Homework 12.

Problem 1. 20 points. A weak transverse coupling.

***** STAR part - 50 points**

Consider a fully uncoupled x and y betatron motion in a storage ring with circumference *C*:

$$\tilde{h}_o = \frac{\pi_1^2 + \pi_3^2}{2} + f(s)\frac{x^2}{2} + g(s)\frac{y^2}{2}$$

described by eigen vectors:

$$\mu_{x,y} = 2\pi Q_{x,y};\ \ Y_x(s) = \begin{bmatrix} \mathrm{w}_x \\ \mathrm{w}'_x + \dfrac{i}{\mathrm{w}_x} \\ 0 \\ 0 \end{bmatrix}; Y_y(s) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{w}_y \\ \mathrm{w}'_y + \dfrac{i}{\mathrm{w}_y} \end{bmatrix}$$

The eigen vectors and tunes are considered to be known. Introduce a week coupling by SQ-quadrupole and solenoidal fields (for torsion equal zero):

$$\delta\tilde{h} = \delta f\frac{x^2}{2} + \delta n\cdot xy + \delta g\frac{y^2}{2} + \delta L\left(x\pi_3 - y\pi_1\right)$$

with

$$\delta n(s) = \frac{e}{2p_o c}\left[\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\right]; \delta L(s) = \frac{e\delta B_s}{2p_o c}; \delta f(s) = \delta g(s) = \delta L^2(s);$$

(a) Write explicitly expressions for new betatron tunes using our developed perturbation method. Show that there is linear term on $\delta n, \delta L$ only in case of coupling resonance when $\mu_x = \pm\mu_y + 2\pi m$.

(b) For the case $\mu_x \neq \mu_y$ write expressions for new Eigen vectors perturbation method developed in class. Normalize them symplecticly.

Solution - use perturbation theory from lecture 18, **Sample III:**

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Here is a short re-collection:

$$\frac{dX}{ds} = \left(\mathbf{D}(s) + \varepsilon\mathbf{D}_1(s)\right)\cdot X = \left(\mathbf{SH}(s) + \varepsilon\mathbf{SH}_1(s)\right)\cdot X$$

$$\frac{d\tilde{Y}_k(s)}{ds} = \mathbf{D}(s)\tilde{Y}_k(s); k = 1,..,n.$$

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k} \left(a_{kj}\tilde{Y}_j + b_{kj}\tilde{Y}_j^* \right) + O(\varepsilon^2);\ k = 1,...,n$$

$$\tilde{Y}_{1k}^{\,*} = \tilde{Y}_k^{\,*} e^{-i\delta\phi_k} + \varepsilon c^*{}_k \tilde{Y}_k + \varepsilon \sum_{j\neq k} \left(a^*{}_{kj}\tilde{Y}_j^* + b^*{}_{kj}\tilde{Y}_j \right) + O(\varepsilon^2);$$

$$\frac{d\tilde{Y}_{1k}}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right)\cdot \tilde{Y}_{1k} + o(\varepsilon^2);$$

leads to

$$\delta\phi_k' \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k' \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k} \left(a_{kj}'\tilde{Y}_j + b_{kj}'\tilde{Y}_j^* \right) = \varepsilon \mathbf{D}_1(s) \tilde{Y}_k e^{i\delta\phi_k}$$

and symplectic orthogonality of the eigen vectors

$$\tilde{Y}_k^{\,*} S \tilde{Y}_j = -\tilde{Y}_k S \tilde{Y}_j^* = 2i\delta_{ik};\ \tilde{Y}_k S \tilde{Y}_j = \tilde{Y}_k^{\,*} S \tilde{Y}_j^{\,*} = 0$$

multiplying by $\tilde{Y}_m^{\,*}S$ or $\tilde{Y}_m S$ from the left yields:

$$-2\delta\phi_k' = \varepsilon \tilde{Y}^*_k \mathbf{S}\mathbf{D}_1(s)\tilde{Y}_k \rightarrow \delta\phi' = \frac{\varepsilon}{2} Y_k^{\,*T}\mathbf{H}_1(s)Y_k;\quad \mathbf{S}\mathbf{D}_1 = -\mathbf{H}_1;$$

$$-2ic' = \tilde{Y}^T_k S\mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \rightarrow c' = \frac{1}{2i} Y_k^{\,T}\mathbf{H}_1(s)Y_k e^{i(2\psi_k+\delta\phi_k)} \cong \frac{1}{2i} Y^T{}_k \mathbf{H}_1 Y_k e^{2i\psi_k}$$

$$2ia_{kj}' = \tilde{Y}^*_j \mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \rightarrow a_{kj}' = \frac{-1}{2i} Y_j^{\,*T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k-\psi_j+\delta\phi_k\right)} \cong \frac{-1}{2i} Y_j^{\,*T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k-\psi_j\right)};\ j\neq k$$

