PHY 564
Advanced Accelerator Physics
Lectures 19 & 20

Beam emittance(s)
and kinematic invariants.
Parameterization of particle's distribution.

Vladimir N. Litvinenko

CENTER for ACCELERATOR SCIENCE AND EDUCATION
Department of Physics & Astronomy, Stony Brook University

Emittance of the beam. For quite a while we were saying words such as emittance or phase space volume occupied by a beam without a rigorous definition what it is? While intuitively we can understand this concept as well as get grip of Liouville theorem and Poincaré invariants, we still lack accurate definition of emittace. To no surprise, there is a number of definitions used for the beam emittances: RMS, core-, 95%, etc… Having something very rigorous would help you to navigate the topic without being lost…

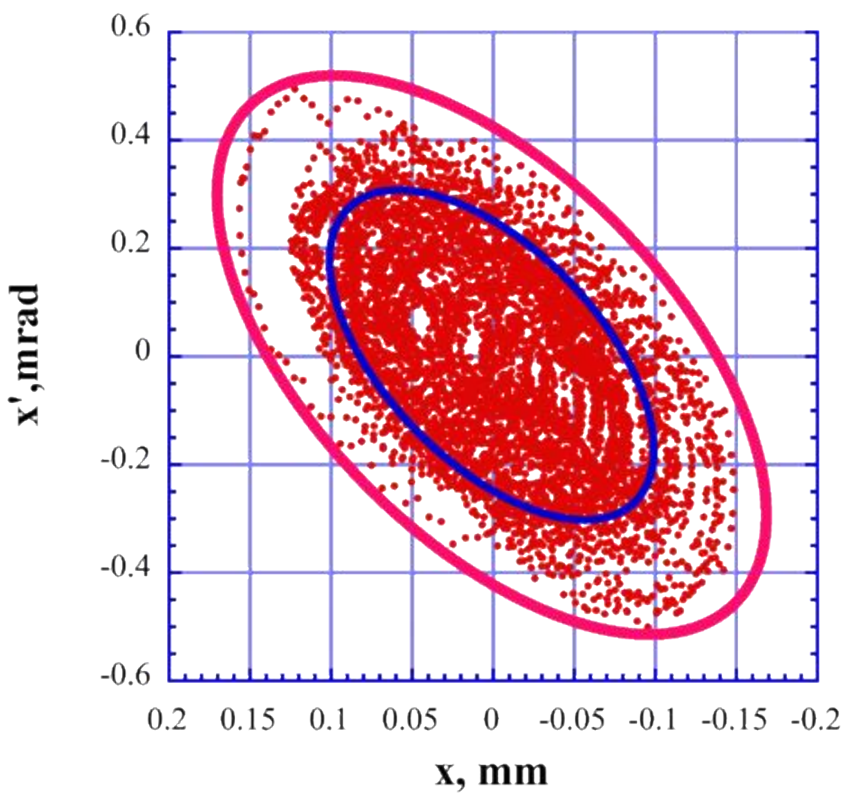
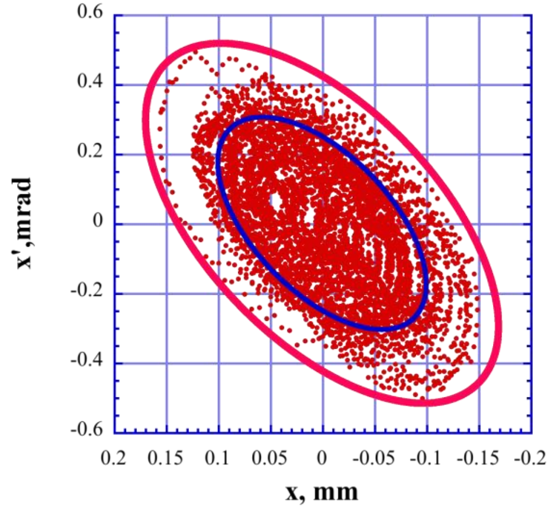


Fig. 1. 1D phase space distribution of particles with RMS-emittance ellipse and one containing all particles found in the plot.



Let's start from uncoupled 1D case. You can find RMS emittance definition in any text-book expressed via determinant of Σ matrix

$$\varepsilon_i^2 = \det\Sigma = \langle q_i^2\rangle\langle p_i^2\rangle - \langle q_i p_i\rangle^2 \, , i = 1,2,...,n,$$

$$\Sigma = \langle X \cdot X^T \rangle = \left\langle \begin{bmatrix} x^2 & xp_x \\ xp_x & p_x^2 \end{bmatrix} \right\rangle ; X^f = \mathbf{M} X^i \rightarrow \tag{1}$$

$$\Sigma^f = \mathbf{M}\Sigma^i\mathbf{M}^T \rightarrow \det\Sigma^f = (\det\mathbf{M})^2 \det\Sigma^i ; \ \varepsilon^{2f} = \varepsilon^{2i} = inv,$$

which is invariant of 1D motion liner Hamiltonian motion - we used det**M**=1.

In order to get to coupled case (e.g. a multi-dimensional linear Hamiltonian motion), let's start from equilibrium distribution in a storage ring. First, we should notice that in a stable ring without damping and diffusion, actions of eigen modes are preserved while phases, in general, are not. For example, nonlinearity of magnetic fields and RF curvature generate tune spread depending on 3 actions. It will spread phases randomly for all three oscillators. In this case one can assume that distribution functions depends only on action:

$$f = f\left(I_1, I_2, I_3\right) = f\left(\frac{\left|Y_1^T SX\right|^2}{2}, \frac{\left|Y_2^T SX\right|^2}{2}, \frac{\left|Y_3^T SX\right|^2}{2}\right). \tag{2}$$

It is even simpler in the case of stationary distribution established by synchrotron radiation damping and quantum fluctuations:

$$f(I,\varphi) = \prod_{k=1}^{3} \frac{1}{2\pi\varepsilon_k} \exp\left[-\frac{I_k}{\varepsilon_k}\right] = \left(\prod_{k=1}^{3} \frac{1}{2\pi\varepsilon_k}\right).\exp\left[-\sum_{k=1}^{3} \frac{I_k}{\varepsilon_k}\right]; \tag{3}$$

with natural substitutions

$$X = \frac{1}{2}\sum_{k=1}^{3}\left(\tilde{a}_k Y_k + \tilde{a}_k^* Y_k^*\right) \rightarrow i\tilde{a}_k = Y_k^{T*}\mathbf{S}X;\ \left|a_k\right|^2 = \left|Y_k^T \mathbf{S}X\right|^2;$$

$$f(X) = \prod_{k=1}^{3} \frac{1}{2\pi\varepsilon_k} \cdot \exp\left[-\sum_{k=1}^{3} \frac{\left|Y_k^T \mathbf{S}X\right|^2}{2\varepsilon_k}\right]; \tag{4}$$

The term in the exponent

$$q(X) = \sum_{k=1}^{3} \frac{\left|Y_k^T \mathbf{S}X\right|^2}{2\varepsilon_k} = \sum_{i,j=1}^{2n} q_{ij} x_i x_j;\ q_{ij} = q_{ji} \tag{5}$$

is a positively defined quadratic form of X components.

Now we should try to find the matrix of quadratic form and we will start from obvious complex form of (4)

$$\sum_{k=1}^{3}\frac{|a_k|^2}{\varepsilon_k}=\frac{1}{2}A^{*T}\Xi^{-1}A=\frac{1}{2}A^{T}\Xi^{-1}A^{*};$$

$$A^T=\left(..,a_k,a_k^{*},...\right);\Xi^{-1}=\begin{bmatrix} \cdots & [0] & [0] \\ [0] & \begin{bmatrix} \varepsilon_k^{-1} & 0 \\ 0 & \varepsilon_k^{-1} \end{bmatrix} & [0] \\ [0] & [0] & \cdots \end{bmatrix}=\begin{bmatrix} \cdots & 0 & 0 \\ 0 & \varepsilon_k^{-1}\mathbf{I} & 0 \\ 0 & 0 & \cdots \end{bmatrix};\mathbf{I}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad (6)$$

$$\mathbf{S}=\begin{bmatrix} \cdots & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \cdots \end{bmatrix};\sigma=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};\Rightarrow \mathbf{S}\Xi=\Xi\mathbf{S}$$

with detailed structure

$$X=\frac{1}{2}\mathbf{U}A\Rightarrow A=2\mathbf{U}^{-1}X; A^{*}=2\mathbf{U}^{*-1}X;\ \mathbf{U}^T\mathbf{S}\mathbf{U}=-2i\mathbf{S};$$

$$2\mathbf{U}^{-1}=-i\mathbf{S}\mathbf{U}^T\mathbf{S}; 2\mathbf{U}^{*-1}=i\mathbf{S}\mathbf{U}^{*T}\mathbf{S};$$

$$q(X)=\frac{1}{2}A^{*T}\Xi^{-1}A=2X^T\left[\left(\mathbf{U}^{*-1}\right)^T\Xi^{-1}\mathbf{U}^{-1}\right]X=\frac{1}{2}X^T\Omega X \quad (7)$$

$$\Omega=4\left[\left(\mathbf{U}^{*-1}\right)^T\Xi^{-1}\mathbf{U}^{-1}\right]=\mathbf{S}\mathbf{U}^{*}\mathbf{S}\Xi^{-1}\mathbf{S}\mathbf{U}^T\mathbf{S}=-\mathbf{S}\mathbf{U}^{*}\Xi^{-1}\mathbf{U}\mathbf{S}=\mathbf{V}^{*}\Xi^{-1}\mathbf{V}^T;$$

$$\mathbf{V}=\mathbf{S}\mathbf{U}=\left[...\mathbf{S}Y_k,\mathbf{S}Y^{*}{}_k...\right]$$

While it is OK to have this in complex form, it would be very nice to express it in real notations. Using the fact that X is real:

$$Y_k = R_k + iQ_k;$$

$$\left|X^T \mathbf{S} Y_k\right|^2 = \left|X^T \mathbf{S}\left(R_k + iQ_k\right)\right|^2 = \left|X^T \mathbf{S} R_k\right|^2 + \left|X^T \mathbf{S} Q_k\right|^2; \mathbf{O} = \left[...R_k, Q_k..\right]$$

$$X = \sum_{k=1}^{n} \left|a_k\right| \left(R_k \cos\psi - Q_k \sin\psi\right) = \mathbf{O}\tilde{B};\ \tilde{B}^T = \left[...\left|a_k\right|\cos\psi, -\left|a_k\right|\sin\psi\right];$$

$$\tilde{B} = \mathbf{O}^{-1} X;\ ;\ \tilde{B}^T \Xi^{-1} \tilde{B} = \sum_{k=1}^{n} \frac{\left|a_k\right|^2}{\varepsilon_k}\left(\cos^2\psi + \sin^2\psi\right) = \sum_{k=1}^{n} \frac{\left|a_k\right|^2}{\varepsilon_k}; \quad , \tag{8}$$

$$\sum_{k=1}^{n} \frac{\left|a_k\right|^2}{\varepsilon_k} = X^T \cdot \left(\mathbf{O}^T\right)^{-1} \Xi^{-1} \mathbf{O}^{-1} \cdot X$$

$$\mathbf{O}^T \mathbf{S} \mathbf{O} = \mathbf{S} \rightarrow \mathbf{O}^{-1} = -\mathbf{S} \mathbf{O}^T \mathbf{S};\ \left(\mathbf{O}^T\right)^{-1} = -\mathbf{S} \mathbf{O} \mathbf{S}$$

we get desirable symmetric form of the stationary distribution:

$$f(X) = \prod_{k=1}^{3} \frac{1}{2\pi\varepsilon_k} \exp\left[-\frac{X^T \cdot \left(\mathbf{O}^T\right)^{-1} \Xi^{-1} \mathbf{O}^{-1} \cdot X}{2}\right]., \tag{9}$$

