Homework 12

Problem 1. 10 points, 2D distribution function and RMS beam sizes

For the case of fully coupled transverse oscillations with eigen vectors

$$Y_1 = \begin{bmatrix} w_{1x} e^{i\varphi_{1x}} \\ \left(u_{1x} + i\frac{q}{w_{1x}} \right) e^{i\varphi_{1x}} \\ w_{1y} e^{i\varphi_{1y}} \\ \left(u_{1y} + i\frac{1-q}{w_{1y}} \right) e^{i\varphi_{1y}} \end{bmatrix}; \quad Y_2 = \begin{bmatrix} w_{2x} e^{i\varphi_{2x}} \\ \left(u_{2x} + i\frac{1-q}{w_{2x}} \right) e^{i\varphi_{2x}} \\ w_{2y} e^{i\varphi_{2y}} \\ \left(u_{2y} + i\frac{q}{w_{2y}} \right) e^{i\varphi_{2y}} \end{bmatrix}$$

and known values of eigen emittances $\varepsilon_{1,2} \equiv I_{1,2} = \frac{\langle a_{1,2}^2 \rangle}{2}$ of stationary Gaussian distribution (solution of Fokker-Plank equation)

(a) **6 points**; Write explicit expression for the distribution function in terms of x, Px, y and Py.

(b) **4 points**; Write expression of the RMS beam sizes

$$\sigma_x = \sqrt{< x^2 >}; \sigma_y = \sqrt{< y^2 >}$$

using beam emittances and necessary components of eigen vectors.

Problem 2. 10 points, 3D distribution function and RMS beam sizes

(a) 5 points: For the case of fully coupled transverse oscillations with eigen vectors

$$Y_k(s) = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + i\frac{q_{kx}}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + i\frac{q_{ky}}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{w}_{k\tau} e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau} + i\frac{q_{k\tau}}{\mathrm{w}_{k\tau}} \right) e^{i\chi_{k\tau}} \end{bmatrix}; k = 1,2,3$$

and known values of eigen emittances $\varepsilon_k \equiv I_k = \frac{\langle a_k^2 \rangle}{2}; k = 1,2,3$ of stationary Gaussian distribution, write expression of the RMS beam sizes

$$\sigma_x = \sqrt{< x^2 >}; \sigma_y = \sqrt{< y^2 >}; \sigma_\tau = \sqrt{< \tau^2 >}$$

using the beam emittances and necessary components of eigen vectors.

(b) 5 points: For the case of slow synchrotron oscillations and approximate expressions for the eigen vectors:

$$Y_k = \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + \dfrac{iq_k}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + \dfrac{i(1-q_k)}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{y}_{k\tau} = \eta^T S Y_{k\beta} \\ 0 \end{bmatrix}; k = 1,2;\ Y_\delta = \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix} = \begin{bmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \\ \chi_\tau \\ 1 \end{bmatrix};$$

and known values of eigen emittances $\varepsilon_k \equiv I_k = \dfrac{\langle a_k^2 \rangle}{2}; k = 1,2$ and RMS values of the relative energy spread $\sigma_\delta = \sqrt{\langle \delta^2 \rangle}$ write expressions for transverse beam sizes:

$$\sigma_x = \sqrt{< x^2 >}; \sigma_y = \sqrt{< y^2 >}$$

Problem 3. 20 points. 1D emittance

For an ensemble or a distribution function of particles 1D geometrical emittance is defined as

$$\varepsilon_y^{\ 2} = \langle y^2 \rangle \langle y'^2 \rangle - \langle yy' \rangle^2;$$

$$\langle g(y,y') \rangle = \frac{\sum_{n=1}^{N_p} g(y_n, y'_n)}{N_p} = \int f(y,y') g(y,y')\, dy\, dy';$$

1. **5 points**: Show that the emittance is invariant to a Canonical linear (symplectic matrix) transformation of

$$\begin{bmatrix} y \\ y' \end{bmatrix} = M \begin{bmatrix} y_1 \\ y'_1 \end{bmatrix}$$

Note: use the fact that $\varepsilon_y^{\ 2} = \det \Sigma; \Sigma = \begin{bmatrix} \langle y^2 \rangle & \langle yy' \rangle \\ \langle yy' \rangle & \langle y'^2 \rangle \end{bmatrix}$; and find transformation rule for the Σ matrix.

2. **15 points**: For one-dimensional betatron (y) distribution find components of eigen vector $\mathbf{w}_y$ and $\mathbf{w'}_y$ generating a given (positively defined)

$$\Sigma = \begin{bmatrix} \langle y^2 \rangle & \langle yy' \rangle \\ \langle yy' \rangle & \langle y'^2 \rangle \end{bmatrix};$$

This operation is called matching the beam into the beam-line optics.