$$-2ib_{kj}' = \tilde{Y}^*_j \mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \rightarrow b_{kj}' = \frac{1}{2i} Y_j^{\,T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k+\psi_j+\delta\phi_k\right)} \cong \frac{1}{2i} Y_j^{\,T}\mathbf{H}_1(s)Y_k e^{i\left(\psi_k+\psi_j\right)};\ j\neq k.$$

$$\delta\phi(s) = \phi_o + \frac{\varepsilon}{2}\int_0^s Y_k^{\,*T}\mathbf{H}_1 Y_k\, d\xi;\ c(s) = c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\,T}\mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)};$$

$$a_{kj} = a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\,*T}\mathbf{H}_1 Y_k e^{i\left(\psi_k-\psi_j+\delta\phi_k\right)}; b_{kj} = b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\,T}\mathbf{H}_1 Y_k e^{i\left(\psi_k+\psi_j+\delta\phi_k\right)};$$

$$\tilde{Y}_{1k} e^{-i(\psi_k+\delta\phi_k)} = Y_k + \varepsilon c_k Y_k^{\,*} e^{-i(2\psi_k+\delta\phi_k)}\left(c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\,T}\mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)} \right) +$$

$$\varepsilon \sum_{j\neq k} \left(\begin{array}{l} Y_j e^{-i\left(\psi_k-\psi_j+\delta\phi_k\right)}\left(a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\,*T}\mathbf{H}_1 Y_k e^{i\left(\psi_k-\psi_j+\delta\phi_k\right)} \right) + \\ Y^*{}_j e^{-i\left(\psi_k+\psi_j+\delta\phi_k\right)}\left(b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\,T}\mathbf{H}_1 Y_k e^{i\left(\psi_k+\psi_j+\delta\phi_k\right)} \right) \end{array} \right) + O(\varepsilon^2)$$

Now we want to have periodic eigen vectors, e.g.

$$\tilde{Y}_{1k}(s+C) = \tilde{Y}_{1k}(s)e^{i\mu_{1k}}; \mu_{1k} = \mu_k + \frac{\varepsilon}{2}\int_0^C Y_k^{\,*T}\mathbf{H}_1 Y_k\, d\xi;$$

we need to choose the initial conditions to make a coefficient looking like:

$$d(s) = e^{-i\theta(s)}\left(d_o - \frac{1}{2i}\int_o^s d\xi f(\xi) e^{i\theta(\xi)}\right);$$

$$\rightarrow \left(d_o + \int_o^s d\xi f(\xi) e^{i\theta(\xi)}\right) = \frac{1}{e^{i\Delta\theta(C)} - 1}\int_o^{s+C} d\xi f(\xi) e^{i\theta(\xi)}.$$

giving us:

$$\tilde{Y}_{1k} e^{-i(\psi_k + \delta\phi_k)} = Y_{1k}(s) = Y_k + \varepsilon \frac{Y_k^{\,*} e^{-i(2\psi_k + \delta\phi_k)}}{2i\left(1 - e^{i(2\mu_k + \delta\mu_k)}\right)} \int_s^{s+C} d\xi Y_k^{\,T} \mathbf{H}_1 Y_k e^{i(2\psi_k + \delta\phi_k)} +$$
$$\varepsilon \sum_{j \neq k} \left(\begin{array}{l} -\dfrac{Y_j e^{i(\psi_j - \psi_k - \delta\phi_k)}}{2i\left(1 - e^{i(\mu_k - \mu_j + \delta\mu_k)}\right)} \displaystyle\int_s^{s+C} d\xi Y_j^{*T} \mathbf{H}_1 Y_k e^{i(\psi_k - \psi_j + \delta\phi_k)} + \\ \dfrac{Y_j^{\,*} e^{-i(\psi_j + \psi_k + \delta\phi_k)}}{2i\left(1 - e^{i(\mu_j + \mu_k + \delta\mu_k)}\right)} \displaystyle\int_s^{s+C} d\xi Y_j^{\,T} \mathbf{H}_1 Y_k e^{i(\psi_k + \psi_j + \delta\phi_k)} \end{array} \right) + O(\varepsilon^2) \qquad (18\text{-}15)$$

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We shall start from writing deviation of the Hamiltonian (desirably in the matrix form):

$$\delta K = \begin{bmatrix} \delta f & 0 & \delta n & \delta L \\ 0 & 0 & -\delta L & 0 \\ \delta n & -\delta L & \delta g & 0 \\ \delta L & 0 & 0 & 0 \end{bmatrix}; \delta D = \begin{bmatrix} 0 & 0 & -\delta L & 0 \\ -\delta f & 0 & -\delta n & -\delta L \\ \delta L & 0 & 0 & 0 \\ -\delta n & \delta L & -\delta g & 0 \end{bmatrix}; \qquad (1)$$

