap.
- This assumption is good for ultra-relativistic electrons/positions which are moving with velocity very close to the speed of light.
- This assumption is also a good approximation for heavy particles (ions or protons) when the energy gain per one accelerator cell is a small portion of the particle's rest mass energy.
- This assumption is violated and can not be used for electron guns, where electrons can accelerate from zero velocity to nearly speed of light. Equation of motion are both time-dependent and non-linear. You can estimate the result, nowadays it is normal to use numerical codes to get all beam dynamics correctly.





113 MHz SRF photo-emission Electron gun for CeC



Multi-cell cavities

- We learned so far that single cell RF cavity has limited accelerating voltage

$$Max(V_{RF}) = \frac{\mathbf{E_o}\lambda_{RF}}{\pi}$$

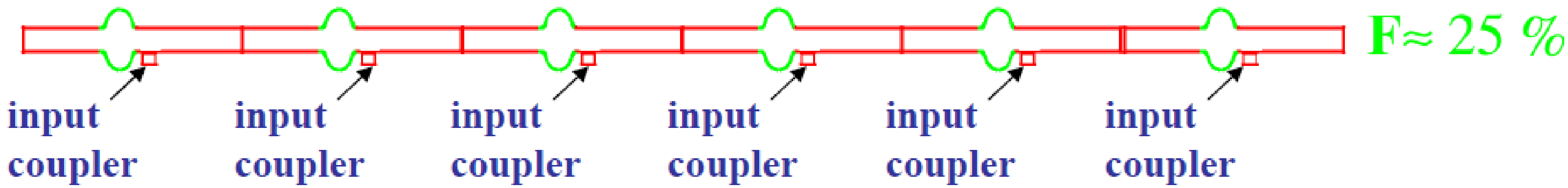
- To gain more energy we can either use more individual cells or use multi-cell cavities
- The first path, while feasible, is expensive (each cavity would need individual transmitter, waveguide, controls, etc.) and less effective – the average accelerating gradient (energy gain per meter of real estate) would be low
- Thus, where the acceleration gradient is important, the accelerator community uses multi-cell cavities

9-cell Tesla design

7-cell

5-cell

Why multi-cell cavities?



higher fill-factor:

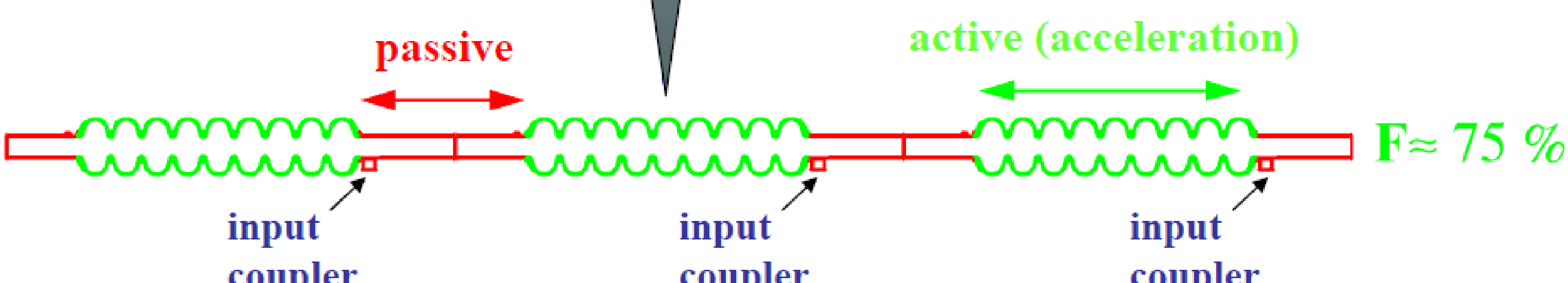
$$F = \frac{\text{active length}}{\text{total length}}$$

