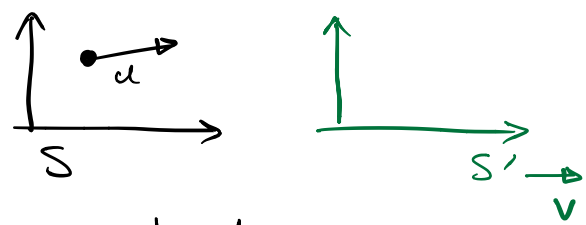
Relativistic Mechanics

Energy and momentum

Transforming velocity from one frame to another

* object has velocity u in frame S



* frame S' is moving with velocity v wrt frame S in $+x$ direction

$$u_x' = \frac{u_x - v}{1 - \frac{uv}{c^2}}$$

for $i = y, z$:
$$u_i' = \frac{u_i}{\gamma\left(1 - u_x \frac{v}{c^2}\right)}$$

Particle momentum: $\bar{p} = \gamma m \bar{u}$ (bar indicates vector)

Particle Energy: $E = \gamma m c^2$

Relations between energy & momentum:

$$E^2 = p^2c^2 + (mc^2)^2$$

$$\frac{E}{p} = \frac{\gamma m c^2}{\gamma m u} = \frac{c^2}{u}$$

normalized relation of p & γ:

$$E^2 = p^2c^2 + (mc^2)^2$$

$$(\gamma m c^2)^2 = p^2c^2 + (mc^2)^2$$

$$\boxed{\gamma^2 = \left(\frac{p}{mc}\right)^2 + 1}$$

e.g. e^- has rest mass $mc^2 = 0.511$ MeV

for particle with $\gamma = 400$,

$p^2 = (400)^2 \cdot (0.511)^2 \, \text{MeV}^2/c^2 \; \cancel{-1}$ ignore

$$p = 400 \cdot 0.511 \text{ MeV}/c$$

$$\boxed{p \cong 200 \text{ MeV}/c}$$

$$E = \gamma m c^2 = 400 \cdot 0.511 \text{ MeV} \cong \boxed{200 \text{ MeV}}$$

Rule of thumb

at high energy, e^- energy & momentum are half the value of γ.

Conservation of energy & momentum example

Lambda (Λ) particle decays to proton (p) & pion (π)

If the lambda particle was initially at rest, what is the energy of proton and pion? $m_\Lambda = 1116 \text{ MeV}/c^2$, $m_p = 938 \text{ MeV}/c^2$, $m_\pi = 140 \frac{\text{MeV}}{c^2}$

$$\Lambda \rightarrow p + \pi$$

$$\sum \bar{P}_i = \sum \bar{P}_f$$

$$0 = \bar{P}_p + \bar{P}_\pi \Rightarrow \quad P_p = P_\pi = P$$

$$\sum E_i = \sum E_f$$

$$E_\Lambda^2 = \underset{0}{\cancel{P_\Lambda^2}} c^2 + (m_\Lambda c^2)^2 \Rightarrow E_\Lambda = m_\Lambda c^2$$

$$E_p^2 = P_p^2 c^2 + (m_p c^2)^2 \Rightarrow E_p^2 = p^2 c^2 + (m_p c^2)^2$$

$$E_\pi^2 = P_\pi^2 c^2 + (m_\pi c^2)^2 \Rightarrow E_\pi^2 = p^2 c^2 + (m_\pi c^2)^2$$

$$E_\Lambda = E_p + E_\pi$$

$$m_\Lambda c^2 - E_p = E_\pi$$

$$\Rightarrow (m_\Lambda c^2)^2 + E_p^2 - 2 E_p m_\Lambda c^2 = E_\pi^2$$

$$\Rightarrow (m_\Lambda c^2)^2 + \cancel{p^2}c^2 + (m_p c^2)^2 - 2E_p m_\Lambda c^2 = \cancel{p^2c^2} + (m_\pi c^2)^2$$

$$\therefore \; +2E_p m_\Lambda c^2 = (m_\Lambda c^2)^2 + (m_p c^2)^2 - (m_\pi c^2)^2$$

$$\therefore E_p = \frac{(m_\Lambda c^2)^2 + (m_p c^2)^2 - (m_\pi c^2)^2}{2m_\Lambda c^2} = \frac{1116^2 + 938^2 - 140^2}{2 \cdot 1116} = \underline{\underline{943 \text{ MeV}}}$$

$$E_\pi = m_\Lambda c^2 - E_p = 172.59 \text{ MeV}$$