In our case we have only 2D case

$$\mu_{x,y} = 2\pi Q_{x,y};\; Y_x(s) = \begin{bmatrix} \mathrm{w}_x \\ \mathrm{w}'_x + \dfrac{i}{\mathrm{w}_x} \\ 0 \\ 0 \end{bmatrix}; Y_y(s) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{w}_y \\ \mathrm{w}'_y + \dfrac{i}{\mathrm{w}_y} \end{bmatrix}$$

First, let's find tune changes

$$\tilde{Y}_x^{*T} \delta\mathbf{H}(\zeta) \tilde{Y}_x = \mathrm{w}^2{}_x \delta f = \beta_x \cdot \delta L^2$$
$$\tilde{Y}_y^{*T} \delta\mathbf{H}(\zeta) \tilde{Y}_y = \mathrm{w}^2{}_y \delta g = \beta_y \cdot \delta L^2$$
$$\delta Q_{x,y} \cong \frac{1}{4\pi}\int_0^C ds \beta_{x,y} \cdot \delta L^2.$$

to prove that they indeed change only in the second order of δL. If one gets to the next (second order) perturbation – there will be also second order term of δn. Still, no first order chance. A special case of the resonance $\mu_x = \pm\mu_y + 2\pi m$ - we will consider later.

To find perturbed eigen vectors using straight-forward convolutions

$$Y_y^{*T}\delta\mathbf{H}Y_x = \delta n \mathrm{w}_y \mathrm{w}_x + \delta L\left(\mathrm{w}_x \mathrm{w}'_y - \mathrm{w}_y \mathrm{w}'_x - \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x} \right) = \left(Y_x^{*T}\delta\mathbf{H}Y_y\right)^*$$

$$Y_y^{T}\delta\mathbf{H}Y_x = \delta n \mathrm{w}_y \mathrm{w}_x + \delta L\left(\mathrm{w}_x \mathrm{w}'_y - \mathrm{w}_y \mathrm{w}'_x + \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x} \right) = Y_x^{T}\delta\mathbf{H}Y_y$$

$$Y_x^{T}\delta\mathbf{H}Y_x = \left(\mathrm{w}_x\delta L\right)^2; Y_y^{T}\delta\mathbf{H}Y_y = \left(\mathrm{w}_2\delta L\right)^2$$

and dropping the last two, since they are of the second order we have

$$Y_1 \cong Y_x + a_x Y_y + b_x Y_y^*$$

$$Y_2 \cong Y_y + a_y Y_x + b_y Y_x^*$$

$$a_x(s) = \frac{-e^{-i(\psi_x-\psi_y)}}{2i\left(e^{i(\mu_x-\mu_y)}-1\right)} \int_s^{s+C} e^{i(\psi_x-\psi_y)}\left(\delta n\cdot \mathrm{w}_y \mathrm{w}_x + \delta L\left(\mathrm{w}_x \mathrm{w}'_y - \mathrm{w}_y \mathrm{w}'_x - \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x} \right)\right) d\xi$$

$$b_x(s) = \frac{e^{-i(\psi_x+\psi_y)}}{2i\left(e^{i(\mu_x+\mu_y)}-1\right)} \int_o^{C} e^{i(\psi_x+\psi_y)}\left(\delta n\cdot \mathrm{w}_y \mathrm{w}_x + \delta L\left(\mathrm{w}_x \mathrm{w}'_y - \mathrm{w}_y \mathrm{w}'_x + \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x} \right)\right) d\xi$$

$$a_x(s) = \frac{-e^{-i(\psi_y-\psi_x)}}{2i\left(e^{i(\mu_y-\mu_x)}-1\right)} \int_o^{C} e^{i(\psi_y-\psi_x)}\left(\delta n\cdot \mathrm{w}_y \mathrm{w}_x + \delta L\left(-\mathrm{w}_x \mathrm{w}'_y + \mathrm{w}_y \mathrm{w}'_x - \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} - \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x} \right)\right) d\xi$$

$$b_x(s) = \frac{e^{-i(\psi_x+\psi_y)}}{2i\left(e^{i(\mu_x+\mu_y)}-1\right)} \int_o^{C} e^{i(\psi_x+\psi_y)}\left(\delta n\cdot \mathrm{w}_y \mathrm{w}_x + \delta L\left(-\mathrm{w}_x \mathrm{w}'_y + \mathrm{w}_y \mathrm{w}'_x - \mathrm{i}\frac{\mathrm{w}_x}{\mathrm{w}_y} + \mathrm{i}\frac{\mathrm{w}_y}{\mathrm{w}_x} \right)\right) d\xi$$