⇒ *lower costs*

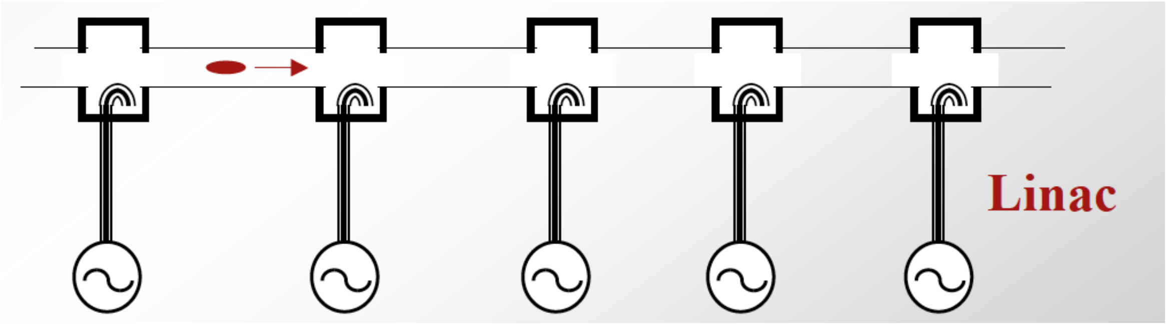
⇒ *better beam*

fewer

- **input couplers**
- **waveguide elements**
- **RF control systems**
- **…**



5-cell linac



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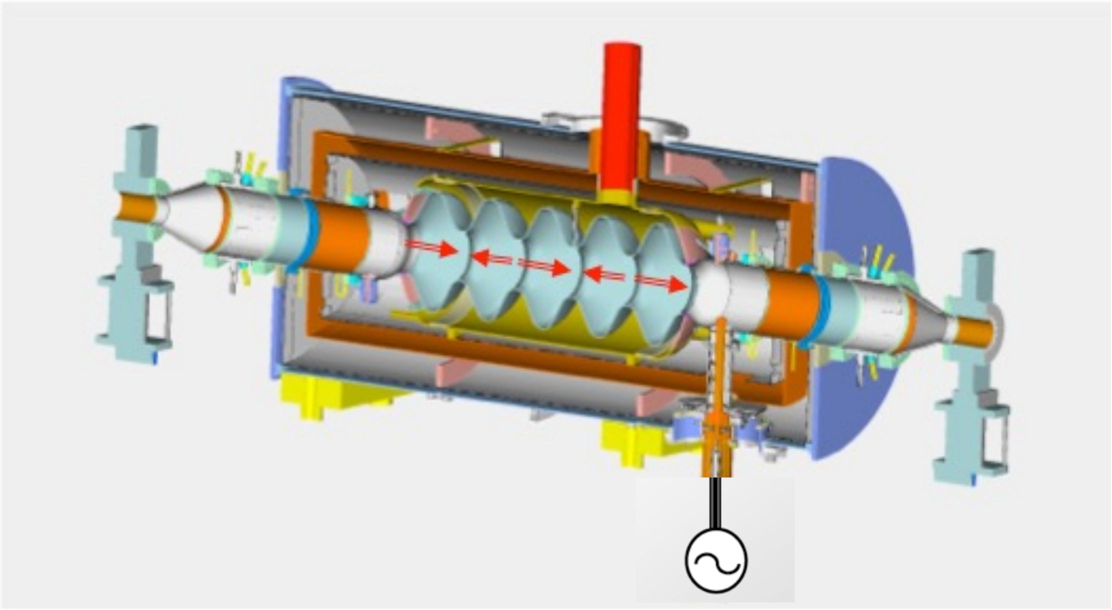
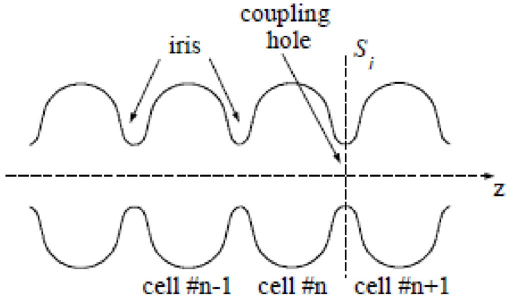
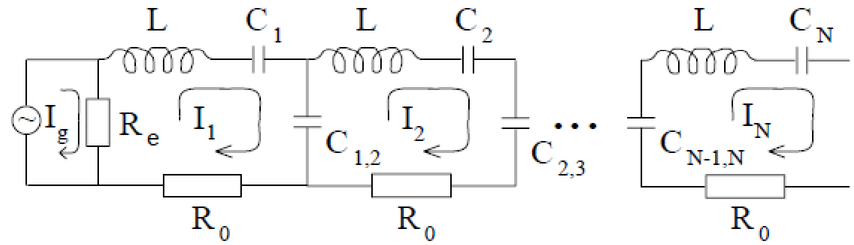


Multi-cell cavities

- Cavity consisting of n-cell is similar to N-coupled linear oscillators or resonant contours
- They all have nearly identical frequencies, but coupling splits then in n modes

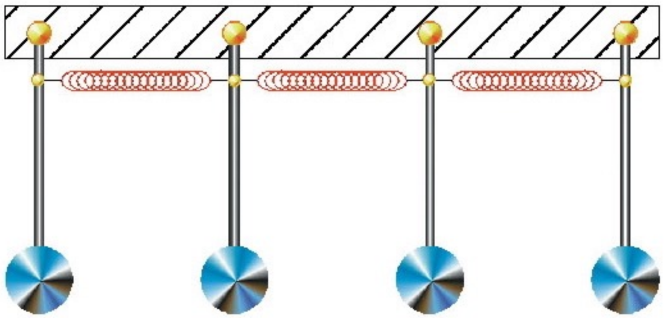




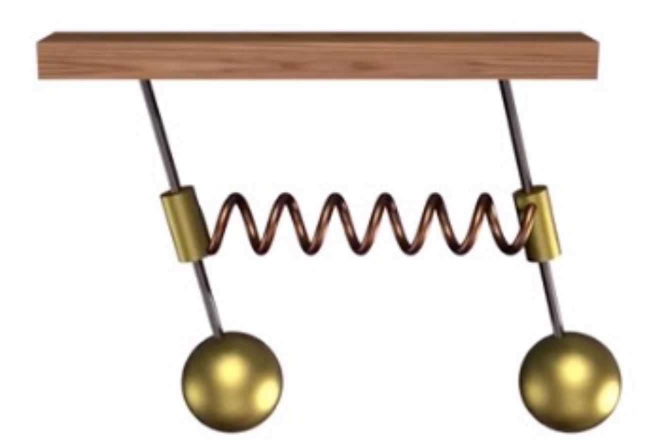
- The width of the pass-band (frequencies of various coupled modes) is determined by the strength of the cell-to-cell coupling *k* and the frequency of the *n*-th mode can be calculated from the dispersion formula

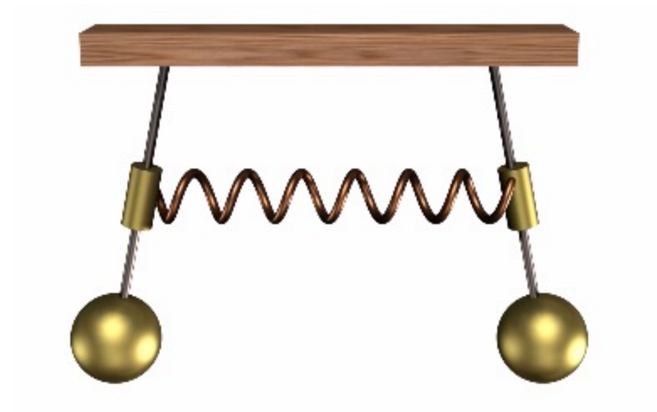
$$\left(\frac{f_n}{f_0}\right)^2 = 1 + 2k\left[1 - \cos\left(\frac{n\pi}{N}\right)\right]$$

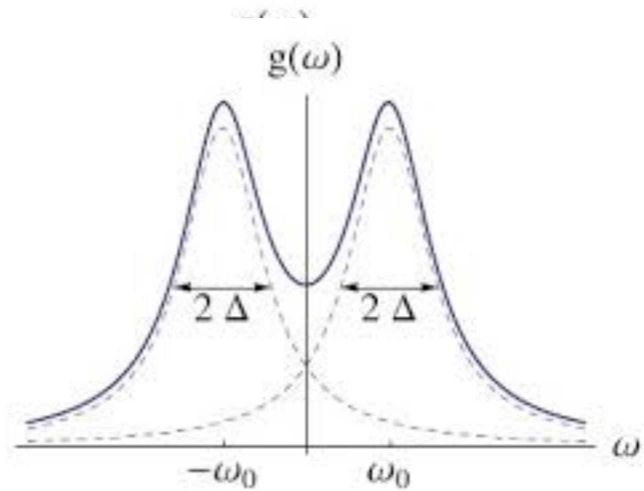
where *N* is the number of cells,
n = *1* ... *N* is the mode number.