Look at 1D case for simplicity:

$$\Xi^{-1} = \varepsilon^{-1}\mathbf{I}; \quad \to \mathbf{O}^{-1T}\Xi^{-1}\mathbf{O}^{-1} = \varepsilon^{-1}\mathbf{O}^{-1T}\mathbf{O}^{-1} = -\varepsilon^{-1}\mathbf{S}\mathbf{O}\mathbf{O}^T\mathbf{S}$$

$$\mathbf{O} = \begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & 1/\mathrm{w} \end{bmatrix}; -\mathbf{S}\mathbf{O} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & 1/\mathrm{w} \end{bmatrix} = \begin{bmatrix} -\mathrm{w}' & -1/\mathrm{w} \\ \mathrm{w} & 0 \end{bmatrix};$$

$$\mathbf{O}^T\mathbf{S} = \begin{bmatrix} \mathrm{w} & \mathrm{w}' \\ 0 & 1/\mathrm{w} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\mathrm{w}' & \mathrm{w} \\ -1/\mathrm{w} & 0 \end{bmatrix};$$

$$-\mathbf{S}\mathbf{O}\mathbf{O}^T\mathbf{S} = \begin{bmatrix} -\mathrm{w}' & -1/\mathrm{w} \\ \mathrm{w} & 0 \end{bmatrix}\begin{bmatrix} -\mathrm{w}' & \mathrm{w} \\ -1/\mathrm{w} & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{w}'^2 + 1/\mathrm{w}^2 & -\mathrm{w}'\mathrm{w} \\ -\mathrm{w}'\mathrm{w} & \mathrm{w}^2 \end{bmatrix}, \tag{10}$$

$$\to x^2\frac{1+\mathrm{w}'^2\mathrm{w}^2}{\mathrm{w}^2} - 2\mathrm{w}'\mathrm{w}\cdot xx' + x'^2\mathrm{w}^2; \alpha = -\mathrm{w}'\mathrm{w}; \beta = \frac{1+\mathrm{w}'^2\mathrm{w}^2}{\mathrm{w}^2}$$

$$f(x,x') = \frac{1}{2\pi\varepsilon}\exp\left[-\frac{x^2 + (\alpha x + \beta x')^2}{2\beta\varepsilon}\right];$$

After a simple manipulations – which we forgo because we will do this for an arbitrary dimensionality- one can easily prove that

$$\det\Sigma = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = \frac{\langle x^2 + (\alpha x + \beta x')^2\rangle}{\beta} = \varepsilon^2$$

determinant of Σ matrix indeed an RMS emittance for Gaussian distributions.

While this is rather "convenient" to stop here – as in many accelerator text-books – for advanced AP course we should expand our studies to find **general moment invariants for linear Hamiltonian systems***.

$$X_f = \mathbf{M}X_i;\ \ \mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}; \tag{11}$$

with f(X) distribution function and define moments as

$$\left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle = \int f(X)x_{i_1}x_{i_2}\cdots x_{i_k}\,dX \leftrightarrow \frac{1}{N}\sum_{i=1}^{N} x_{i_1}x_{i_2}\cdots x_{i_k}; \tag{12}$$

with

$$\begin{aligned}\left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle^i &= \int f^i(X)x_{i_1}x_{i_2}\cdots x_{i_k}\,dX \leftrightarrow \frac{1}{N}\sum_{i=1}^{N}\left(x_{i_1}x_{i_2}\cdots x_{i_k}\right)^i\\ \left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle^f &= \int f^f(X)x_{i_1}x_{i_2}\cdots x_{i_k}\,dX \leftrightarrow \frac{1}{N}\sum_{i=1}^{N}\left(x_{i_1}x_{i_2}\cdots x_{i_k}\right)^f;\end{aligned} \tag{13}$$

Liouville's theorem requires that phase space density is preserved:

$$f^f(X^f) = f^i(X^i) \Leftrightarrow f^f(X) = f^i(\mathbf{M}^{-1}X); \tag{14}$$

and

$$\begin{aligned}\left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle^f &= \int f^i(\mathbf{M}^{-1}X)x_{i_1}x_{i_2}\cdots x_{i_k}\,dX;\\ X = \mathbf{M}\breve{X} \rightarrow \left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle^f &= \int f^i(\breve{X})(\mathbf{M}\breve{x})_{i_1}(\mathbf{M}\breve{x})_{i_2}\cdots(\mathbf{M}\breve{x})_{i_k}\,d\breve{X}\\ \left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle^f &= m_{i_1j_1}m_{i_2j_2}\cdots m_{i_kj_k}\left\langle x_{j_1}x_{j_2}\cdots x_{j_k}\right\rangle^i\end{aligned} \tag{15}$$

Dragt suggest following compact form

$$\begin{aligned}\left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle^f &= \left(\overset{k}{\otimes}\mathbf{M}\right)\left\langle x_{j_1}x_{j_2}\cdots x_{j_k}\right\rangle^i\\ \overset{k}{\otimes}\mathbf{M} &\equiv \mathbf{M}\otimes\mathbf{M}\otimes\cdots\otimes\mathbf{M}\end{aligned} \tag{15}$$

We identified the k-th order moments which are elements of *k*-th order tensor:

$$\begin{aligned}\mathbf{X}^{(k)} \Leftrightarrow \mathbf{X}^{(k)}{}_{i_1i_2\ldots i_k} &= \left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle\\ \mathbf{X}^{(k)f} &= \left(\overset{k}{\otimes}\mathbf{M}\right)\mathbf{X}^{(k)i}\end{aligned} \tag{16}$$

** for most of derivations we are following Alex J Dragt et all, Phys. Rev. A, **45**, 2572, 1992.*

Kinematic invariants. *A. General concepts*

Motivation – as we had seen in uncoupled or 1D case (eq,(1))

$$\varepsilon_i^2 = \langle q_i^2\rangle\langle p_i^2\rangle - \langle q_i p_i\rangle^2 = inv,\ i = 1,2,\ldots,n \tag{17}$$

determinant of Σ matrix is invariant of 1D motion. Symplecticity conditions for transport matrix 1D case gives one invariant – its unit determinant.

An n-dimensional linear Hamiltonian has symplectic transport matrix with n(2n-1) conditions on its coefficients and one expect to have n(2n-1) independent (but in general case not necessarily all non-zero!) invariants. The corresponding invariant of motion is 1D emittance defined as (1) or (17).

In 2D case symplecticity of transport matrix gives 6 condition and we should expect 6 invariants of motion. In 3D case we have 15 conditions and should expect corresponding number of invariants.

For decoupled motion in 3D case we would have 3 conserved emittances as three invariants. All other invariants, which could be non-zero for coupled motion, are simply zeros in this case – and can be ignored. This is why accelerator physicists are trying as much as possible to stay away from coupling…

Let's now look for generalized invariants of linear Hamiltonian system. Suppose we have a kinematic invariant function:

$$I\left(\left(\overset{k}{\otimes}\mathbf{M}\right)\mathbf{X}^{(k)}\right) = I\left(\mathbf{X}^{(k)}\right)\quad \forall\, \mathbf{M} \in \left\{\mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}\right\} \tag{18}$$

Let's define equivalence classes of k-th order moments: two moments are equivalent if and only if they are connected by symplectic transformation:

$$\mathbf{X}^{(k)\prime} \sim \mathbf{X}^{(k)} \Leftrightarrow \mathbf{X}^{(k)\prime} = \left(\overset{k}{\otimes}\mathbf{M}\right)\mathbf{X}^{(k)}\ \&\ \mathbf{M} \in \mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S} \tag{19}$$

Define set of equivalent k-th order moments

$$\left[\mathbf{X}^{(k)}\right] \Leftrightarrow \mathbf{X}^{(k)\prime} \in \left[\mathbf{X}^{(k)}\right] \rightarrow \mathbf{X}^{(k)\prime} \sim \mathbf{X}^{(k)} \tag{20}$$

From (20) we conclude that the kinematic invariant function is a class function:

$$I\left(\mathbf{X}^{(k)\prime}\right) = I\left(\mathbf{X}^{(k)}\right)\ if\ \ \mathbf{X}^{(k)\prime} \sim \mathbf{X}^{(k)} \rightarrow I = I\left(\left[\mathbf{X}^{(k)\prime}\right]\right) \tag{21}$$

B. Quadratic moment invariants

Consider a quantity of

$$I_2^{(n)}\left(\left[\mathbf{X}^{(2)}\right]\right) = tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^n\right];\ \mathbf{X}_{ij}^{(2)} = \left\langle x_i x_j \right\rangle. \tag{21}$$

Let's show that $I_2^{(n)}$ is indeed a kinematic invariant:

$$\begin{gathered}(\mathbf{M}\otimes\mathbf{M})\,\mathbf{X}^{(2)} = \mathbf{M}\mathbf{X}^{(2)}\mathbf{M}^T;\ tr[\mathbf{ABC}] = tr[\mathbf{BCA}];\\ \mathbf{X}^{(2)} = (\mathbf{M}\otimes\mathbf{M})\,\mathbf{X}^{(2)} \to I_2^{(n)}\left(\left[(\mathbf{M}\otimes\mathbf{M})\,\mathbf{X}^{(2)}\right]\right) = tr\left[\left(\mathbf{M}\mathbf{X}^{(2)}\mathbf{M}^T\mathbf{S}\right)^n\right] = \\ tr\left[\left(\mathbf{X}^{(2)}\mathbf{M}^T\mathbf{S}\mathbf{M}\right)^n\right] = tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^n\right] = I_2^{(n)}\left(\left[\ \mathbf{X}^{(2)}\right]\right)\#\end{gathered} \tag{22}$$

Hence, there is infinite number of quadratic moment invariants, but all odd order invariants are simple zeros: odd number invariant contains odd number of **S**, which is asymmetric. In contrast, $\mathbf{X}^{(2)}$ is symmetric by definition. Hence:

$$tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^n\right] = tr\left[\left(\mathbf{X}^{(2)}\right)^n\mathbf{S}^n\right] = tr\left[\left(\mathbf{X}^{(2)}\right)^n\mathbf{S}^n\right]^T = (-1)^n\, tr\left[\left(\mathbf{X}^{(2)}\right)^n\mathbf{S}^n\right] \tag{23}$$

We can calculate $I_2^{(2)}$ directly:

$$\begin{gathered}n = 2 \to tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^2\right] = \\ -2\left(\sum_{i=1}^{3}\left(\left\langle q_i^2\right\rangle\left\langle p_i^2\right\rangle - \left\langle q_i p_i\right\rangle^2\right) + 2\sum_{i\neq j}\left(\left\langle q_i q_j\right\rangle\left\langle p_i p_j\right\rangle - \left\langle q_i p_j\right\rangle\left\langle p_i q_j\right\rangle\right)\right)\end{gathered} \tag{24}$$

It is clearly generalization of the 1D emittance definition, but it is not eigen emittances! It just a single number out of 3! It is definitely possible to write expressions for $I_2^{(4)}$ and $I_2^{(6)}$: the first will cover one page, the second quite a few!