It is easy to see that attained expression is periodic. We can show that new set is symplectic-orthogonal for a general case - it is not more complicated that for specific case. We just need to re-write (8-15) in a compact form.

$$Y_{1k} = Y_k + \varepsilon\left(c_k Y_k^* + \sum_{j\neq k}\left(a_{kj}Y_j + b_{kj}Y^*{}_j\right)\right);$$

$$Y_{1k}^* = Y_k^* + \varepsilon\left(c^*{}_k Y_k + \varepsilon\sum_{j\neq k}\left(a_{kj}^* Y^*{}_j + b_{kj}^* Y_j\right)\right);$$

With obvious $Y_{1k}^T\mathbf{S}Y_{1k} \equiv 0$ let's check first normal pairs $Y^*{}_{1k}, Y_{1k}$:

$$Y_{1k}^{*T}\mathbf{S}Y_{1k} = Y_k^{*T}\mathbf{S}Y_k + \varepsilon\left(c_k Y_k^{*T}\mathbf{S}Y_k^* + \sum_{j\neq k}\left(a_{kj}Y_k^{*T}\mathbf{S}Y_j + b_{kj}Y_k^{*T}\mathbf{S}Y_j^*\right)\right); Y_k^*$$

$$+\varepsilon\left(c^*{}_k Y_k^T\mathbf{S}Y_k +^* \sum_{j\neq k}\left(a_{kj}^* Y_j^{*T}\mathbf{S}Y_k + b_{kj}^* Y_j\mathbf{S}Y_k\right)\right) + O\left(\varepsilon^2\right) = 2i$$

where all term is red are equal zero. One can also evaluate second order term to be

$$O\left(\varepsilon^2\right) = 2i\varepsilon^2\sum_{j\neq k}\left(\left|a_{kj}\right|^2 - \left|b_{kj}\right|^2\right)$$

Let's now look at $Y_{1k}^T\mathbf{S}Y_{1m}; m\neq k$

$$Y_{1k} = Y_k + \varepsilon\left(c_k Y_k^{\ *} + \sum_{j\neq k}\left(a_{kj} Y_j + b_{kj} Y^*_{\ j}\right)\right);$$

$$Y_{1m} = Y_k + \varepsilon\left(c_m Y_m^{\ *} + \sum_{j\neq k}\left(a_{mj} Y_j + b_{mj} Y^*_{\ j}\right)\right);$$

$$Y_{1k}^{\ T}\mathbf{S}Y_{1m} = Y_k^{\ T}\mathbf{S}Y_m + \varepsilon\left(c_m Y_k^{\ T}\mathbf{S}Y_m^{\ *} + \sum_{j\neq k}\left(a_{mj} Y_k^{\ T}\mathbf{S}Y_j + b_{mj} Y_k^{\ T}\mathbf{S}Y^*_{\ j}\right)\right) +$$

$$\varepsilon\left(c_k Y_k^{\ *T}\mathbf{S}Y_m + \sum_{j\neq k}\left(a_{kj} Y_j^{\ T}\mathbf{S}Y_m + b_{kj} Y_j^{\ *T}\mathbf{S}Y_m\right)\right) + O\left(\varepsilon^2\right);$$

$$Y_k^{\ T}\mathbf{S}Y^*_{\ j} = -2i\delta_{kj}; Y_j^{\ *T}\mathbf{S}Y_m = 2i\delta_{jm}$$

$$Y_{1k}^{\ T}\mathbf{S}Y_{1m} = 2i\varepsilon\left(b_{km} - b_{mk}\right);$$

$$b_{km} = b_{mk} = \frac{e^{-i(\psi_m+\psi_k)}}{2i\left(1-e^{i(\mu_m+\mu_k)}\right)}\int_s^{s+C} d\xi Y_j^{\ T}\mathbf{H}_1 Y_k e^{i(\psi_k+\psi_m)}$$

Again, red term are equal zero and can be dropped. Similar equation work for $Y_{1k}^{\ *T}\mathbf{S}Y_{1m}; m\neq k$

$$Y_{1m}^{\ *T}\mathbf{S}Y_{1k} = 2i\varepsilon\left(a_{mk}^{\ *} - a_{kn}\right);$$

$$a_{km} = a_{mk}^{\ *} = \frac{-e^{i(\psi_m-\psi_{kk})}}{2i\left(1-e^{i(\mu_k-\mu_m)}\right)}\int_s^{s+C} d\xi Y_j^{\ *T}\mathbf{H}_1 Y_k e^{i(\psi_k-\psi_m)}$$

Note, that this is all approximation with error of $O\left(\varepsilon^2\right)$. Naturally, not to have any errors, we should get the transport matrices for the motion and then do everything perfectly – but not analytically!