Two coupled oscillators: 0-mode and π-mode

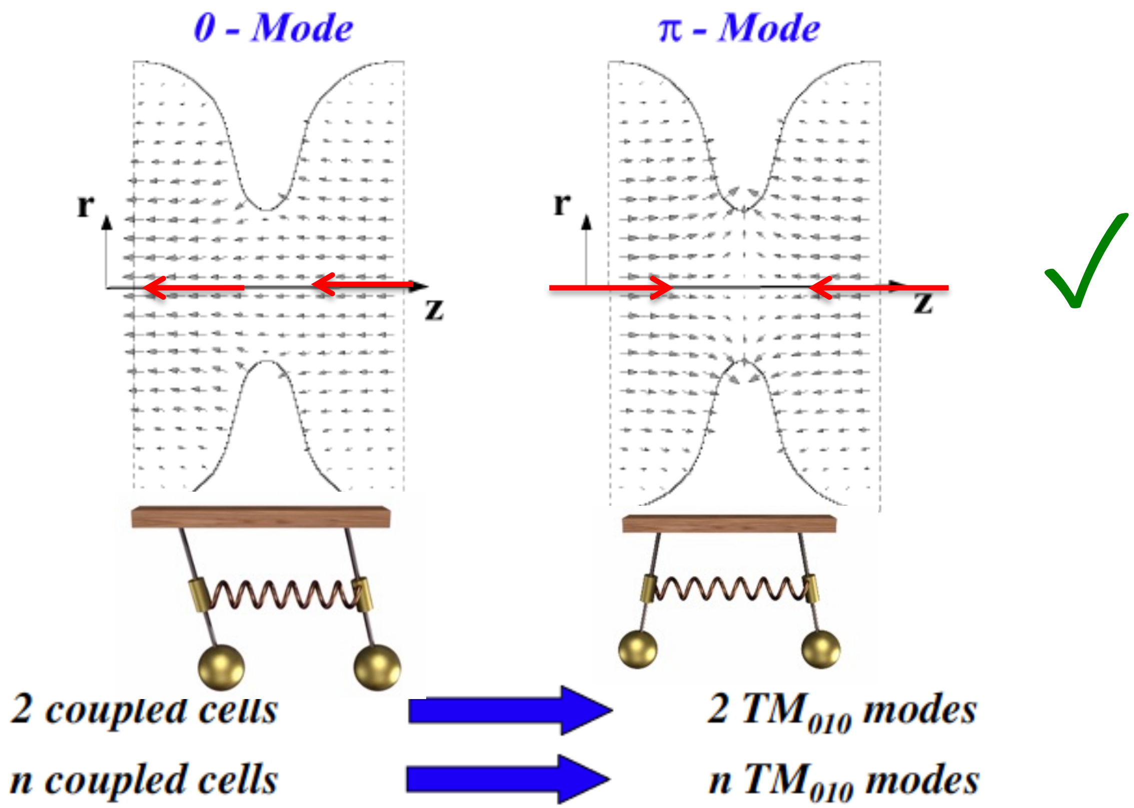






Multi-cell cavities: coupled oscillators

- Several cells can be connected together to form a multi-cell cavity
- Coupling of TM_{010} modes of the individual cells via the iris causes them to split
- 0-mode does not give any advantages – all cavities have the same direction of the field…
- π-mode is of special interest for us:
  - electric field has opposite directions on neighboring cells
  - particle passes through accelerating voltage in a cell in half of RF period
  - when particle crosses to the next cell – it sees again accelerating voltage



PHY 564
Advanced Accelerator Physics
Lectures 11 and 12
Linear accelerators
and RF systems for storage rings

Vladimir N. Litvinenko

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BROOKHAVEN NATIONAL LABORATORY

Stony Brook University

Traveling and standing wave structures

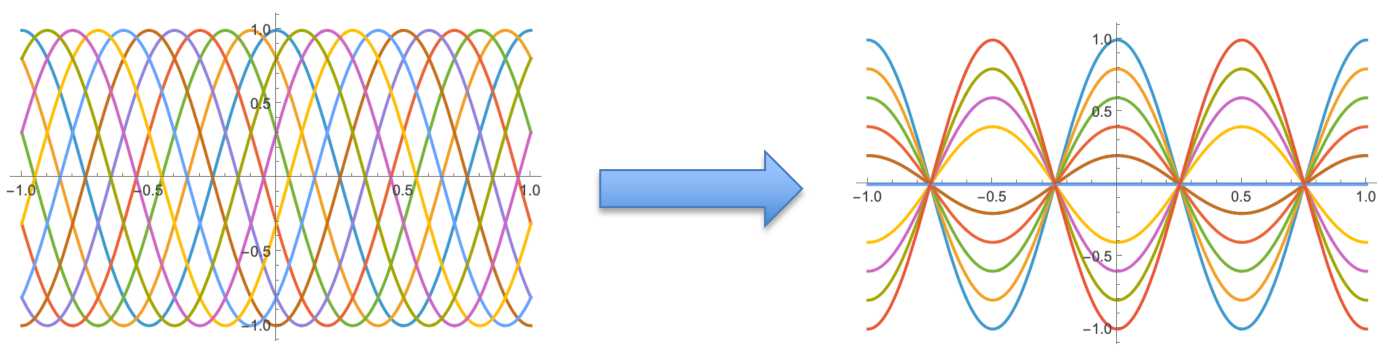
- We know very well structure of plane EM wave propagating with speed of light

$$\vec{E}_{\pm} = Re\left(\vec{E}_o \cdot e^{i\left(\omega_o t \mp k_o z\right)}\right) \equiv Re\left(\vec{E}_o \cdot e^{\pm i k_o (z \mp ct)}\right);\ k_o = \frac{\omega_o}{c};\ E_z = 0;$$

- Combination of contra-propagating waves results in a standing wave

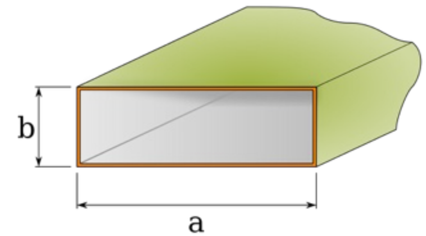
$$\vec{E}_{sw} = Re\left(\vec{E}_o \cdot e^{i\left(\omega_o t \mp k_o z\right)} + \vec{E}_o e^{i\left(\omega_o t + k_o z\right)}\right) = 2 \cdot \cos\left(k_o z\right) \cdot Re\left(\vec{E}_o \cdot e^{i\omega_o t}\right)$$

- Unfortunately, these waves are not useful for accelerators – E and B are perpendicular to direction of propagation.