Much more natural step is to determine number of independent invariants is to study properties of the form. Let's classify $\mathbf{X}^{(2)}$ according to its equivalency class:

$$\Sigma \overset{def}{\equiv} \mathbf{X}^{(2)};\ \ \mathbf{X}^{(2)\prime} = \mathbf{M}\mathbf{X}^{(2)}\mathbf{M}^T \Leftrightarrow \Sigma' = \mathbf{M}\Sigma\mathbf{M}^T \tag{25}$$

and we claim that any of them has its "normal" form.

Theorem: Given a set of quadratic forms $\Sigma = \mathbf{X}^{(2)}$ there exists a symplectic matrix transferring it to a special form with

$$\langle q_i q_i \rangle = \langle p_i p_i \rangle = \lambda_i > 0; \quad \langle x_i x_{j\neq i} \rangle = 0; \tag{25}$$

Our Σ matrix

$$\Sigma = \left[\Sigma_{ij}\right]; \; \Sigma_{ij} = \langle x_i x_j \rangle = \int x_i x_j f(X) dX; \tag{26}$$

is obviously symmetric matrix. We need to prove that it is also positively defined!

Lemma 1. Since $f(X)$ is density of particles in the phase space, it must be positively defined, e.g. $f(X) \geq 0$. Suppose that $f(X)$ is continuous at some point X_o and $f(X_o) > 0$ - then X_{ij} is positively defined. *Proof*: Since $f(X)$ is non-zero and continuous at X_o, there exists a ball

$$B_\varepsilon = \left\{X, |X - X_o| \leq \varepsilon\right\} \to f(X) \geq \delta > 0; \tag{27}$$

Let Z be any non-zero vector and

$$(Z, \Sigma Z) = \sum_{i,j} z_i \; \Sigma_{ij} z_j = \int \sum_{i,j} z_i x_i x_j z_j f(X) dX;$$

$$\sum_{i,j} z_i x_i x_j z_j = \left(\sum_i z_i x_i\right)^2 \to (Z, \Sigma Z) = \int \left(\sum_i z_i x_i\right)^2 f(X) dX; \tag{28}$$

$$f(X) \geq \delta \to (Z, \Sigma Z) \geq \delta \int \left(\sum_i z_i x_i\right)^2 dX > 0 \;\; \#.$$

Note: it is even easier for individual particles:

$$\Sigma = \left[\Sigma_{ij}\right]; \; \Sigma_{ij} = \langle x_i x_j \rangle = \frac{1}{N} \sum_{k=1}^{N} x_i^k x_j^k;$$

$$Z^T \Sigma Z = \frac{1}{N} \sum_{k=1}^{N} \sum_{i,j} z_i x_i^k x_j^k z_j = \frac{1}{N} \sum_{k=1}^{N} \left(\sum_i z_i x_i^k\right)^2 > 0 \#. \tag{29}$$

This is exactly definition of positively defined matrix. Proven#.

Lemma 2. Consider a Hamiltonian defined as:

$$H(Z) = \frac{1}{2} Z^T \Sigma Z = \frac{1}{2} X_{ij} z_i z_j; \tag{30}$$

which is positively definite. Hence, there exists c>0

$$H(Z) \square c\|Z\|^2 \tag{31}$$

Set $\|N\|$=1 and find minimum of $H(N)$ - since sphere $\|N\|$=1 is compact it has to have a minimum, which is greater than zero. The rest is just scaling:

$$Z = \|Z\| \cdot \frac{Z}{\|Z\|} \to H(Z) = \|Z\|^2 H\left(\frac{Z}{\|Z\|}\right); \left\|\frac{Z}{\|Z\|}\right\| = 1. \tag{32}$$

Consider two matrices

$$\begin{gathered} \mathbf{M}^{-1} = -\mathbf{S}\mathbf{M}^T\mathbf{S};\ \Sigma' = \mathbf{M}\Sigma\mathbf{M}^T;\ \ \mathbf{T} = \mathbf{S}\Sigma;\ \ \mathbf{T}' = \mathbf{S}\Sigma'; \\ \to \Sigma = -\mathbf{S}\mathbf{T};\ \mathbf{T}' = \mathbf{S}\mathbf{M}\Sigma\mathbf{M}^T = \left(\mathbf{M}^T\right)^{-1}\mathbf{T}\mathbf{M}^T \end{gathered} \tag{33}$$

e.g. matrices $\mathbf{T}$ and $\mathbf{T}'$ are similar and have the same eigen values. None of them equal zero, otherwise determinant of $\mathbf{T}$ is equal zero – but it is not possible since it equal to determinant of $\mathbf{\Sigma}$, which is positively defined with not zero determinant!

Lemma 3. Spectrum of is **T** purely imaginary pairs. Its eigen vectors form a basis and bring **T** to diagonal form, even in case of non-distinct eigen values.

Proof: Consider a Hamiltonian equations

$$Z' = \{Z, H(Z)\} = \mathbf{S}\Sigma \cdot Z = \mathbf{T} \cdot Z \to Z(s) = e^{\mathbf{T}s} Z(0) \tag{34}$$

Let's consider that matrix **T** can be brought to Jordan normal form

$$\mathbf{T} = \mathbf{A}\Lambda\mathbf{A}^{-1} \to e^{\mathbf{T}s} = \mathbf{A}e^{\Lambda s}\mathbf{A}^{-1} \tag{35}$$

The matrix $\exp(\Lambda s)$ has also normal form, which we studied in the Sylvester formulae class. The set of eigen values is a set of {λ, -λ} pairs. If one of the eigen values, λ_k , is not purely imaginary, than we should have either $\exp(\lambda_k s)$ or $\exp(-\lambda_k s)$ growing exponentially with

$$\|Z(s)\| = \|e^{\mathbf{T}s}\| \|Z(0)\| □ \; □ □ \; H(Z(s)) > c\|Z(s)\|^2 □ \; □ \tag{36}$$

which is in contradiction with the simple fact that energy is conserved for s-independent Hamiltonian:

$$H(Z(s)) = H(Z(0)) = const \tag{37}$$

Similarly, if some of Jordan block has dimension >1 (e.g. matrix **N** is not diagonal!), we would have an elements proportional to *s* at least in first power:

$$\|Z(s)\| \propto \|s^n e^{\lambda s}\| \|Z(0)\| \to \infty \Rightarrow H(Z(s)) > c\|Z(s)\|^2 \to \infty \tag{38}$$

which again contradicts energy conservation. Proven#.

$$AT^{N}A^{-1} = ATA^{-1}AT....TA^{-1} = \left(ATA^{-1}\right)^{n}$$

Thus, **T** can be diagonalized with all imaginary eigen values and linearly independent eigen vectors:

$$\{\lambda_i, -\lambda_i\}, i = 1,2...; \operatorname{Im}\lambda_i = \varepsilon_i > 0;$$
$$\mathbf{T}\cdot\Upsilon_i = \lambda_i \Upsilon_i; \mathbf{T}\cdot\Upsilon_i^* = \lambda_i^* \Upsilon_i^*; \tag{39}$$

Let's introduce a new angular inner product with matrix **K**:

$$\mathbf{K} = -i\mathbf{S};\ \ \mathbf{K}^\dagger = \left(\mathbf{K}^T\right)^* = \mathbf{K};\ \mathbf{S}^T = -\mathbf{S};$$
$$\langle A,B\rangle \equiv A^{*T}\mathbf{K}B;\ \langle A,B\rangle^* = -\langle A,B\rangle; \tag{40}$$
$$A^T\mathbf{S}B = \left(A^T\mathbf{S}B\right)^T = -B^T\mathbf{S}A \rightarrow \langle A,B\rangle^{T*} = \langle B,A\rangle;$$

and use it for eigen vectors

$$\mathbf{S}\Sigma\cdot\Upsilon_i = \lambda_i\Upsilon_i; \lambda_i = i\varepsilon_i \rightarrow \Sigma\cdot\Upsilon_i = -\lambda_i\mathbf{S}\Upsilon_i = \varepsilon_i\mathbf{K}\Upsilon_i$$
$$\mathbf{K}\Upsilon_i = \frac{1}{\varepsilon_i}\Sigma\cdot\Upsilon_i \rightarrow \Upsilon_i^\dagger\mathbf{K}\Upsilon_i = \langle\Upsilon_i,\Upsilon_i\rangle = \frac{1}{\varepsilon_i}\Upsilon_i^\dagger\Sigma\cdot\Upsilon_i; \varepsilon_i > 0. \tag{41}$$
$$\Upsilon_i = \mathcal{R}_i + i\mathcal{Q}_i;\ \ \Upsilon_i^\dagger\Sigma\cdot\Upsilon_i = \mathcal{R}_i^T\Sigma\cdot\mathcal{R}_i + \mathcal{Q}_i^T\Sigma\cdot\mathcal{Q}_i > 0$$
$$\langle\Upsilon_i,\Upsilon_i\rangle > 0$$

To prove that

$$\langle\Upsilon_i,\Upsilon_j\rangle = -i\Upsilon_i{}^{*T}\,\mathbf{S}\Upsilon_j = 0;\ \lambda_i \neq \lambda_j$$
$$\Sigma = -\mathbf{S}\mathbf{T} \rightarrow \Sigma^T - \Sigma = 0 \rightarrow \mathbf{T}^T\mathbf{S} + \mathbf{S}\mathbf{T} = 0; \tag{42}$$
$$\Upsilon_i^{*T}\left(\mathbf{T}^T\mathbf{S} + \mathbf{S}\mathbf{T}\right)\Upsilon_j = \left(\lambda_j - \lambda_i\right)\Upsilon_i{}^{*T}\,\mathbf{S}\Upsilon_j = 0\,\#$$

is easy. Similarly

$$\langle\Upsilon^*{}_i,\Upsilon_j\rangle = -i\Upsilon_i^T\mathbf{S}\Upsilon_j - 0;\ \mathbf{T}^T\mathbf{S} + \mathbf{S}\mathbf{T} = 0;$$
$$\Upsilon_i^T\left(\mathbf{T}^T\mathbf{S} + \mathbf{S}\mathbf{T}\right)\Upsilon_j = \left(\lambda_j + \lambda_i\right)\Upsilon_i{}^{*T}\,\mathbf{S}\Upsilon_j = 0\,\# \tag{43}$$

Lemma 5. Starting with vector Υ_i, one can construct vectors Υ_j such that

$$\begin{aligned}
&(1)\quad \mathbf{T}\Upsilon_j = \lambda_j \Upsilon_j = i\varepsilon_j \Upsilon_j,\ \ \varepsilon_j > 0; \\
&(2)\quad \left\langle \Upsilon_j, \Upsilon_k \right\rangle = 2\delta_{ik}; \\
&(3)\quad \left\langle \Upsilon_j, \Upsilon^*_k \right\rangle = 0.
\end{aligned} \tag{44-46}$$

Proof. In simplest case of distinct eigen values, it is coming from previous lemma plus simple normalization of the vectors.