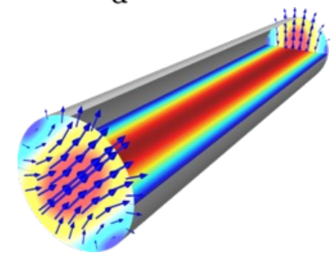
Traveling and standing wave structures

- We can create *z*-components of EM field using transmission line for TM_{nm} or TE_{nm} modes

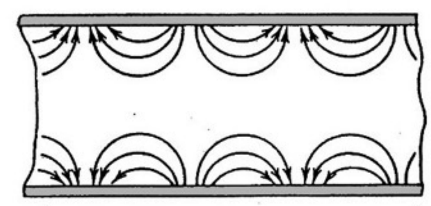


$$\vec{E}_{\pm} = Re\left(\vec{E}_{nm}\cdot e^{i(\omega_o t \mp k_z z)}\right) \equiv Re\left(\vec{E}_{nm}\cdot e^{\pm i k_z (z \mp ct)}\right);\ k_z = \sqrt{k_o^2 - k_{nm}^2};$$

- Waves wit frequencies above the cut-off $\omega_c = ck_{nm}$ will propagate in the waveguide



- Naturally, you can construct a standing wave

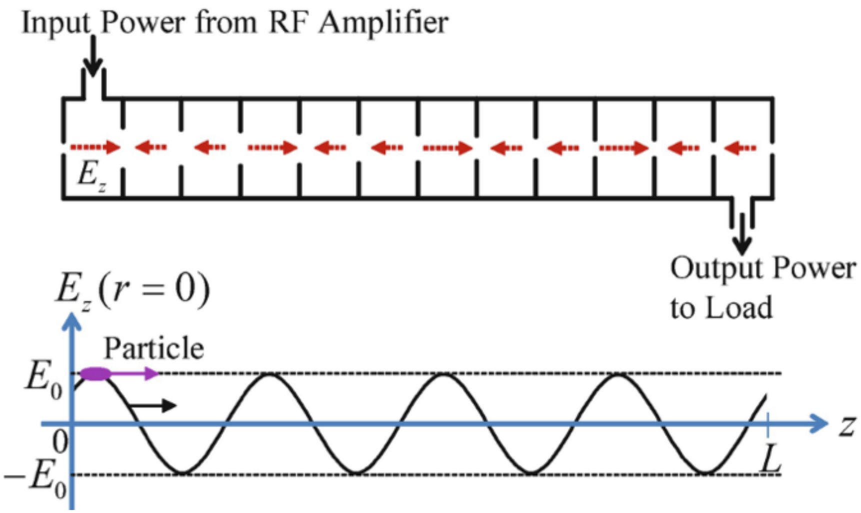


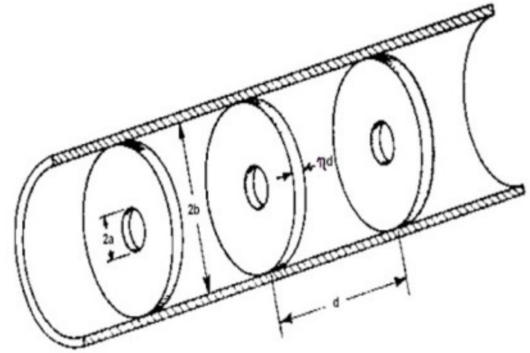
$$\vec{E}_{sw} = Re\left(\vec{E}_{nm}\cdot e^{i(\omega_o t \mp k_z z)} + \vec{E}_{nm} e^{i(\omega_o t + k_z z)}\right) = 2\cdot\cos(k_z z)\cdot Re\left(\vec{E}_{nm}\cdot e^{i\omega_o t}\right)$$

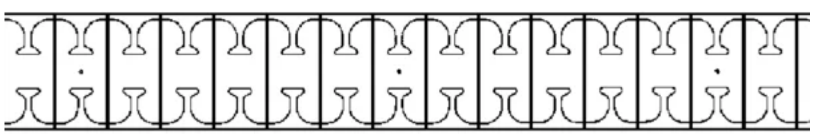
- Unfortunately, the phase of the wave in the waveguide is faster than the speed of light and particles can not keep-up with the wave

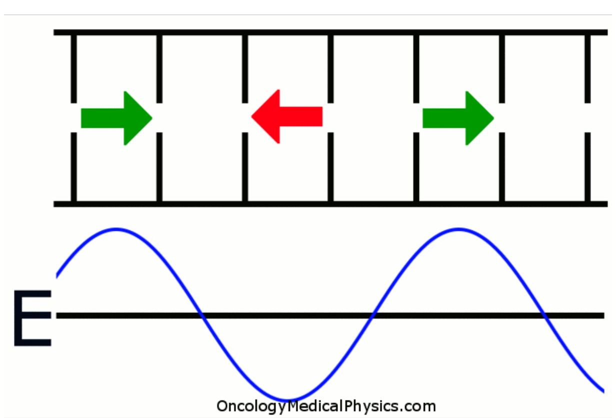
$$v_{z\,ph} = \frac{\omega_o}{k_z} = c\cdot\frac{k_o}{\sqrt{k_o^2 - k_{nm}^2}} > c \rightarrow \left\langle \vec{E}_{\pm}(t, \pm vt)\right\rangle \sim \left\langle Re\left(e^{i(k_o \pm k_z)ct}\right)\right\rangle = 0$$

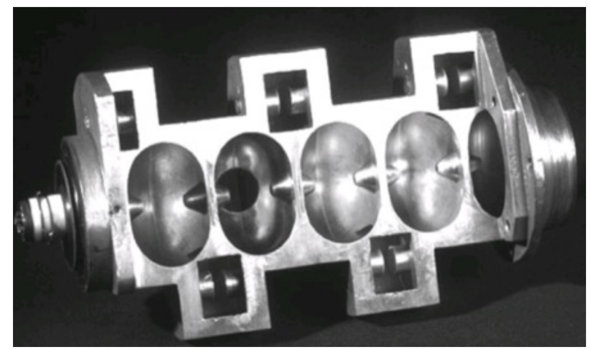
The problem is solved by introducing irises/semi-isolated cells to slow the EM wave

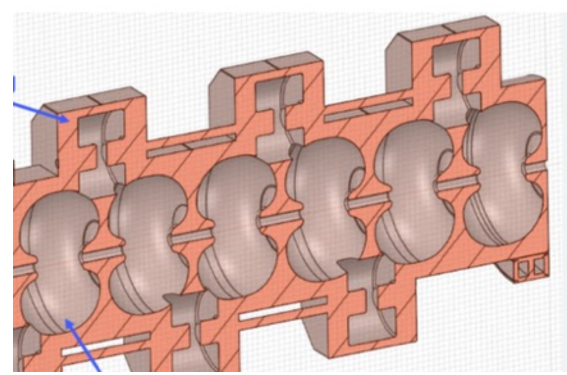






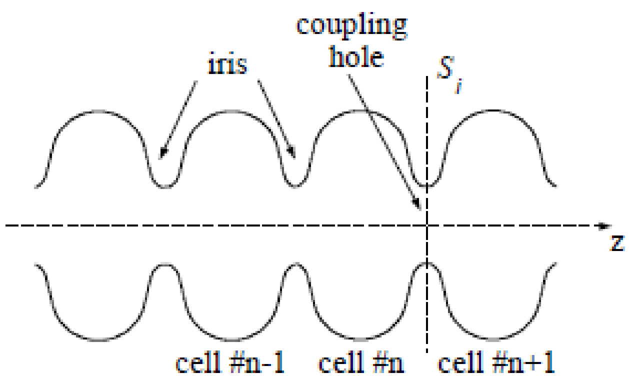


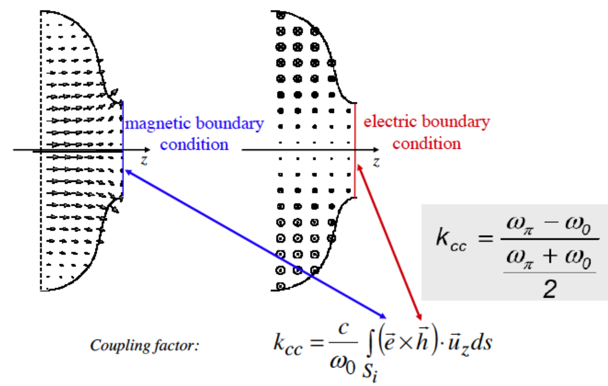


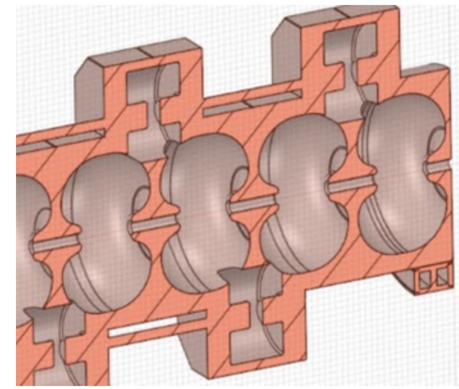


Multi-cell cavities - coupling

- Each isolated cavities (cells) has a resonant frequencies at the desired mode, which are tuned to ω_o.
- EM fields in the cells are coupled through opening in irises or slots at the cell's perimeter



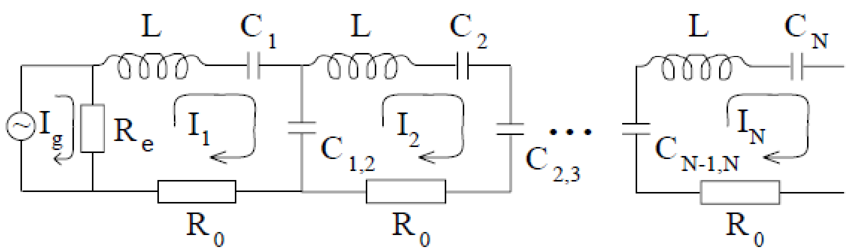


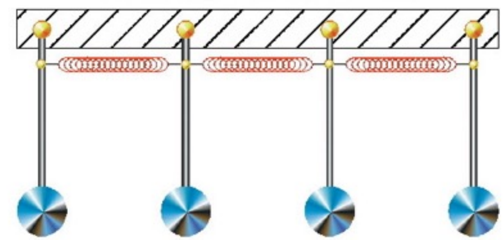


- While there are several methods to estimate coupling, using EM codes serves us best
- It is worth mentioning that larger iris provides for stronger coupling and better uniformity of the field
- But increasing the iris reduces the electric field on axis (shunt impedance) and reduces accelerating gradient of such accelerator - hence, there is a compromise

Multi-cell cavities - coupling

- There are too many engineering details in designs of the multi-cell cavities to include in this course
- This includes type of the coupling between cells (inductive, capacitive or mixed). For example, the slot coupling on the sides of the cavities can also results in the phase delay.
- Engineers like using coupled resonant circuits to describe the system, but there is an excellent analogy between the multi-cell cavity and set of coupled oscillators which is closer my heart





$$\ddot{x}_1 + \omega_o^2 \cdot x_1 = \nu \cdot x_2$$

$$\ldots\ldots$$

$$\ddot{x}_n + \omega_o^2 \cdot x_n = \nu \cdot (x_{n-1} + x_{n+1})$$

$$\ldots$$

$$\ddot{x}_N + \omega_o^2 \cdot x_N = \nu \cdot x_{N-1}$$

$$\ddot{x}_1 + \omega_o^2 \cdot x_1 = \nu \cdot (x_o + x_2)\,;\; x_o = 0;$$

$$\ldots\ldots$$

$$\ddot{x}_n + \omega_o^2 \cdot x_n = \nu \cdot (x_{n-1} + x_{n+1})$$

$$\ldots$$

$$\ddot{x}_N + \omega_o^2 \cdot x_N = \nu \cdot (x_{N-1} + x_{N+1})\,;\; x_{N+1} = 0;$$

we intentionally decided not assume the relative sign of coupling from up- and down-stream cells

Multi-cell cavities - coupling

- Solution of this equations is known

$$x_n = a \cdot e^{i(\omega t - n\cdot\theta + \varphi_o)} \Rightarrow a \cdot \cos\omega t \cdot \sin(n\cdot\theta);$$

$$(\omega_0^2 - \omega^2) a \cdot e^{i(\omega t - n\cdot\theta)} = \nu \cdot a \cdot e^{i(\omega t - n\cdot\theta)} \cdot \left(e^{i\theta} \pm e^{-i\theta}\right);$$

$$\frac{\omega}{\omega_o} = \sqrt{1 - \alpha\cdot\cos\theta};\ \alpha = \frac{2\nu}{\omega_0^2} << 1;\ \theta_i = 2\pi \cdot \left(m + \frac{i}{N}\right);$$

- For the traveling field in the cavity, this solution would imply that structure of the field in each cell is reputed itself with a phase shift

$$\vec{E}_o\left(z + d, \vec{r}_\perp\right) = \vec{E}_o\left(z, \vec{r}_\perp\right)\cdot e^{-i\theta};\quad \kappa = \frac{\theta}{d} > 0;$$

$$\vec{E}\left(z, \vec{r}_\perp, t\right) = Re\sum_{n=1}^{N} \vec{E}_c\left(\vec{r}_\perp, z - n\cdot d\right)\cdot e^{i(\omega t - \kappa\cdot z)};\ \vec{E}_c\left(\vec{r}_\perp, z\right) = \begin{Bmatrix} E_{cell},\ 0<z<d \\ 0,\quad z<0\ or\ z>d \end{Bmatrix}$$

- Expanding field in the cell we get

$$\vec{E}_c\left(\vec{r}_\perp, z\right) = \sum_{n=-\infty}^{n=\infty} \vec{E}_n \cdot e^{-in\cdot k_c z};\ k_c = \frac{2\pi}{d} \Rightarrow \vec{E}\left(z, \vec{r}_\perp, t\right) = Re\sum_{n=-\infty}^{n=\infty} \vec{E}_n \cdot e^{i\left(\omega t - \left(\kappa + nk_c\right)\cdot z\right)}$$

with a set of phase velocities $v_{ph} = \omega/(\kappa + nk_c)$ that can fit the particle's velocity for positive or zero *n*.