The proof is for arbitrary case. Let's consider a degeneracy of λ_k of order h (in 3D case it is either 2 or 3). Since matrix is diagonalized, there is h linearly independent eigen vectors

$$\begin{gathered}
\mathbf{T}\Upsilon_k^m = \lambda_k \Upsilon_k^m = i\varepsilon_k \Upsilon_k^m,\ \varepsilon_k > 0 \to \left\langle \Upsilon_k^m, \Upsilon_k^m \right\rangle > 0;\ k = 1,...,h; \\
\breve{\Upsilon} = \sum_m \alpha_m \Upsilon_k^m \to \mathbf{T}\Upsilon = \lambda_k \breve{\Upsilon}.
\end{gathered} \tag{47}$$

Let's construct first eigen vector perpendicular to the rest using (seen to be called Gram-Schmidt) following procedure:

$$\begin{aligned}
&\breve{\Upsilon}_k^1 = \Upsilon_k^1; \\
&\breve{\Upsilon}_k^2 = \Upsilon_k^2 - \frac{\left\langle \breve{\Upsilon}_k^1, \Upsilon_k^2 \right\rangle}{\left\langle \breve{\Upsilon}_k^1, \breve{\Upsilon}_k^1 \right\rangle} \breve{\Upsilon}_k^1; \quad \left\langle \breve{\Upsilon}_k^2, \breve{\Upsilon}_k^1 \right\rangle = 0; \\
&\breve{\Upsilon}_k^3 = \Upsilon_k^3 - \sum_{m=1}^{2} \frac{\left\langle \breve{\Upsilon}_k^m, \Upsilon_k^3 \right\rangle}{\left\langle \breve{\Upsilon}_k^m, \breve{\Upsilon}_k^m \right\rangle} \breve{\Upsilon}_k^m;\ \left\langle \breve{\Upsilon}_k^l, \breve{\Upsilon}_k^3 \right\rangle = 0;\ l = 1,2 \\
&.... \\
&\breve{\Upsilon}_k^h = \Upsilon_k^h - \sum_{m=1}^{h-1} \frac{\left\langle \breve{\Upsilon}_k^m, \Upsilon_k^3 \right\rangle}{\left\langle \breve{\Upsilon}_k^m, \breve{\Upsilon}_k^m \right\rangle} \breve{\Upsilon}_k^m;\ \left\langle \breve{\Upsilon}_k^l, \breve{\Upsilon}_k^h \right\rangle = 0;\ l = 1,2,..,h-1
\end{aligned} \tag{48}$$

$$\hat{\Upsilon}_k^m = 2\frac{\breve{\Upsilon}_k^m}{\left\langle \breve{\Upsilon}_k^m, \breve{\Upsilon}_k^m \right\rangle} \to \left\langle \hat{\Upsilon}_k^m, \hat{\Upsilon}_k^m \right\rangle = 2 \tag{49}$$

which makes complete set of symplectically normalized and mutually orthogonal eigen vectors. We then simply remunerate these vectors in continuous sequence to drop and extra index. This ends the proof #.

These eigen vectors are definitely complex with non-zero real and imaginary part

$$\begin{gathered}\left\langle \Upsilon_k, \Upsilon_k \right\rangle = -i\Upsilon_k^{*T}\mathbf{S}\Upsilon_k = 2;\ \Upsilon_k = \mathcal{R}_k + i\mathcal{Q}_k;\ \Upsilon_k^{*T} \equiv \Upsilon_k^{\dagger}; \\ A^T\mathbf{S}A \equiv 0; \Rightarrow \mathcal{R}_k^T\mathbf{S}\mathcal{Q}_k \equiv \left(\mathcal{R}_k, \mathbf{S}\mathcal{Q}_k\right) = 1 = -\mathcal{Q}_k^T\mathbf{S}\mathcal{R}_k.\end{gathered} \tag{50}$$

– otherwise their symplectic product would be equal zero!

Lemma 6. One can construct symplectic matrix Θ from $\mathcal{Q}_k, \mathcal{R}_k$ that bring the matrix **Σ** to diagonal form with all positive identical pairs of diagonal elements

$$\Theta^T \boldsymbol{\Sigma} \Theta = diag\left\{\varepsilon_1, \varepsilon_1 \ldots \varepsilon_n, \varepsilon_n\right\} = \begin{bmatrix} \ldots & 0 & 0 \\ 0 & \begin{bmatrix} \varepsilon_i & 0 \\ 0 & \varepsilon_i \end{bmatrix} & 0 \\ 0 & 0 & \ldots \end{bmatrix} \tag{51}$$

Proof. Let's construct Θ in the following way:

$$\Theta = \left[\mathcal{R}_1, \mathcal{Q}_1 .. \mathcal{R}_k, \mathcal{Q}_k ... \mathcal{R}_n, \mathcal{Q}_n\right] \Rightarrow \Theta^T \mathbf{S} \Theta = \mathbf{S}. \tag{52}$$

From definition of matrix $\mathbf{T}$ we have:

$$\Sigma = -\mathbf{ST} \rightarrow \Sigma\Theta = -\mathbf{ST}\Theta;\ \mathbf{T}\Upsilon_k = i\varepsilon_k \Upsilon_k;$$

$$\mathcal{R}_k = \Upsilon_k + \Upsilon_k^*; i\mathcal{Q}_k = \Upsilon_k - \Upsilon_k^*;$$

$$\mathbf{T}\mathcal{R}_k = i\varepsilon_k\left(\Upsilon_k - \Upsilon_k^*\right) = -\varepsilon_k \mathcal{Q}_k; \mathbf{T}\mathcal{Q}_k = \varepsilon_k\left(\Upsilon_k + \Upsilon_k^*\right) = \varepsilon_k \mathcal{R}_k;$$

$$\Sigma\Theta = \left[\varepsilon_1 \mathbf{S}\mathcal{Q}_1, -\varepsilon_1 \mathbf{S}\mathcal{R}_1 .... \varepsilon_n \mathbf{S}\mathcal{Q}_n, -\varepsilon_n \mathbf{S}\mathcal{R}_n\right]$$

$$\Xi = \Theta^T \Sigma \Theta = \left[\mathcal{R}_1, \mathcal{Q}_1 ...., \mathcal{R}_n, \mathcal{Q}_n\right]^T \left[\varepsilon_1 \mathbf{S}\mathcal{Q}_1, -\varepsilon_1 \mathbf{S}\mathcal{R}_1 .... \varepsilon_n \mathbf{S}\mathcal{Q}_n, -\varepsilon_n \mathbf{S}\mathcal{R}_n\right]$$

$$\begin{bmatrix} ... \\ \mathcal{R}_k^T \\ \mathcal{Q}_k^T \\ ... \end{bmatrix} \left[.... \varepsilon_j \mathbf{S}\mathcal{Q}_j, -\varepsilon_j \mathbf{S}\mathcal{R}_j .... \right] = \left[\Xi_{kj}\right]$$

$$\Xi_{kj} = \varepsilon_j \begin{bmatrix} \mathcal{R}_k^T \mathbf{S}\mathcal{Q}_j & -\mathcal{R}_k^T \mathbf{S}\mathcal{R}_j \\ \mathcal{Q}_k^T \mathbf{S}\mathcal{Q}_j & -\mathcal{Q}_k^T \mathbf{S}\mathcal{R}_j \end{bmatrix} = \varepsilon_k \delta_{kj} = \varepsilon_k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Xi = \begin{bmatrix} ... & 0 & 0 \\ 0 & \begin{bmatrix} \varepsilon_k & 0 \\ 0 & \varepsilon_k \end{bmatrix} & 0 \\ 0 & 0 & ... \end{bmatrix}$$

This ends the proof.

It also identifies how one define emittances of arbitrary particles distribution of particles in 6D phase space as well as initial values for eigen vectors.

$$\Sigma = \left[\left\langle x_i x_j \right\rangle\right]; \Theta^T \mathbf{S} \Theta = \mathbf{S}$$

$$\Xi = \Theta^T \Sigma \Theta = \begin{bmatrix} \ldots & 0 & 0 & 0 \\ 0 & \varepsilon_k & 0 & 0 \\ 0 & 0 & \varepsilon_k & 0 \\ 0 & 0 & 0 & \ldots \end{bmatrix} \rightarrow \Sigma = \left(\Theta^T\right)^{-1} \Xi \Theta^{-1} = \mathbf{S}\Theta\mathbf{S}\Xi\mathbf{S}\Theta^T\mathbf{S}; \qquad (53)$$

$$\Sigma = \left(\Theta^T\right)^{-1} \Xi \Theta^{-1} \Leftrightarrow \Xi = \Theta^T \Sigma \Theta$$

Before going to connect this parameterization of this quadratic moment of particle's distribution with parameterization of particle's motion, let's give the answer on question of how many if kinematic invariants (21)

$$I_2^{(m)}\left(\left[\mathbf{X}^{(2)}\right]\right) = tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^m\right] \equiv tr\left[\left(\Sigma \cdot \mathbf{S}\right)^m\right] \qquad (54)$$

are functionally independent.

Remembering that all odd-order invariant are zeros, using diagonalized form (52) we can write non-zero even order invariants as:

$$\Sigma = \left(\Theta^T\right)^{-1} \Xi \Theta^{-1};\ \left(\Theta^T\right)^{-1} \mathbf{S}\Theta^{-1} = \mathbf{S};\ \mathbf{S}\Xi = \Xi\mathbf{S};\ \mathbf{S}^2 = -\mathbf{I} \Rightarrow$$

$$\left(\Sigma \cdot \mathbf{S}\right)^{2m} = \left(\Theta^T\right)^{-1} \Xi \underbrace{\Theta^{-1}\mathbf{S}\left(\Theta^T\right)^{-1}}_{\mathbf{S}} \Xi \Theta^{-1}\mathbf{S} \ldots \left(\Theta^T\right)^{-1} \Xi \underbrace{\Theta^{-1}\mathbf{S}\left(\Theta^T\right)^{-1}}_{\mathbf{S}} \Xi \Theta^{-1}\mathbf{S} =$$

$$(-1)^{m-1}\left(\Theta^T\right)^{-1} \Xi^{2n}\mathbf{S}\Theta^{-1}\mathbf{S};$$

$$tr\left[\left(\Sigma \cdot \mathbf{S}\right)^m\right] = (-1)^{m-1} tr\left[\left(\Theta^T\right)^{-1} \Xi^{2m}\mathbf{S}\Theta^{-1}\mathbf{S}\right] = (-1)^{m-1} tr\left[\Xi^{2m}\mathbf{S}\underbrace{\Theta^{-1}\mathbf{S}\left(\Theta^T\right)^{-1}}_{\mathbf{S}}\right] = \tag{55}$$

$$(-1)^{m-1} tr\left[\Xi^{2m}\mathbf{S}^2\right] = (-1)^m tr\left[\Xi^{2m}\right] = (-1)^m \sum_{k=1}^{n} \varepsilon_k^{2m};$$

$$I_2^{(m)}(\Sigma) = (-1)^m \sum_{k=1}^{n} \varepsilon_k^{2m}$$

e.g. for n-dimensional system all invariants are functions of n emittances, or to be exactly their squares ε_k^2. Thus only n out of infinite number of invariants (21) are functionally independents. For 3D case,

$$I_2^{(m)}(\Sigma) = (-1)^m \left(\varepsilon_1^{2m} + \varepsilon_2^{2m} + \varepsilon_3^{2m}\right) \tag{56}$$

and only three of them are functionally independent. For us it is easiest to use three values of eigen emittances.