- The dispersion relations between RF frequency and k-factor is defined above

$$x_n = a \cdot e^{i(\omega t - n\cdot\theta + \varphi_o)} \Rightarrow a \cdot \cos\omega t \cdot \sin(n\cdot\theta);$$

$$(\omega_0^2 - \omega^2) a \cdot e^{i(\omega t - n\cdot\theta)} = \nu \cdot a \cdot e^{i(\omega t - n\cdot\theta)} \cdot \left(e^{i\theta} \pm e^{-i\theta}\right);$$

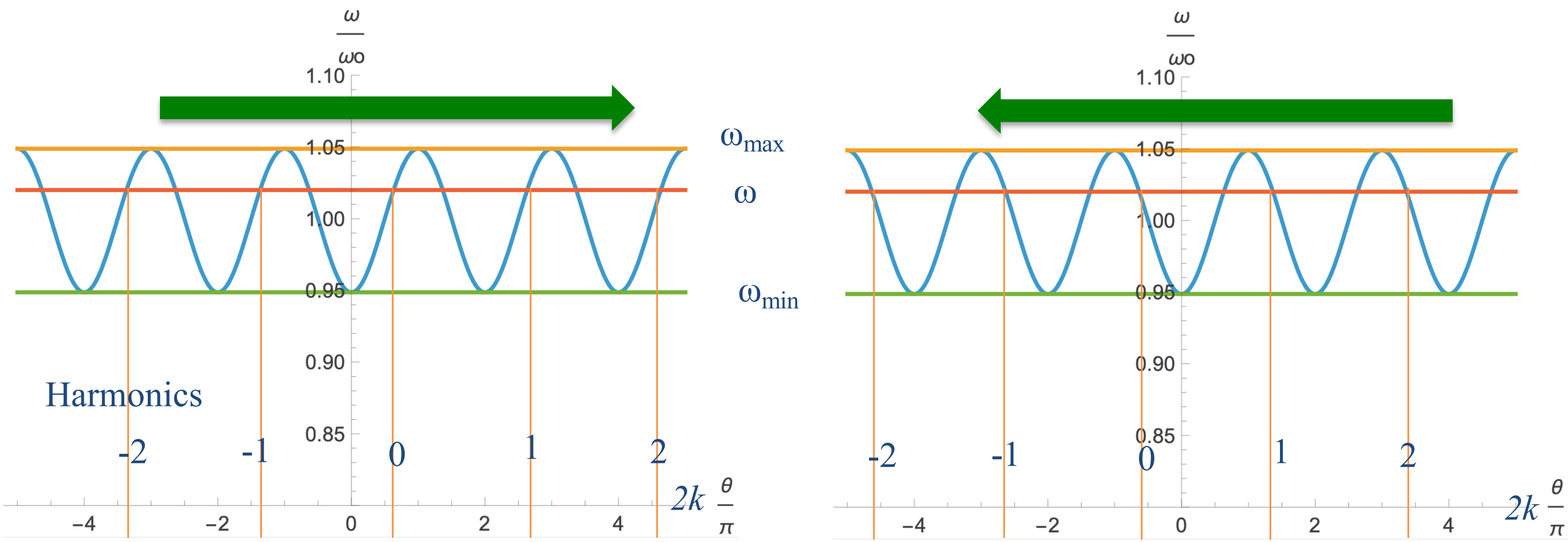
$$\frac{\omega}{\omega_o} = \sqrt{1 - \alpha\cdot\cos\theta};\ \alpha = \frac{2\nu}{\omega_0^2} << 1;\ \theta_i = 2\pi \cdot \left(m + \frac{i}{N}\right);$$

Multi-cell cavities - coupling

- What it means that we can choose frequency which is matched with velocity of the particles

$$\frac{\omega}{\omega_o}=\sqrt{1-\alpha\cdot\cos\left(\frac{k_o d}{\beta}-2\pi\cdot n\right)}=\frac{\omega}{\omega_o}=\sqrt{1-\alpha\cdot\cos\left(\frac{k_o d}{\beta}\right)}$$

and to use these dispersion relations to identify maximum and minimal frequencies that can propagate in this structure and *k*-values(and of cause β) for our design frequency ω

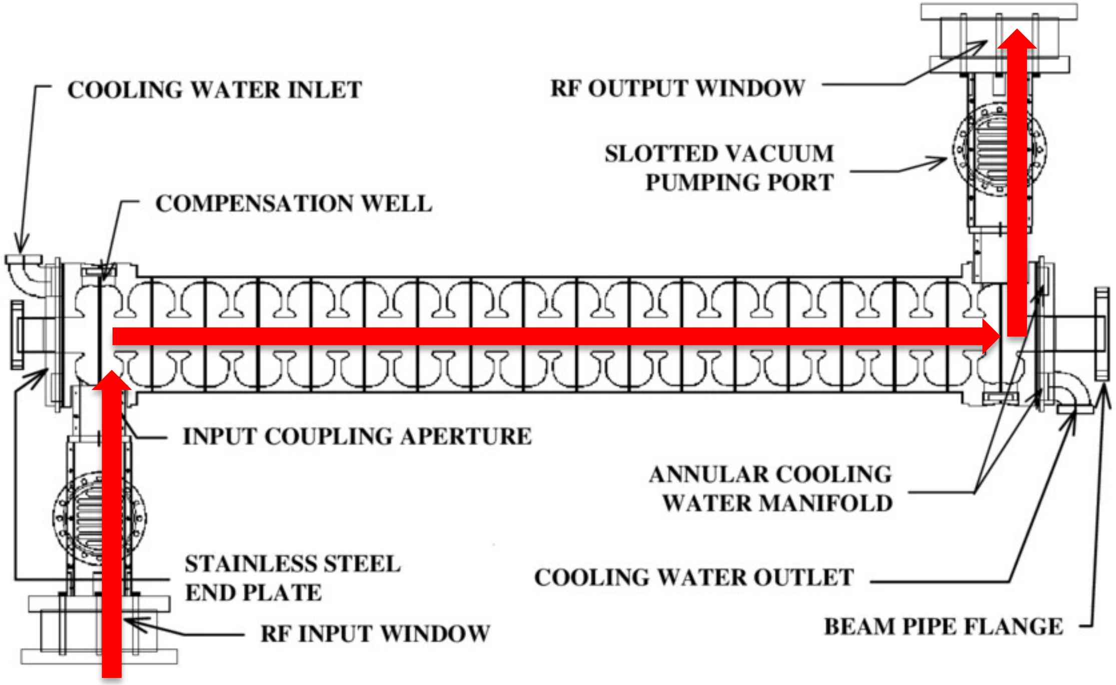


Forward wave

Backward wave

- The direction of the propagation is defined by group velocity $v_g=\frac{d\omega}{dk_z}$ which is the same for all harmonics

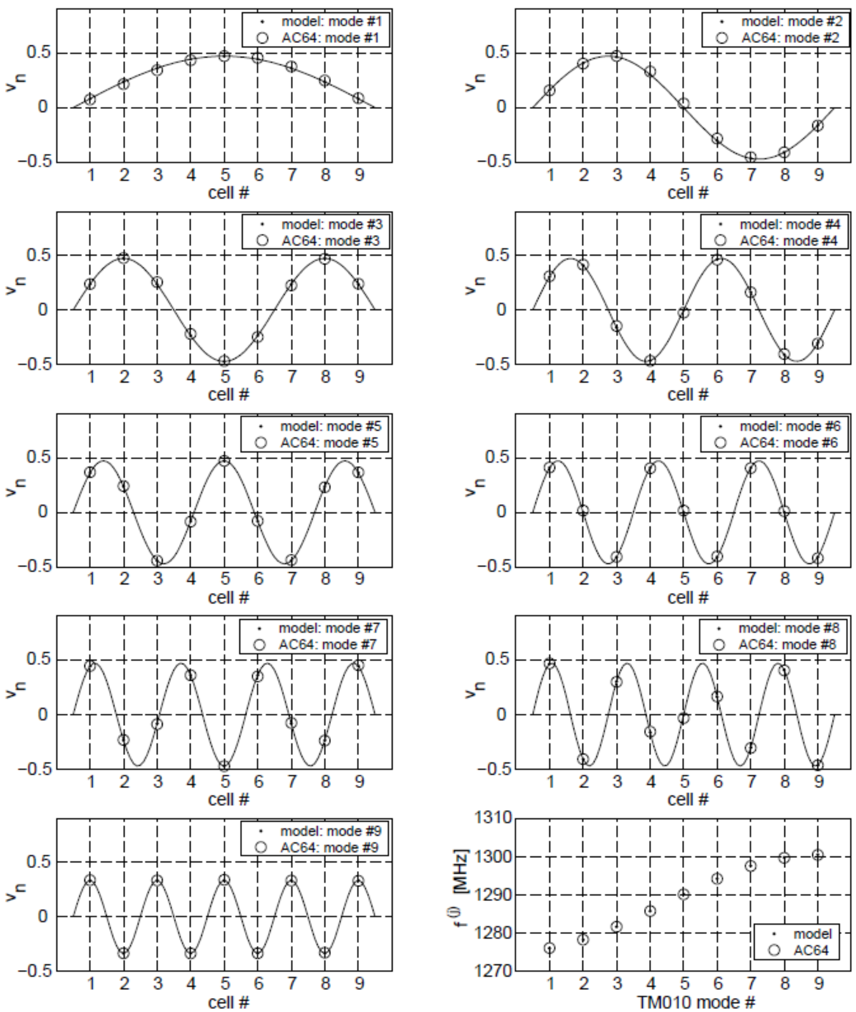
Standing and traveling wave multi-cell cavities: the main difference is e flow of RF power



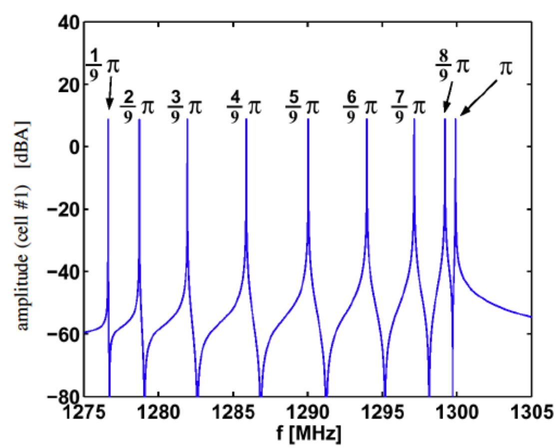
$$\vec{E}(z,\vec{r}_{\perp},t) = Re\left(\sum_{n=-\infty}^{n=\infty} \vec{E}_{nf}\cdot e^{i\left(\omega t-\left(\kappa+nk_{c}\right)\cdot z\right)} + \sum_{n=-\infty}^{n=\infty} \vec{E}_{nb}\cdot e^{i\left(\omega t+\left(\kappa+nk_{c}\right)\cdot z\right)}\right) =$$

$$\Rightarrow 2Re\sum_{n=-\infty}^{n=\infty} e^{i\omega t}\vec{E}_{nf}\cos\left(\kappa+nk_{c}\right)\cdot z = E_{sw}(z)\cdot\cos(\omega t+\varphi)$$

Real multi-cell cavity

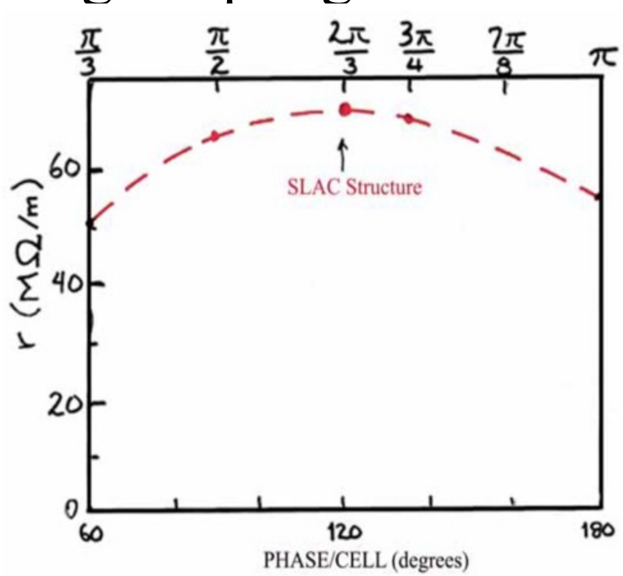


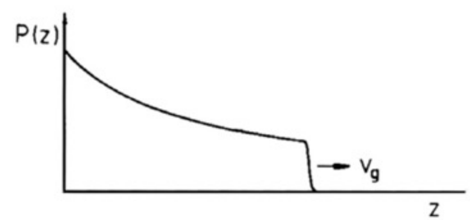
- Figure shows an example of calculated eigenmodes amplitudes in a 9-cell TESLA cavity compared to the measured amplitude profiles. Also shown are the calculated and measured eigenfrequencies.
- A longer cavity with more cells has more modes in the same frequency range, hence the reduction in frequency difference between adjacent modes. The number of cells is usually a result of the accelerating structure optimization.
- The accelerating mode for SC cavities is usually the **p**-mode, which has the highest frequency for electrically coupled structures.
- The same considerations are true for HOMs.