$$\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3\,. \tag{57}$$

End of lecture 18

$$\Sigma = \left[\left\langle x_i x_j \right\rangle \right] = \left\langle X \circ X^T \right\rangle$$

We will discuss later invariants of higher order moments, but now let's focus on connecting matrix Θ and its components of eigen vectors (52) with parameterization we had used for storage ring. To make this connection, let's calculate second order moment, Σ-matrix, for Gaussian distribution of n oscillators:

$$f(X) = \prod_{k=1}^{n} \frac{1}{2\pi\varepsilon_k} \exp\left[-\frac{X^T\left[\left(\mathbf{O}^T\right)^{-1} \Xi^{-1} \mathbf{O}^{-1} \right] X}{2} \right]., \tag{58}$$

to show that

$$\Sigma = \left[\Sigma_{ij} \right];\ \Sigma_{ij} = \int x_i x_j f(X) dX; f(X) = \prod_{k=1}^{n} \frac{1}{2\pi\varepsilon_k} \exp\left[-\frac{X^T \cdot \left(\mathbf{O}^T\right)^{-1} \Xi^{-1} \mathbf{O}^{-1} \cdot X}{2} \right]$$

$$\Sigma = \int \left[\Sigma_{ij} \right] f(X) dX \tag{59}$$

$$\Rightarrow \Sigma = \mathbf{O} \Xi \mathbf{O}^T$$

It can be done by changing variable under the integral to n oscillators and taking the integral

$$outer - product:\ \left[X \cdot X^T \right]_{ij} = x_i x_j;$$

$$X = \mathbf{O}\breve{A}; \breve{A}^T = \left[..a_k \cos\varphi_k, -a_k \sin\varphi_k ... \right]$$

$$\mathbf{O} = \left[...R_k, Q_k ... \right];\ \ \mathbf{O}^T \mathbf{S} \mathbf{O} = \mathbf{S};\ \left[X \cdot X^T \right] = \mathbf{O} \left[\breve{A} \cdot \breve{A}^T \right] \mathbf{O}^T \tag{60}$$

$$\Sigma = \int \left[X \cdot X^T \right] f(X) dX = \mathbf{O} \left(\int \left[\breve{A} \cdot \breve{A}^T \right] f(X) dX \right) \mathbf{O}^T$$

$$f(X) = \left(\prod_{k=1}^{n} \frac{1}{2\pi\varepsilon_k} \right) \exp\left[-\frac{X^T \left(\mathbf{O}^T\right)^{-1} \Xi^{-1} \mathbf{O}^{-1} X}{2} \right] = \left(\prod_{k=1}^{n} \frac{1}{2\pi\varepsilon_k} \right) \exp\left[-\frac{\breve{A}^T \Xi^{-1} \breve{A}}{2} \right]$$

with trivial follow up by making Canonical (unit determinant) transformation of variables:

$$\Sigma = \int \left[X \cdot X^T \right] f(X) dX = \mathbf{O}\left(\int \left[\breve{A} \cdot \breve{A}^T \right] f(X) dX \right) \mathbf{O}^T$$

$$X \to \breve{A} \to \int ... dX = \int ... \det \mathbf{O}\, d\breve{A} = \int ... \det \mathbf{O}\, d\breve{A} = \int ... \prod_{k=1}^{n} d\varphi_k d\left(\frac{a_k^2}{2} \right);$$

$$\breve{A}^T \Xi^{-1} \breve{A} = \sum_k \frac{a_k^2}{\varepsilon_k} \left(\cos^2 \varphi_k + \sin^2 \varphi_k \right) = \sum_k \frac{a_k^2}{\varepsilon_k} \tag{61}$$

$$\exp\left[-\frac{\breve{A}^T \Xi^{-1} \breve{A}}{2} \right] = \prod_{k=1}^{n} \exp\left[-\frac{a_k^2}{2\varepsilon_k} \right]$$

$$\Sigma = \mathbf{O}\left(\int \left[\breve{A} \cdot \breve{A}^T \right] \prod_{k=1}^{n} \left(\frac{1}{2\pi\varepsilon_k} \exp\left[-\frac{a_k^2}{2\varepsilon_k} \right] d\varphi_k d\left(\frac{a_k^2}{2} \right) \right) \right) \mathbf{O}^T$$

Now it is good time to look onto the inner product $\left[\breve{A}\cdot\breve{A}^T\right]$ under the integral

$$
\begin{gathered}
\left[\breve{A}\cdot\breve{A}^T\right]_{ij}=\breve{A}_i\breve{A}_j;\ \breve{A}^T=\left[\ldots a_k\cos\varphi_k,-a_k\sin\varphi_k\ldots\right];\\
\left[\breve{A}\cdot\breve{A}^T\right]=\begin{bmatrix}\ldots\\ a_i\cos\varphi_i\\ -a_i\sin\varphi_i\\ \ldots\end{bmatrix}\bullet\left[\ldots a_j\cos\varphi_j-a_j\sin\varphi_j\ldots\right]\\
\breve{A}_{2i-1}\breve{A}_{2j-1}=a_ia_j\cos\varphi_i\cos\varphi_j;\breve{A}_{2i}\breve{A}_{2j}=a_ia_j\sin\varphi_i\sin\varphi_j;\\
\breve{A}_{2i-1}\breve{A}_{2j}=-a_ia_j\cos\varphi_i\sin\varphi_j;\ \breve{A}_{2i-1}\breve{A}_{2j};\breve{A}_{2i}\breve{A}_{2j-1}=-a_ia_j\sin\varphi_i\cos\varphi_j;\\
\int_0^{2\pi}\ldots\iint\breve{A}_i\breve{A}_j\prod_{k=1}^n d\varphi_k=(2\pi)^n\delta_{ij}\frac{a_m{}^2}{2};\ m=\mathrm{int}\left(\frac{i+1}{2}\right);\\
{\color{blue}\Sigma}=\mathbf{O}\left(\int\begin{bmatrix}\ldots&0&0&0\\ 0&\frac{a_k{}^2}{2}&&0\\ 0&0&\frac{a_k{}^2}{2}&0\\ 0&0&0&\ldots\end{bmatrix}\prod_{k=1}^3\left(\frac{1}{\varepsilon_k}\exp\left[-\frac{a_k^2}{2\varepsilon_k}\right]d\frac{a_k^2}{2}\right)\right)\mathbf{O}^T
\end{gathered}
\tag{62}
$$

with finish line as:

$$\Sigma = \mathbf{O}\begin{bmatrix} \dots & 0 & 0 & 0 \\ 0 & \alpha_i & & 0 \\ 0 & 0 & \alpha_i & 0 \\ 0 & 0 & 0 & \dots \end{bmatrix}\mathbf{O}^T; \quad \alpha_i = \int \frac{a_i^2}{2}\prod_{k=1}^{n}\left(\frac{1}{2\pi\varepsilon_k}\exp\left[-\frac{a_k^2}{2\varepsilon_k}\right]d\frac{a_k^2}{2}d\varphi_k\right)$$

$$\alpha_i = \int \frac{a_i^2}{2}\prod_{k=1}^{n}\left(\frac{1}{\varepsilon_k}\exp\left[-\frac{a_k^2}{2\varepsilon_k}\right]d\frac{a_k^2}{2}\right) = \varepsilon_i \int \xi_i \prod_{k=1}^{n}\left(\exp\left[-\xi_k\right]d\xi_k\right);\ \xi_k = \frac{a_k^2}{2\varepsilon_k} \in \{0,+\infty\};$$

$$\alpha_i = \varepsilon_i\left(\int_0^\infty \xi_i e^{-\xi_i}\,d\xi_i\right)\prod_{k\neq i}\int_0^\infty e^{-\xi_k}\,d\xi_k;\int_0^\infty e^{-\xi_k}\,d\xi_k = 1;\left(\int_0^\infty \xi_i e^{-\xi_i}\,d\xi_i\right) = 1;\alpha_i = \varepsilon_i \tag{63}$$

$$\Sigma = \mathbf{O}\begin{bmatrix} \dots & 0 & 0 & 0 \\ 0 & \varepsilon_i & & 0 \\ 0 & 0 & \varepsilon_i & 0 \\ 0 & 0 & 0 & \dots \end{bmatrix}\mathbf{O}^T = \mathbf{O\Xi O}^T\,\#.$$

and the same time we have from (53)

$$\Sigma = \left(\Theta^T\right)^{-1}\Xi\Theta^{-1} \tag{64}$$

Comparing (63) with (64) finally give us relations between eigen vectors and Σ matrix and our parameterization for periodic systems:

$$\Sigma = \left(\Theta^T\right)^{-1} \Xi \Theta^{-1} = \mathbf{O} \Xi \mathbf{O}^T \rightarrow$$
$$\mathbf{O} = \left(\Theta^T\right)^{-1} = -\mathbf{S}\Theta\mathbf{S}; \tag{65}$$

Hence, we closed the circle: Any arbitrary **Σ** matrix can be brough to diagonal form

$$\Sigma = \mathbf{O}\Xi\mathbf{O}^T; \; \Xi = \begin{bmatrix} \ldots & 0 & 0 & 0 \\ 0 & \varepsilon_i & & 0 \\ 0 & 0 & \varepsilon_i & 0 \\ 0 & 0 & 0 & \ldots \end{bmatrix} \tag{66}$$

with real symplectic matrix **O** which can be used as definition of eigen vectors for any beam distribution. At the same time, Gaussian distribution in a storage ring (or a periodic system) using parameterization (in real notations)

$$f(X) = \prod_{k=1}^{3} \frac{1}{2\pi\varepsilon_k} \exp\left[-\frac{X^T\left[\left(\mathbf{O}^T\right)^{-1}\Xi^{-1}\mathbf{O}^{-1}\right]X}{2}\right], \tag{67}$$

will generate **Σ** matrix in eq. (65). Hence, we established one to one correspodnece between various defintions of emittance.

One can change appearances of phase space projections into 1D phase space (frequently called emittance exchange between different degrees of freedom), but can not modify neither the values of the eigen emittances nor of their product. In contrast with eigen emittances, eigen vectors can be multiplied by a complex exponent without modifying the result (50) and (31)

$$\Upsilon_k \to \breve{\Upsilon}_k = \Upsilon_k e^{i\varphi_k} \Leftrightarrow \langle \breve{\Upsilon}_k, \breve{\Upsilon}_j \rangle = \langle \Upsilon_k, \Upsilon_j \rangle; \\ \langle \Upsilon_k, \Upsilon_j \rangle = -i\Upsilon_k^{*T}\mathbf{S}\Upsilon = 2\Upsilon_k^{*T}\mathbf{S}\Upsilon = 2i. \tag{68}$$

which is essentially flexibility of separating oscillation phase from phase of the oscillator. This flexibility includes multiplication by *-1,* e.g. changing sign. It can be also seen from the fact that both **O** and Θ appear in binary pairs in Σ-matrix and Gaussian distribution. It means that changing sign does not change neither the matrix of the distribution. For example, we can select sign of any desirable element in $\mathcal{R}$.