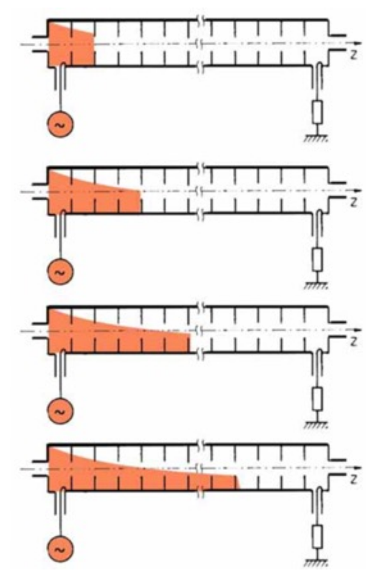


SLAC cell

- Since traveling wave linac need a lot of power to operate, they operate in a pulsed mode
- The power comes from one side of the linac and propagating with the group velocity and takes time to fill the entire cavity (excite all oscillators to the same amplitude) before accelerating beams
- Group velocity depends on the coupling $\frac{d\omega}{dk_z} \sim \alpha$ and relatively strong coupling is needed in traveling wave structures







Transverse focusing

- Most of accelerating cavities – because of simplicity and convenience – have azimuthal symmetry
- Let's consider a linac with z-electric field on axis

$$E_z = Re\left(E_{zo}(z) \cdot e^{i\omega t}\right)$$

- and using absence of the charges

$$div\vec{E} = \frac{\partial E_z}{\partial z} + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = -\frac{\partial E_z}{\partial z}$$

- connect transverse derivatives of the field. For axial symmetry

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = \frac{\partial E_r}{\partial r} = -\frac{1}{2}\frac{\partial E_z}{\partial z}$$

and from second Maxwel equation

$$curl\vec{B} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} \Rightarrow 2\frac{\partial B_\theta}{\partial r} = \frac{1}{c}\frac{\partial E_s}{\partial t}$$

yielding first order of field extensions

$$E_x = -\frac{x}{2}\frac{\cdot \partial E_z}{\partial z};\ E_y = -\frac{y}{2}\frac{\partial E_z}{\partial z};\ B_\theta = \frac{r}{2c}\frac{\partial E_z}{\partial t};$$

$$\frac{dP_r}{ds} = \frac{e \cdot r}{2}\left(\frac{\partial E_z}{c\,\partial t} - \frac{1}{\beta_o}\frac{\partial E_z}{\partial z}\right) = \frac{e \cdot r}{2} Re\left(ik_o E_{zo} - \frac{1}{\beta_o}\frac{\partial E_{zo}}{\partial z}\right)e^{i\omega t};\ r = x,y$$

Standing wave

- For standing wave is cavities with symmetric cells (typical)

$$E_z = E_{zo}(z)\cos(\omega t + \varphi) = E_0 \cdot \cos(\omega t + \varphi) \cdot \sum_{n=1}^{\infty} a_n \cos\left(n k_c z\right)$$

- we get "rational" expression for the force

$$E_r = \frac{k_c \cdot r}{2} E_0 \cdot \cos(\omega t + \varphi) \cdot \sum_{n=1}^{\infty} n \cdot a_n \sin\left(n k_c z\right);\ B_\theta = -\frac{k_o \cdot r}{2} E_0 \cdot \sin(\omega t + \varphi) \cdot \sum_{n=1}^{\infty} a_n \cos\left(n k_c z\right);$$

$$\frac{dP_r}{dz} = -\frac{eE_0 \cdot r}{2} \sum_{n=1}^{\infty} a_n \left(k_o \sin(\omega t + \varphi) \cdot \cos\left(n k_c z\right) - n k_c \cdot \cos(\omega t + \varphi) \cdot \sin\left(n k_c z\right)\right)$$

- And for traveling wave

$$E_z = E_0 \cdot \sum_{n=1}^{\infty} a_n \cos\left(\omega t + \varphi_n - \left(\kappa + n k_c\right) z\right)$$

$$\frac{dP_r}{dz} = -eE_0 \cdot r \sum_{n=1}^{\infty} a_n \frac{k_o + \left(\kappa + n k_c\right)}{2} \sin\left(\omega t + \varphi - \left(\kappa + n k_c\right) z\right)$$

Average focusing

- Analytical approximations are possible only simplest case with significant approximations, such as constant RF phase, small variations of the particle's momentum.
- One important case is of relatively weak focusing and short cells size - in other words very large k_c.
- This case required a little bit of preparation: so called motion in fast oscillating field

$$\frac{dP_r}{dz} = r \cdot \sum_n a_n \sin\left(nk_c + \varphi_n\right)$$

- Perturbation of second order gives

$$\left\langle \frac{dP_r}{dz} \right\rangle = \frac{r}{k_c^2} \cdot \sum_{n=1}^{\infty} \frac{a_n^2}{n^2}$$

- Putting coefficient from previous page gives us averaged focusing field in an axially symmetric linacs

How it is done

$$t=\frac{z}{\mathrm{v}};\Delta E=q\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos(\omega_0 t+\varphi)dz=q\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\left(\omega_0\frac{z}{\mathrm{v}}+\varphi\right)dz$$

$$\cos\left(\omega_0\frac{z}{\mathrm{v}}+\varphi\right)=\cos(\varphi)\cos\left(\omega_0\frac{z}{\mathrm{v}}\right)-\sin(\varphi)\sin\left(\omega_0\frac{z}{\mathrm{v}}\right)$$

$$\Delta E=q\cos(\varphi)\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\left(\omega_0\frac{z}{\mathrm{v}}\right)dz-q\sin(\varphi)\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\sin\left(\omega_0\frac{z}{\mathrm{v}}\right)dz$$

$$V_c=\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\left(\omega_0\frac{z}{\mathrm{v}}\right)dz;V_s=\int_{-\infty}^{\infty}\mathbf{E_o}(z)\sin\left(\omega_0\frac{z}{\mathrm{v}}\right)dz$$

$$V_{RF}=\sqrt{V_s^{\,2}+V_c^{\,2}};\tan(\varphi_o)=\frac{V_c}{V_s};$$

$$\Delta E=qV_{RF}\cos(\varphi+\varphi_o)$$

For particle moving with constant velocity all cavities are described by accelerating voltage and phase! Nothing else

Average z-motion

- If we can assume that energy gain in a cell is given by

$$\Delta E = eV \cdot \cos\varphi \qquad \Delta\delta = \frac{eV}{E_o} \cdot \cos\varphi$$

- and time of flight change

$$\Delta\tau = \frac{d}{(\gamma\beta)^2} \cdot \delta \Rightarrow \Delta t = -\frac{d}{c(\gamma\beta)^2} \cdot \delta \Rightarrow \Delta\varphi = \omega\,\Delta t = -\frac{k_o d}{2} \cdot \delta$$

- gives us cell-averaged equations of motions

$$\frac{d\varphi}{dz} = -\frac{k_o}{(\gamma\beta)^2} \cdot \delta; \; \frac{d\delta}{dz} = \frac{eE_c}{\gamma mc^2} \cos\varphi$$

- i.e. a well known pendulum equation: this is for the next class