We are now fully equipped to connect set of eigen vectors (59) with parameterization of linearized motion at any given location s_o in our accelerator:

$$\mathbf{O} = [..R_k, Q_k..] = (\Theta^T)^{-1} = -\mathbf{S}\Theta\mathbf{S} = -[.\mathbf{S}\mathcal{R}_k, \mathbf{S}\mathcal{Q}_k..]\mathbf{S} = [.\mathbf{S}\mathcal{Q}_k, -\mathbf{S}\mathcal{R}_k..]; \\ Y_k = R_k + iQ_k = \mathbf{S}\mathcal{Q}_k - i\mathbf{S}\mathcal{R}_k = -i\mathbf{S}\Upsilon_k; \; Y_k^* = i\mathbf{S}\Upsilon_k^*; \\ Y_k^{*T}\mathbf{S}Y_k = \Upsilon^{*T}{}_k\mathbf{S}^T\mathbf{S}\mathbf{S}\Upsilon_k == \Upsilon^{*T}{}_k\mathbf{S}\Upsilon_k = 2i; \\ Y_k = -i\mathbf{S}\Upsilon_k \;\; \# \tag{69}$$

When the parameterization eigen vectors are defined at s_o, we can propagate them according to already established rules using transport matrix:

$$\tilde{Y}_k(s) = \mathbf{M}(s_o|s)Y_k(s_o). \tag{70}$$

Making a dedicated transport channel to have a specific form (again, defined with flexibility of phase advance (59)), for example to fit it with one in a periodic lattice, injection into a storage ring or for a special device (a wiggler or interaction region for beam collisions), is called matching. Traditionally, when the energy if the beam is fixed, it is reduced to matching transverse eigen vectors using magnetic elements – e.g. 2D or 4D phase space problem. But it also involve matching transverse dispersion functions and bunch length.

But in modern accelerators, such as energy recovery linacs or sophisticated beam manipulation system with emittance exchange, matching can involve all six components in the phase space.

How to calculate the Σ matrix and connect it with parameterization

In practice particle's displacements are taken from the position of reference particle (orbit) and if beam as a whole is displaced

$$\langle x_i \rangle \neq 0 \tag{71}$$

its center will execute oscillation (or at least collective motion) in the beam-line. If the position in the phase of the beam centroid can be corrected (or used as the reference!), we can remove the average displacement and use more traditional definition of the correlation matrix:

$$\begin{gathered}\Sigma = \left[\Sigma_{ij}\right];\ \langle x_i \rangle = \sum_{k=1}^{N} x_i^k; \\ \Sigma_{ij} = \left\langle \left(x_i - \langle x_i \rangle\right)\left(x_j - \langle x_j \rangle\right)\right\rangle = \frac{1}{N}\sum_{k=1}^{N}\left(x_i^k - \langle x_i \rangle\right)\left(x_j^k - \langle x_j \rangle\right).\end{gathered} \tag{72}$$

with the rest of treatment being identical to the above. To find a set of eigen vectors suitable to describe the actual 6D beam distribution we need to find eigen values of supporting matrix $\mathbf{T} = \mathbf{S}\Sigma$ by solving cubic equation on squares of its eigen values (they come in (λ,-λ) pairs):

$$\begin{gathered}\det\left(\mathbf{S}\Sigma - \lambda\mathbf{I}\right) \equiv \det\left(\mathbf{S}\Sigma - \lambda\mathbf{I}\right)^T = \det\left(\Sigma^T\mathbf{S}^T - \lambda\mathbf{I}\right) = (-1)^{2n}\det\left(\Sigma\mathbf{S} + \lambda\mathbf{I}\right) = \det\left(\mathbf{S}\Sigma + \lambda\mathbf{I}\right) \equiv \\ \det\left[\mathbf{T} - \lambda\mathbf{I}\right] = \det\left[\mathbf{S}\Sigma + \lambda\mathbf{I}\right] = \prod_{k=1}^{3}\left(\lambda - i\varepsilon_k\right)\left(\lambda + i\varepsilon_k\right) = \prod_{k=1}^{3}\left(\lambda^2 + \varepsilon_k^{\ 2}\right);\ \varepsilon_k > 0.\end{gathered} \tag{73}$$

The we need to find full set of eigen vectors of matrix $\mathbf{T}$ by picking them from columns of following matrices (beware that some of them can be zero!):

$$\left(\mathbf{T} - i\varepsilon_k\mathbf{I}\right)\prod_{\kappa_j \neq \kappa_k}\left(\mathbf{T}^2 - \varepsilon_j^2\mathbf{I}\right) \tag{74}$$

and follow the Gram-Schmidt procedure to find the set of symplectically orthogonal eigen vectors. These eigen vectors will give the parameterization of the beam (69) and ε_k will give three eigen emittance of the beam. We proved that these eigen emittance cannot be changed in any linear Hamiltonian transport (even though can be spoiled in non-linear one!) and are invariants of motion. Their product of eigen emittance is called 3D emittance of the beam

$$\varepsilon_{3D} = \varepsilon_1\varepsilon_2\varepsilon_3 = \sqrt{\det\Sigma} \tag{75}$$

When all three eigen emittance are distinct, the set of eigen vectors (69) are well defined – e.g. we can not build a combination of two or eigen vectors to be an eigen vector. Only their phases are flexible – it is simply translates into initial phases of oscillations.

Situation is different if two or more eigen emittances are equal. In this case we had follow the Gram-Schmidt procedure and select a different non-zero as our first eigen vector and follow the procedure to generate alternative set of symplectically orthogonal set:

$$
\begin{aligned}
&\breve{\Upsilon}_k^1 = \sum_{j=1}^{h} \alpha_n \Upsilon_k^j;\\
&\breve{\Upsilon}_k^2 = \Upsilon_k^2 - \frac{\langle \breve{\Upsilon}_k^1, \Upsilon_k^2 \rangle}{\langle \breve{\Upsilon}_k^1, \breve{\Upsilon}_k^1 \rangle} \breve{\Upsilon}_k^1; \quad \langle \breve{\Upsilon}_k^2, \breve{\Upsilon}_k^1 \rangle = 0;\\
&\breve{\Upsilon}_k^3 = \Upsilon_k^3 - \sum_{m=1}^{2} \frac{\langle \breve{\Upsilon}_k^m, \Upsilon_k^3 \rangle}{\langle \breve{\Upsilon}_k^m, \Upsilon_k^m \rangle} \breve{\Upsilon}_k^m; \ \langle \breve{\Upsilon}_k^l, \breve{\Upsilon}_k^3 \rangle = 0; \, l = 1,2\\
&....\\
&\breve{\Upsilon}_k^h = \Upsilon_k^h - \sum_{m=1}^{h-1} \frac{\langle \breve{\Upsilon}_k^m, \Upsilon_k^3 \rangle}{\langle \breve{\Upsilon}_k^m, \Upsilon_k^m \rangle} \breve{\Upsilon}_k^m; \ \langle \breve{\Upsilon}_k^l, \breve{\Upsilon}_k^h \rangle = 0; \, l = 1,2,..,h-1
\end{aligned}
\tag{76}
$$

It means that in this case we have an additional flexibility of choosing the beam-defined eigen vectors.

Let's consider, for concreteness, 3D coupled beam with two equal eigen emittances, ε_1, *h=2*:

$$
\breve{\Upsilon}_1^1 = \Upsilon_1^1 + \alpha \Upsilon_1^2 \, ; \ \breve{\Upsilon}_1^2 = \Upsilon_1^2 - \frac{\langle \breve{\Upsilon}_1^1, \Upsilon_1^2 \rangle}{\langle \breve{\Upsilon}_1^1, \breve{\Upsilon}_1^1 \rangle} \breve{\Upsilon}_1^1; \quad \langle \breve{\Upsilon}_1^2, \breve{\Upsilon}_1^1 \rangle = 0; \tag{77}
$$

where we can make at least one component of $\breve{\Upsilon}_k^1$ equal zero. In the case of maximum degeneration of *h=3*, we can zero two components of $\breve{\Upsilon}_1$:

$$
\breve{\Upsilon}_1 = \Upsilon_1 + \alpha \Upsilon_2 + \beta \Upsilon_3; \ \breve{\Upsilon}_2 = \Upsilon_2 - \frac{\langle \breve{\Upsilon}_1, \Upsilon_2 \rangle}{\langle \breve{\Upsilon}_1, \breve{\Upsilon}_1 \rangle} \breve{\Upsilon}_1; \breve{\Upsilon}_3 = \Upsilon_3 - \frac{\langle \breve{\Upsilon}_1, \Upsilon_3 \rangle}{\langle \breve{\Upsilon}_1, \breve{\Upsilon}_1 \rangle} \breve{\Upsilon}_1 - \frac{\langle \breve{\Upsilon}_2, \Upsilon_3 \rangle}{\langle \breve{\Upsilon}_2, \breve{\Upsilon}_2 \rangle} \breve{\Upsilon}_2; \tag{78}
$$

Let's look again at what we will get as the result of beam-based parameterization.

1D case it is rather simple for selecting phase in (59) to have coordinate component in Q:

$$Y=\begin{bmatrix} \mathrm{w} \\ \mathrm{w}'+i/\mathrm{w} \end{bmatrix}; R=\begin{bmatrix} \mathrm{w} \\ \mathrm{w}' \end{bmatrix}; Q=\begin{bmatrix} 0 \\ 1/\mathrm{w} \end{bmatrix}; \mathbf{U}=\begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & 1/\mathrm{w} \end{bmatrix}. \tag{79}$$

In case of higher dimensions (two and above) this choice is not obvious, since any of eigen vector component in general can be zero. The only one invariant of motion is beam emittance.

In 2D case for x-y or x-τcoupling in general has 6 invariants:

$$Y_k(s)=\begin{bmatrix} \mathrm{w}_{kx}e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx}+i\dfrac{q_{kx}}{\mathrm{w}_{kx}}\right)e^{i\chi_{kx}} \\ \mathrm{w}_{ky}e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky}+i\dfrac{q_{ky}}{\mathrm{w}_{ky}}\right)e^{i\chi_{ky}} \end{bmatrix}; \text{ or for x-}\tau\text{coupling } Y_k(s)=\begin{bmatrix} \mathrm{w}_{kx}e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx}+i\dfrac{q_{kx}}{\mathrm{w}_{kx}}\right)e^{i\chi_{kx}} \\ \mathrm{w}_{k\tau}e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau}+i\dfrac{q_{k\tau}}{\mathrm{w}_{k\tau}}\right)e^{i\chi_{k\tau}} \end{bmatrix}; k=1,2 \tag{80}$$

with conditions

$$Y_k^{\,T}SY_j=0;\quad Y^{*}{}_j^{\,T}SY_k=2i\delta_{kj}; \tag{81}$$

resulting in partial conditions

$$\begin{aligned} &q_{kx}+q_{ky}=1;\ k=1,2\rightarrow q_{1x}=q_{2y}=q;\ q_{2x}=q_{1y}=1-q; \\ &or \\ &q_{kx}+q_{k\tau}=1;\ k=1,2\rightarrow q_{1x}=q_{2\tau}=q;\ q_{2x}=q_{1\tau}=1-q. \end{aligned} \quad (82$$

In the case of degenerated emittances, we can make one of the elements in Y_1 zero. Since we also have flexibility to numerate eigen vectors (e.g. 1<->2), we can decide to zero w_{1y}, which makes

$$q_{2x}=q_{1y}=0$$

The last equation is nothing else but conservation of phase space projection (including sign! – q can be negative or larger than 1!) on two 1D phase spaces for each oscillator – you may still remember one of Poincare invariants:

$$\begin{aligned} &\sum_{i=1}^{n}\iint dq_i dP^i=\iint dxdP^x+\iint dydP^y=inv \\ &or \\ &\sum_{i=1}^{n}\iint dq_i dP^i=\iint dxdP^x+\iint d\tau dP^\tau=inv \end{aligned} \tag{83}$$

In 3D case has 15 invariants of motion

$$Y_k(s) = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + i\dfrac{q_{kx}}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + i\dfrac{q_{ky}}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{w}_{k\tau} e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau} + i\dfrac{q_{k\tau}}{\mathrm{w}_{k\tau}} \right) e^{i\chi_{k\tau}} \end{bmatrix}; k = 1,2,3 \tag{84}$$

$$Y_k^{\ T} S Y_j = 0; \quad Y^{*}{}_j^{\ T} S Y_k = 2i\delta_{kj}; \tag{85}$$

or

$$\begin{gathered} q_{kx} + q_{ky} + q_{k\tau} = 1; \ k = 1,2,3 \\ \sum_{k=1}^{3} q_{kx} = \sum_{k=1}^{3} q_{ky} = \sum_{k=1}^{3} q_{k\tau} = 1; \end{gathered} \tag{87}$$

six condition only 5 of which are independent. It leaves 4 independent parameters in (87).

In 3D case we have following well-know Poincaré invariants:

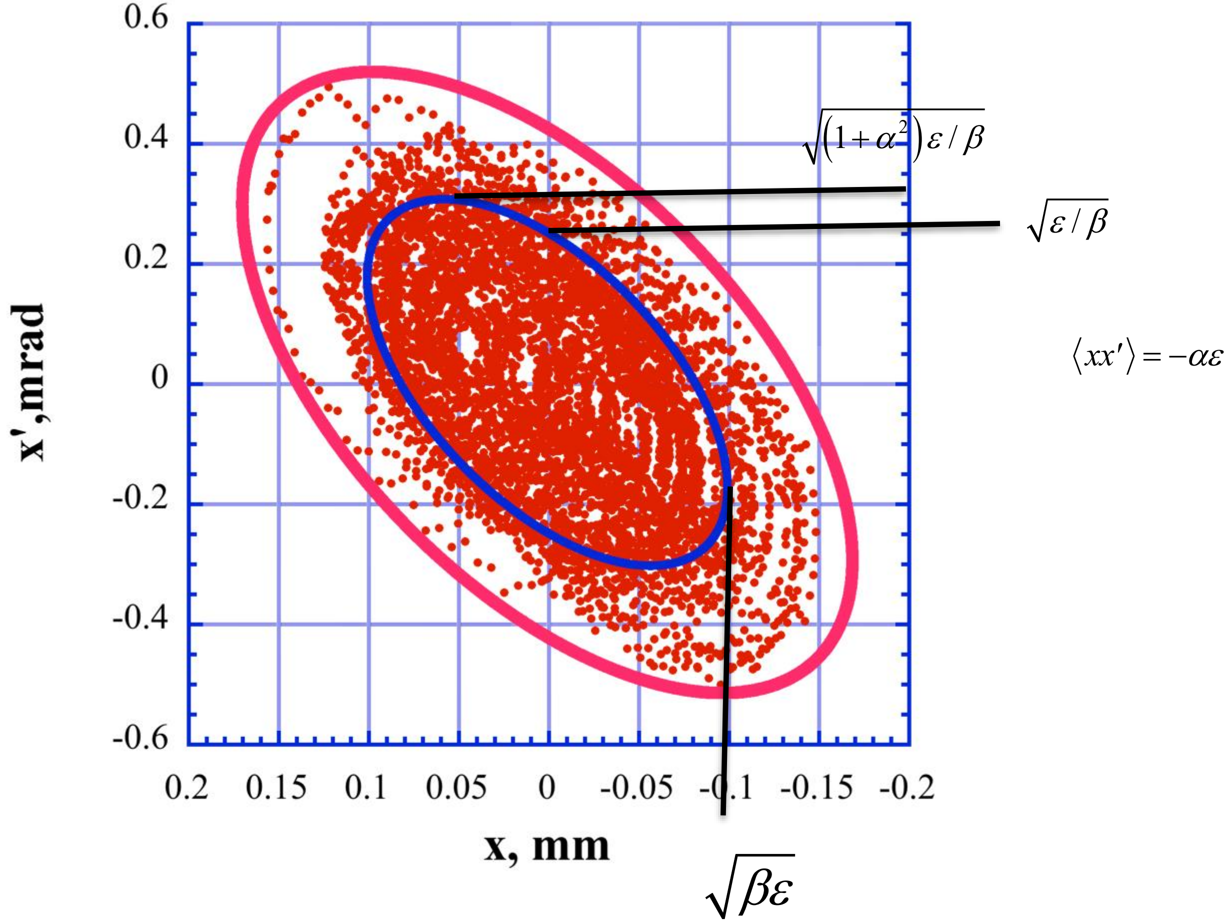
$$\sum_{i=1}^{n} \iint dq_i dP^i = \iint dx dP^x + \iint dy dP^y + \iint d\tau dP^\tau = inv$$

$$\sum_{i \neq j} \iint \iint dq_i \, dP^i \, dq_j \, dP^j = \tag{88}$$

$$\iint \iint dx \, dP^x \, dy \, dP^y + \iint \iint dx \, dP^x \, d\tau \, dP^\tau + \iint \iint d\tau \, dP^\tau \, dy \, dP^y = inv$$

e.g. conservation is sum of projections.

Trivial example: x-x' distribution defines the ellipse and emittance of the 1D uncouple motion.



We found n independent invariants – eigen emittances for n-dimensional linear Hamiltonian system.

$$I_2^{(n)}\left(\left[\mathbf{X}^{(2)}\right]\right)=tr\left[(\Xi\mathbf{S})^n\right]=I_2^{(n)}\left(\varepsilon_1,\ldots,\varepsilon_n\right);$$

$$\Xi\mathbf{S}=\mathbf{S}\Xi\Leftrightarrow\begin{bmatrix}\ldots & 0 & 0\\ 0 & \varepsilon_k I_k & 0\\ 0 & 0 & \ldots\end{bmatrix}\begin{bmatrix}\ldots & 0 & 0\\ 0 & \sigma & 0\\ 0 & 0 & \ldots\end{bmatrix}=\begin{bmatrix}\ldots & 0 & 0\\ 0 & \sigma & 0\\ 0 & 0 & \ldots\end{bmatrix}\begin{bmatrix}\ldots & 0 & 0\\ 0 & \varepsilon_k I_k & 0\\ 0 & 0 & \ldots\end{bmatrix}; \tag{89}$$

$$I_2^{(2n)}\left(\left[\mathbf{X}^{(2)}\right]\right)=tr\left[(\Xi\mathbf{S})^{2n}\right]=tr\left[\Xi^{2n}\mathbf{S}^{2n}\right]=(-1)^n tr\left[\Xi^{2n}\right]=2(-1)^n\sum_{k=1}^{n}\varepsilon_k{}^{2n}$$

Where are the rest of *n(2n-1) - n* invariants (12 of 15 in 3D case) of motion?

Higher order invariants

Invariants made of a fixed order moments are called pure. Mixed invariants can be constructed from moments of various orders.

1. Pure invariants. Let's consider following quantities

$$I_{2m}^{(n)}\left(\mathbf{X}^{(2m)}\right)=tr\left[\left\{\mathbf{X}^{(2m)}\left(\overset{m}{\otimes}\mathbf{S}\right)\right\}^n\right] \tag{90}$$

and

$$I_{2m+1}^{(2n)}\left(\mathbf{X}^{(2m+1)}\right)=tr\left[\left\{\mathbf{X}^{(2m+1)}\left(\overset{m}{\otimes}\mathbf{S}\right)\mathbf{X}^{(2m+1)}\left(\overset{m+1}{\otimes}\mathbf{S}\right)\right\}^n\right] \tag{91}$$

Examples on such quantities with index summations is

$$tr\left[\left\{\mathbf{X}^{(4)}\left(\overset{2}{\otimes}\mathbf{S}\right)\right\}^2\right]=\mathbf{X}^{(4)}{}_{i1,i2,i3,i4}\mathbf{S}_{i3,k3}\mathbf{S}_{i4,k4}\mathbf{X}^{(4)}{}_{k1,k2,k3,k4}\mathbf{S}_{k1,i1}\mathbf{S}_{k2,i2} \tag{92}$$

$$tr\left[\left\{\mathbf{X}^{(3)}\left(\overset{1}{\otimes}\mathbf{S}\right)\mathbf{X}^{(3)}\right\}\left(\overset{2}{\otimes}\mathbf{S}\right)^2\right]=\mathbf{X}^{(3)}{}_{i1,i2,i3,}\mathbf{S}_{i3,k3}\mathbf{X}^{(3)}{}_{k1,k2,k3,}\mathbf{S}_{k1,i1}\mathbf{S}_{k2,i2} \tag{93}$$

 Naturally, summation is assumed for any repeated index…

Need to prove that $I_{2m}^{(n)}\left(\mathbf{X}^{(2m)}\right)$ and $I_{2m+1}^{(2n)}\left(\mathbf{X}^{(2m+1)}\right)$ are kinematic invariants under symplectic transformations

$$\mathbf{X}^{(k)} \to \left(\overset{k}{\otimes}\mathbf{M}\right)\mathbf{X}^{(k)} \tag{94}$$

Then

$$I_{2m}^{(n)}\left(\left(\overset{2m}{\otimes}\mathbf{M}\right)\mathbf{X}^{(2m)}\right) = tr\left(\left\{\left(\overset{2m}{\otimes}\mathbf{M}\right)\mathbf{X}^{(2m)}\left(\overset{m}{\otimes}\mathbf{S}\right)\right\}\right)^{n} =$$
$$tr\left(\left\{\mathbf{X}^{(2m)}\left(\overset{m}{\otimes}\mathbf{M}^{T}\mathbf{S}\mathbf{M}\right)\right\}\right)^{n} = I_{2m}^{(n)}\left(\mathbf{X}^{(2m)}\right) \tag{95}$$

To prove this one need to use the definitions bellow and tr(ABC)=tr(BCA)

$$\left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle^{f} = \int f^{i}(\mathbf{M}^{-1}X)x_{i_1}x_{i_2}\cdots x_{i_k}\,dX;$$
$$X = \mathbf{M}\breve{X} \to \left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle^{f} = \int f^{i}(\breve{X})\left(\mathbf{M}\breve{x}\right)_{i_1}\left(\mathbf{M}\breve{x}\right)_{i_2}\cdots\left(\mathbf{M}\breve{x}\right)_{i_k}\,d\breve{X} \tag{96}$$
$$\left\langle x_{i_1}x_{i_2}\cdots x_{i_k}\right\rangle^{f} = m_{i_1 j_1}m_{i_2 j_2}\cdots m_{i_k j_k}\left\langle x_{j_1}x_{j_2}\cdots x_{j_k}\right\rangle^{i}$$

The proof is based at the same idea that we used to prove invariance of second order invariants:

$$(\mathbf{M}\otimes\mathbf{M})\,\mathbf{X}^{(2)} = \mathbf{M}\mathbf{X}^{(2)}\mathbf{M}^T;\; tr[\mathbf{ABC}] = tr[\mathbf{BCA}];$$

$$\mathbf{X}^{(2)} = (\mathbf{M}\otimes\mathbf{M})\,\mathbf{X}^{(2)} \to I_2^{(n)}\left(\left[(\mathbf{M}\otimes\mathbf{M})\,\mathbf{X}^{(2)}\right]\right) = tr\left[\left(\mathbf{M}\mathbf{X}^{(2)}\mathbf{M}^T\mathbf{S}\right)^n\right] = \tag{97}$$

$$tr\left[\left(\mathbf{X}^{(2)}\mathbf{M}^T\mathbf{S}\mathbf{M}\right)^n\right] = tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^n\right] = I_2^{(n)}\left(\left[\ \mathbf{X}^{(2)}\right]\right),$$

by observing that

$$\left(\mathbf{M}^T\mathbf{S}\mathbf{M}\right)_{ij} \equiv m_{ki}\mathbf{S}_{kl}m_{lj} = \mathbf{S}_{ij} \tag{98}$$

and use it on any combination appeared in (92) or (93) when we add symplectic transformation. The last step is to apply rather obvious: tr(ABC)=tr(BCA)

$$tr[ABC] = \sum_{i,k,j} a_{ik}b_{kj}c_{ji} \equiv \sum_{i,k,j} b_{kj}c_{ji}a_{ik} = tr[BCA] \tag{99}$$

which makes necessary one more application of (98) and

$$\mathbf{S}^2 = -\mathbf{I},\; \mathbf{S}_{ij}\mathbf{S}_{jk} = -\delta_{ik} \tag{100}$$

throughout this tedious, but other ways starlight forward exercise.

Similar technique gives:

$$I_{2m+1}^{(2n)}\left(\left(\overset{2m+1}{\otimes}\mathbf{M}\right)\mathbf{X}^{(2m+1)}\right) = I_{2m+1}^{(2n)}\left(^{(2m+1)}\right) \tag{101}$$

Not all of these invariants are useful:

$$
\begin{aligned}
&(a)\ I^{(2n+1)}_{2(2m+1)} = 0\,; \\
&(b)\ I^{(1)}_{4} = 0; \\
&(c)\ I^{(2(2n+1))}_{2m+1} = 0.
\end{aligned}
\tag{102}
$$

Conditions (a) and (c) are result from fact that $\mathbf{X}^{(2m)}$ is symmetric tensor relative to all indices and asymmetric $\mathbf{S}$ appears in odd power. Case (b) is more interesting because it has even number of $\mathbf{S}$ but its indices summed against symmetric indices of $\mathbf{X}^{(2m)}$:

$$\sum_{i,k} x_i x_k \mathbf{S}_{ik} \equiv 0$$

Some of non-zero high order invariants for 1D case are shown bellow

$$
\begin{aligned}
I^{(4)}_{3}\left(\mathbf{X}^{(3)}\right) = {}& \langle q_1^3\rangle^2 \langle p_1^3\rangle^2 - 3\langle q_1^2 p_1\rangle^2 \langle q_1 p_1^2\rangle^2 + 4\langle q_1^3\rangle \langle q_1 p_1^2\rangle^3 \\
& + 4\langle q_1^2 p_1\rangle^3 \langle p_1^3\rangle - 6\langle q_1^3\rangle \langle q_1^2 p_1\rangle \langle q_1 p_1^2\rangle \langle p_1^3\rangle;
\end{aligned}
\tag{103}
$$

$$
I^{(2)}_{4}\left(\mathbf{X}^{(4)}\right) = \langle q_1^4\rangle \langle p_1^4\rangle + 3\langle q_1^2 p_1^2\rangle^2 - 4\langle q_1^3 p_1\rangle \langle q_1 p_1^3\rangle
\tag{104}
$$

and

$$
\begin{aligned}
I^{(3)}_{4}\left(\mathbf{X}^{(4)}\right) = {}& \langle q_1^4\rangle \langle p_1^4\rangle \langle q_1^2 p_1^2\rangle - \langle q_1^4\rangle \langle q_1 p_1^3\rangle^2 - \langle q_1^2 p_1^2\rangle^3 \\
& - \langle q_1^3 p_1\rangle^2 \langle p_1^4\rangle + 2\langle q_1^3 p_1\rangle \langle q_1 p_1^3\rangle \langle q_1^2 p_1^2\rangle;
\end{aligned}
\tag{104}
$$

and it clearly indicates the complexity of them as well removing desire to calculate them for 3D case…

There is alternative derivation of these invariants using properties of Lie algebras, which we plan to learn about later in the course.

Now, just few words about **mixed invariants**: they are build from W-blocks as follows:

$$I_{m_1,\ \ldots m_k}^{(n_1,\ \ldots n_k)} = tr\left[\left(W^{(m_1)}\right)^{n_1} \cdots \left(W^{(m_k)}\right)^{n_k}\right]; W^{(m)} = \mathbf{X}^{(2m)}\left(\overset{m}{\otimes}\mathbf{S}\right); \tag{105}$$

If m_j is odd, the mixed invariant is zero, unless corresponding n_j is even. Also, they are zero unless $\sum_{j=1}^{k} n_j m_j = 4N$. An example of mixed invariants:

$$I_{1,2}^{(2,1)} = \left\langle q_1^2 \right\rangle \left\langle p_1 \right\rangle^2 - 2\left\langle q_1 p_1 \right\rangle \left\langle q_1 \right\rangle \left\langle p_1 \right\rangle + \left\langle p_1^2 \right\rangle \left\langle q_1 \right\rangle^2 \tag{106}$$

which is non-zero for off-center beam with non-zero $\left\langle p_1 \right\rangle$ or/and $\left\langle q_1 \right\rangle$.

Graphic representation of invariants: each node represents $\mathbf{X}^{(k)}$, where k is number of line emanating from this node. Each line connects represents none-zero invariant.

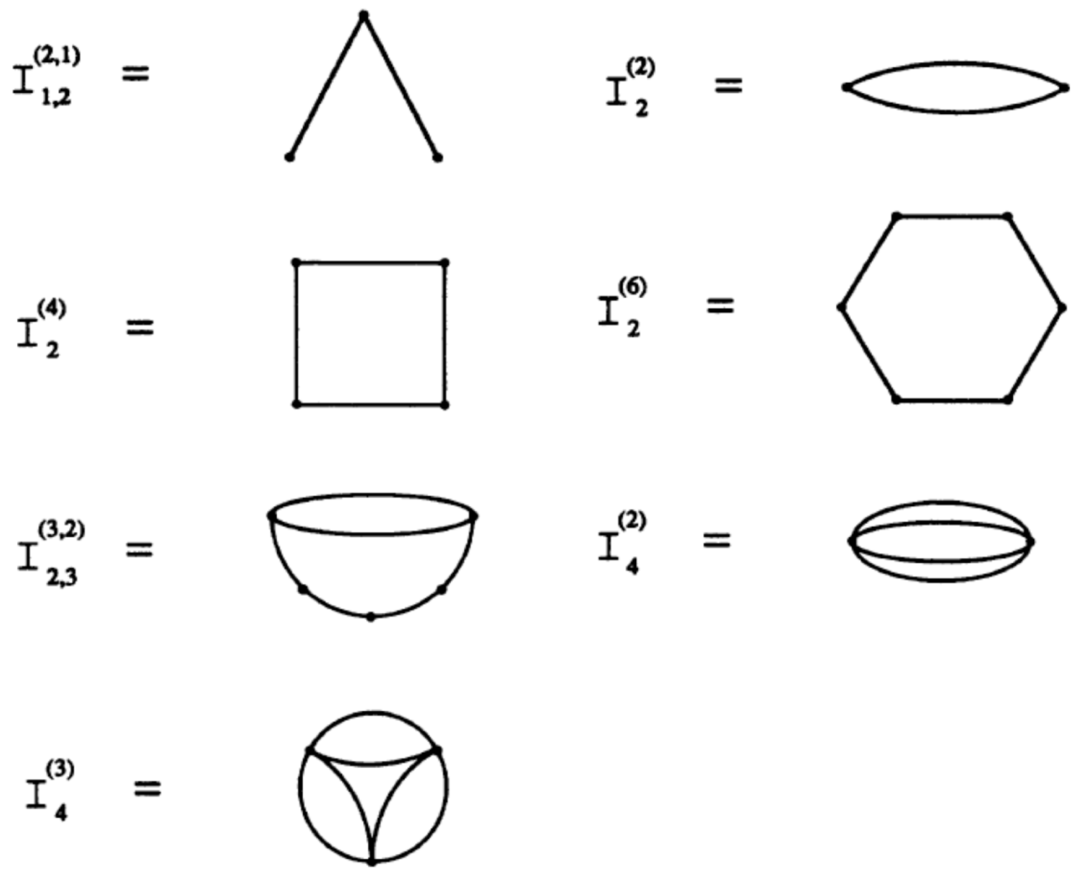


FIG. 1. Diagrammatic representation of moment invariants.

What we learned in 2 classes

- We studies some of best-known kinematic invariants of motion in linear Hamiltonian systems – eigen "RMS" emittances
- We define classes of invariants, including those coming from quadratic form (Σ-matrix) of phase space particles positions
- We found eigen "RMS" emittances them by transforming the quadratic form (Σ-matrix) using a symplectic transformation Θ in the phase space to positively defined double-degenerated diagonal matrix
- The diagonal terms are nothing else that eigen emittances which are invariants of motion
- We than compared our finding with parameterization we used for the describing particles motion – using a Gaussian distribution we got for a storage ring with synchrotron radiation - and found relation between the parameterization and the symplectic matrix Θ : $\mathbf{O} = \left[\ldots \mathrm{Re}\, Y_k, \mathrm{Im}\, Y \ldots \right] = \left(\Theta^T \right)^{-1} == -\mathbf{S}\Theta\mathbf{S}$
- This provided us with additional way of determining parameterization of particle's motion in any piece of accelerator, not only in period systems
- We gave a brief look into algebra of higher order forms and corresponding invariants but stopped short of determining how many of them are independent.